Khayyam J. Math. 5 (2019) no. 2, 132–138
DOI:10.22034/kjm.2019.88428

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

CONVERGENCE OF OPERATORS WITH CLOSED RANGE

P. SAM JOHNSON[1*] AND S. BALAJI [2]

Communicated by A.K. Mirmostafaee

Abstract. Izumino has discussed a sequence of closed range operators (T_n) that converges to a closed range operator T on a Hilbert space to establish the convergence of $T_n^\dagger \to T^\dagger$ for Moore-Penrose inverses. In general, if $T_n \to T$ uniformly and each T_n has a closed range, then T need not have a closed range. Some sufficient conditions have been discussed on T_n and T such that T has a closed range whenever each T_n has a closed range.

1. Introduction

Many of the concrete applications of mathematics in science and engineering, eventually result in a problem involving operator equations. This problem can be usually represented as an operator equation

$$Tx = y, \tag{1.1}$$

where $T : X \to Y$ is a linear or nonlinear operator (between certain function spaces or Euclidean spaces) such as a differential operator or an integral operator or a matrix. The spaces X and Y are linear spaces endowed with certain norms on them. Solving linear equations with infinitely many variables is a problem of functional analysis, while solving equations with finitely many variables is one of the main themes of linear algebra.

The normed space of all bounded linear operators from a normed space X to a normed space Y is denoted by $\mathcal{B}(X, Y)$. We write $\mathcal{B}(X)$ for $\mathcal{B}(X, Y)$ when $X = Y$. If $T \in \mathcal{B}(X, Y)$, we denote the kernel of T by $N(T)$ and the range of T by $R(T)$. The problem of solving equation (1.1) is well-posed if it asserts existence and

Date: Received: 20 July 2018; Revised: 24 March 2019 Accepted: 27 May 2019.
*Corresponding author.
2010 *Mathematics Subject Classification.* Primary 47A05; Secondary 47B33.
Key words and phrases. Frechet spaces, closed range operators, Moore–Penrose inverses.

uniqueness of a solution of (1.1) and the continuous dependence of the solution on the data y. It is well-known that the problem of solving the operator equation (1.1) is essentially well-posed if $R(T)$ is closed. The study of operators with closed range on Hilbert spaces predominantly appears to pervade the literature dealing with the Moore-Penrose inverse. Closed rangeness of operators has been discussed for restrictions [1], compositions [2, 8, 9], compact perturbations and factorizations [13], and so on. It has been found useful in applications; see [3]. Moreover, a recent research is going on to analyze closed rangeness of operators on Hilbert C^*-modules; see [5, 14].

Let H and K be Hilbert spaces. If $T \in \mathcal{B}(H, K)$ with a closed range, then $T^\dagger$ is the unique linear operator in $\mathcal{B}(K, H)$ satisfying

(1) $TT^\dagger T = T$;
(2) $T^\dagger TT^\dagger = T^\dagger$;
(3) $TT^\dagger = (TT^\dagger)^*$;
(4) $T^\dagger T = (T^\dagger T)^*$.

The operator $T^\dagger$ is called the Moore-Penrose inverse of T. The convergence of closed range operators on a Hilbert space has been discussed in [7] to establish the convergence of Moore-Penrose inverses. If T_n and T have closed ranges and $\|T_n - T\| \to 0$, then the following conditions are equivalent:

(1) $\|T_n^\dagger - T^\dagger\| \to 0$.
(2) $\|T_n T_n^\dagger - TT^\dagger\| \to 0$.
(3) $\|T_n^\dagger T_n - T^\dagger T\| \to 0$.
(4) $\sup \|T_n^\dagger\| < \infty$.

It is well known that an operator T has a closed range if and only if its Moore-Penrose inverse $T^\dagger$ exists. Topological properties of the set of all bounded linear operators between Hilbert spaces with closed range have been studied by considering certain natural metrics on the set. Also using homogeneous structure of closed range operators, several other equivalent conditions are given in [4]. If a sequence (T_n) converges to T uniformly and each T_n has a closed range, then T need not have a closed range in general (see Example 3.1). In section 2, some characterizations for closed range operators between Frechet and Banach spaces are given. The third and final sections of the paper are devoted to find conditions for operators T_n and T between Banach spaces such that T has a closed range whenever each T_n has a closed range.

2. Preliminaries

Banach's closed range theorem [15] states that if X and Y are Banach spaces and if $T \in \mathcal{B}(X, Y)$, then $R(T)$ is closed in Y if and only if $R(T^*)$ is closed in X^*. Some characterizations for operators to have a closed range are given in [6, 10, 11]. In this section, we give characterizations of closed range continuous operators between Frechet spaces and between Banach spaces. A Frechet space is a complete metrizable topological vector space, see [12].

Theorem 2.1. *Let X and Y be Frechet spaces and let $T : X \to Y$ be a continuous linear operator. Then $R(T)$ is closed in Y if and only if, for a given open* *neighborhood U of 0 in X, there is an open neighborhood V of 0 in Y such that, for a given $x \in X$ with $Tx \in V$, there is an element $y \in U$ satisfying $Tx = Ty$.*

Proof. Suppose that $R(T)$ is closed in Y. Write $N = N(T)$ and $X' = X/N$ is a quotient space with the quotient topology. Define $T' : X' \to R(T)$ by $T'(x+N) = T(x)$. Then T' is a one-to-one continuous linear operator from X' onto $R(T)$; hence T'^{-1} is continuous by the open mapping theorem. Let $\pi : X \to X'$ be a quotient mapping.

Now fix an open neighborhood U of 0 in X. Then $\pi(U) = U + N = U'$ (say) is an open neighborhood of $0 + N$ in X'. Then there is an open neighborhood V of 0 in Y such that $R(T) \cap V \subseteq T'(U') = T'(\pi(U)) = T'(U+N) = T(U)$. Thus, for a given $x \in X$ with $Tx \in V$, there is an element $y \in U$ such that $Tx = Ty$. This proves one part.

Conversely assume that, for a given open neighborhood U of 0 in X, there is an open neighborhood V of 0 in Y such that, for a given $x \in X$ with $Tx \in V$, there is $y \in U$ satisfying $Tx = Ty$.

Let (U_n) be a sequence of balanced open neighborhoods of 0 which form a local base at 0 in X such that $U_{n+1} + U_{n+1} \subseteq U_n$ for every n. For each U_n, let us find an open neighborhood V_n of 0 in Y such that if $Tx \in V_n$ for some $x \in X$, then $Tx = Ty$ for some $y \in U_n$. Without loss of generality, we assume that $\{V_n : n = 1, 2, \ldots\}$ is a local base at 0 in Y such that $V_{n+1} + V_{n+1} \subseteq V_n$ for every n.

Fix $y_0 \in \overline{R(T)}$. Find a sequence (x_n') in X such that $Tx_n' \to y_0$ as $n \to \infty$ and $Tx_{n+1}' - Tx_n' \in V_n$ for every n. For every n, find $x_n \in U_n$ such that $Tx_n = Tx_{n+1}' - Tx_n' \in V_n$. Then

$$\begin{aligned}\sum_{n=1}^{m} Tx_n &= (Tx_2' - Tx_1') + (Tx_3' - Tx_2') + \cdots + (Tx_{m+1}' - Tx_m') \\ &= Tx_{m+1}' - Tx_1' \to y_0 - Tx_1' \text{ as } m \to \infty.\end{aligned}$$

Thus $\sum_{n=1}^{\infty} Tx_n$ converges to $y_0 - Tx_1'$. Also for $m < n$, we have

$$\begin{aligned} x_m + x_{m+1} + x_{m+2} + \cdots + x_n &\in U_m + U_{m+1} + U_{m+2} + \cdots + U_{n-1} + U_n \\ &\subseteq U_m + U_{m+1} + \cdots + U_{n-2} + U_{n-1} + U_{n-1} \\ &\subseteq U_m + U_{m+1} + \cdots + U_{n-3} + U_{n-2} + U_{n-2} \\ &\ \ \vdots \\ &\subseteq U_m + U_m \subseteq U_{m-1}. \end{aligned}$$

This proves that $\sum_{n=1}^{\infty} x_n$ converges to x_0, say, in the Frechet space X, and hence $\sum_{n=1}^{\infty} Tx_n$ converges to Tx_0 in Y. Therefore $Tx_0 = y_0 - Tx_1' = \sum_{n=1}^{\infty} Tx_n$, so that $y_0 = Tx_0 + Tx_1' \in R(T)$. This proves that $R(T)$ is closed in Y. □

Corollary 2.2. *Let $T : X \to Y$ be a continuous linear operator from a Frechet space X into a Frechet space Y. Then T has a closed range in Y if and only if* *for every sequence* (y_n) *in* $R(T)$ *that converges to* 0*, there is a sequence* (x_n) *in* X *which also converges to* 0 *such that* $Tx_n = y_n$ *for every* n.

Theorem 2.3. *Let* X *and* Y *be Banach spaces and let* $T \in \mathcal{B}(X, Y)$*. Then* $R(T)$ *is closed in* Y *if and only if there is a constant* $c > 0$ *such that, for given* $x \in X$*, there is an element* $y \in X$ *such that* $Tx = Ty$ *and* $\|y\| \leq c\|Tx\|$.

Proof. Suppose that $R(T)$ is closed in Y. Write $N = T^{-1}(0) = N(T)$ and let $X' = X/N$ be the quotient space with the quotient norm. Define $T' : X' \to R(T)$ by $T'(x + N) = Tx$ for $x \in X$. Then T' is a well defined one-to-one continuous linear operator from X' onto $R(T)$. Therefore, by the open mapping theorem, there exists a constant $c' > 0$ such that $\|x + N\| \leq c'\|T'(x + N)\|$ for every $x \in X$. That is, $\|x + N\| \leq c'\|Tx\|$ for every $x \in X$. Take $c = c' + 1$. Then for given $x \in X$, if $Tx \neq 0$, then there is an element $z \in N$ such that $\|x + z\| \leq \|x + N\| + \|Tx\| \leq c'\|Tx\| + \|Tx\| = c\|Tx\|$. In this case, we take $y = x + z$, so that $\|y\| \leq c\|Tx\|$. If $Tx = 0$, then we take $y = 0$, so that $\|y\| \leq c\|Tx\|$. Thus for given $x \in X$, there is $y \in X$ such that $Tx = Ty$ and $\|y\| \leq c\|Tx\|$.

Conversely assume that, for given $x \in X$, there is $y \in X$ such that $Tx = Ty$ and $\|y\| \leq c\|Tx\|$ for some fixed $c > 0$. Fix $y_0 \in \overline{R(T)}$, the closure of $R(T)$ in Y. Then there is a sequence (x_n) in X such that $\|x_n\| \leq c\|Tx_n\|$ and $\|(y_0 - Tx_1 - Tx_2 - \cdots - Tx_{n-1}) - Tx_n\| \leq \frac{1}{2^{n+2}}$ for every $n = 1, 2, 3, \ldots$. Then $\frac{1}{c}\|x_n\| \leq \|Tx_n\| \leq \|y_0 - Tx_1 - Tx_2 - \cdots - Tx_n\| + \|y_0 - Tx_1 - Tx_2 - \cdots - Tx_{n-1}\| \leq \frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} \leq \frac{1}{2^n}$. Therefore, the series $\sum_{n=1}^{\infty} x_n$ converges to x_0, say, in X and the series $\sum_{n=1}^{\infty} Tx_n$ converges to y_0. Since T is continuous, $\sum_{n=1}^{\infty} Tx_n$ converges to $T(\sum_{n=1}^{\infty} x_n) = Tx_0$. Therefore $y_0 = Tx_0 \in R(T)$. This proves that $R(T)$ is closed in Y. □

3. Main Results

Izumino [7] discusses a sequence of closed range operators (T_n) which converges to a closed range operator T on a Hilbert space to establish the convergence of $T_n^\dagger \to T^\dagger$ for Moore-Penrose inverses. In general, if $\|T_n - T\| \to 0$ and each T_n has a closed range then T need not have a closed range. This section is devoted to find conditions on T_n and T such that T has a closed range whenever each T_n has closed range.

The following example shows that the limit of (T_n) need not have a closed range even each T_n has a closed range and the convergence is uniform.

Example 3.1. Let $X = Y = \ell_2$. Define $T_n : X \to Y$ by

$$T_n(x_1, x_2, x_3, \ldots) = \Big(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots, \frac{x_n}{n}, 0, 0, \ldots\Big)$$

and $T : X \to Y$ by

$$T(x_1, x_2, x_3, \ldots) = \Big(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\Big).$$

Then $\|T_n - T\| \to 0$, each T_n is of finite rank. Hence each T_n is a closed range operator. But the limit T does not have a closed range in Y because T is a compact operator.

Theorem 3.2. *Let $T_n, T \in \mathcal{B}(X,Y)$, where X and Y are Banach spaces. Suppose that $N(T_n) = N(T)$ for all n and that $\|T_n x - Tx\| \to 0$ for every $x \in X$. Suppose that for each $n \in \mathbb{N}$ and given any x, there is some x_n in X such that $T_n x_n = T_n x$ and $\|T_n x\| \geq \gamma \|x_n\|$ for some constant $\gamma > 0$. Then T has a closed range in Y.*

Proof. Without loss of generality, we assume that $N(T_n) = N(T) = \{0\}$, by passing to $X/N(T)$. Fix $x \in X$. If $Tx = 0$, then $x = 0$ and hence $\|Tx\| \geq \frac{\gamma}{2}\|x\|$. Suppose $Tx \neq 0$, so that $x \neq 0$ and find n such that $\|Tx - T_n x\| < \frac{\gamma}{2}\|x\|$. Then, we have $\|Tx\| \geq \|T_n x\| - \|Tx - T_n x\| \geq \gamma\|x\| - \frac{\gamma}{2}\|x\| = \frac{\gamma}{2}\|x\|$. This shows that T also has a closed range in Y. □

Lemma 3.3. *Suppose that X and Y are Banach spaces and that $T_n \in \mathcal{B}(X,Y)$ for all n. Suppose $\|T_n - T_{n+1}\| \to 0$ as $n \to \infty$ and assume that there is a constant $\gamma > 0$ such that for each $n \in \mathbb{N}$ and each $x \in X$, there is some x_n in X such that $T_n x_n = T_n x$ and $\|T_n x\| \geq \gamma\|x_n\|$. Further assume that $R(T_n) \subsetneq R(T_{n+1})$ or $R(T_n) \supsetneq R(T_{n+1})$ for every n. Then there is an integer n_0 such that $R(T_n) = R(T_{n_0})$ for all $n \geq n_0$.*

Proof. If $R(T_n)$ is strictly contained in $R(T_{n+1})$, then by Riesz's lemma, there is an element $x \in X$ such that $\|T_{n+1}x\| = 1$ and $\inf\limits_{y \in X} \|T_n y - T_{n+1}x\| \geq 1 - \frac{1}{2}$. To this x, find $x' \in X$ such that $T_{n+1}x = T_{n+1}x'$ and $\|T_{n+1}x\| \geq \gamma\|x'\|$, so that $\|x'\| \leq \frac{1}{\gamma}$.

Thus, if $R(T_n)$ is strictly contained in $R(T_{n+1})$, then there is an element $x' \in X$ such that $\|T_n x'\| = 1, \|x'\| \leq \frac{1}{\gamma}$ and $\|T_n x' - T_{n+1}x'\| \geq \frac{1}{2}$, so that $\|T_n - T_{n+1}\| \geq \frac{\gamma}{2}$.

Similarly, if $R(T_n)$ is strictly containing $R(T_{n+1})$, then we have $\|T_n - T_{n+1}\| \geq \frac{\gamma}{2}$. Since $\|T_n - T_{n+1}\| \to 0$, there should be an integer n_0 such that $R(T_n) = R(T_{n_0})$ for every $n \geq n_0$. □

Corollary 3.4. *Assume the hypothesis of Lemma 3.3. Suppose that $T \in \mathcal{B}(X,Y)$, $\|T_n x - Tx\| \to 0$ for each $x \in X$, and that each T_n is compact. Then T is compact and T has a closed range in Y.*

Proof. By the previous Lemma 3.3, there is an integer n_0 such that $R(T_n) = R(T_{n_0})$ for all $n \geq n_0$. Since T_{n_0} is compact, $R(T_{n_0})$ is of finite dimension. Since $T_n x \to Tx$ for every $x, R(T) \subseteq R(T_{n_0})$, hence $R(T)$ is of finite dimension. □

Lemma 3.5. *Assume the hypothesis of Lemma 3.3. Suppose that Y is a Hilbert space and that $R(T_n) + R(T_{n+1})$ is closed for each $n \in \mathbb{N}$. Then there is an integer n_0 such that $R(T_n) = R(T_{n_0})$ for all $n \geq n_0$.*

Proof. Suppose that $R(T_n)$ is strictly contained in $R(T_n) + R(T_{n+1})$. Then by Riesz's lemma, there is an element $T_n x_0 + T_{n+1}y_0$ in $R(T_n) + R(T_{n+1})$ such that $\|T_n x_0 + T_{n+1}y_0\| = 1$ and $\|T_n x - (T_n x_0 + T_{n+1}y_0)\| \geq \frac{1}{2}$ for all $x \in X$ and such that $T_{n+1}y_0 \in R(T_n)^{\perp}$. In that case $\|T_{n+1}y_0\| \leq \|T_n x_0 + T_{n+1}y_0\| = 1$.

For this y_0, there is an element z_0 in X such that $T_{n+1}y_0 = T_{n+1}z_0$ and $1 \geq \|T_{n+1}y_0\| \geq \gamma\|z_0\|$. Thus $\|T_{n+1}z_0 - T_n z_0\| \geq \frac{1}{2}$, where $\|z_0\| \leq \frac{1}{\gamma}$. Thus $\|T_{n+1} -$

$T_n\| \geq \frac{\gamma}{2}$, if $R(T_n)$ is strictly contained in $R(T_n)+R(T_{n+1})$. Similarly, one can prove that if $R(T_{n+1})$ is strictly contained in $R(T_n)+R(T_{n+1})$, then $\|T_{n+1}-T_n\| \geq \frac{\gamma}{2}$.

Since $\|T_n - T_{n+1}\| \to 0$, there should be an integer n_0 such that $R(T_n) = R(T_{n+1}) = R(T_n) + R(T_{n+1})$ for every $n \geq n_0$. □

Corollary 3.6. *Assume the hypothesis of Lemma 3.5. Suppose that $T \in \mathcal{B}(X,Y)$, $\|T_nx - Tx\| \to 0$ for each $x \in X$ and that each T_n is compact. Then T is compact and T has a closed range in Y.*

Proof. It is similar to the proof of Corollary 3.4. □

Theorem 3.7. *Let $T_n, T \in \mathcal{B}(X,Y)$, where Y is a Banach space and X is reflexive. Suppose that there is a constant $\gamma > 0$ such that for each $n \in \mathbb{N}$ and $x \in X$, there is an element x_n in X such that $\|T_nx\| \geq \gamma\|x_n\|$ and $T_nx = T_nx_n$. Assume that $\|T_nx - Tx\| \to 0$ for each $x \in X$ and $\|T_n^*f - T^*f\|$ for each $f \in Y^*$. Then T has a closed range in Y.*

Proof. Fix $x \in X$. By assumption, for each $n \in \mathbb{N}$, there is $x_n \in X$ such that $\gamma\|x_n\| \leq \|T_nx\| \leq \|T_nx - Tx\| + \|Tx\|$. Therefore (x_n) is a bounded sequence in X. Since X is reflexive, there is a subnet $(x_{n_\delta})_{\delta\in D}$ of (x_n), which converges weakly to some y in X. Fix $f \in Y^*$ arbitrarily. Since $f \circ T \in X^*$, so $f(Tx_{n_\delta}) \to f(Ty)$. The condition $\sup\limits_{\|x\|\leq 1} |fT_nx - fTx| \to 0$ is equivalent to $\|T_n^*f - T^*f\| \to 0$. Also

$$\begin{aligned} |fTx - fTx_{n_\delta}| &\leq |fTx - fT_{n_\delta}x_{n_\delta}| + |fT_{n_\delta}x_{n_\delta} - fTx_{n_\delta}| \\ &= |fTx - fT_{n_\delta}x| + |fT_{n_\delta}x_{n_\delta} - fTx_{n_\delta}| \to 0. \end{aligned}$$

Thus $fTx_{n_\delta} \to fTx$. Since $f \in Y^*$ is arbitrary, $Tx = Ty$. Moreover $|g(x_{n_\delta})| \to |g(y)|$, for every $g \in X^*$, and hence $\limsup\limits_{n\to\infty} \|x_{n_\delta}\| \geq \|y\|$, which implies that $\gamma\|y\| \leq \|Tx\|$, where $Tx = Ty$. Therefore T has a closed range in Y. □

Example 3.8. Let $X = Y = \ell_2$. Define $T_n : X \to Y$ by

$$T_n(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_n, 0, 0, \ldots).$$

Let $T = I$ be the identity operator. Then $\|T_n - T\|$ does not converge to 0 and $\|T_nx - Tx\| \to 0$ for every $x \in X$, and

$$\sup_{\|x\|\leq 1} |fT_nx - fTx| \to 0,$$

for every $f \in Y^* = \ell_2$. Note that T_n and T have closed ranges in Y.

Example 3.9. Let $X = Y = \ell_1$. Define T_n and T as in Example 3.8. Then $\|T_n - T\|$ does not converge to 0 and $\|T_nx - Tx\| \to 0$ for every $x \in X$, and

$$\sup_{\|x\|\leq 1} |fT_nx - fTx|$$

does not converge to 0 with $f = (1, 1, 1, \ldots) \in Y^* = \ell_\infty$. Here T_n and T have closed ranges in Y, but X and Y are not reflexive.

We propose the following conjecture that is based on the previous examples: Let X and Y be Banach spaces and let $T_n, T \in \mathcal{B}(X,Y)$. Suppose $\|T_nx - Tx\| \to 0$ for every $x \in X$ and $\|T_n^*f - T^*f\| \to 0$ for every $f \in Y^*$. Suppose that there is a constant $\gamma > 0$ such that for each $n \in \mathbb{N}$ and $x \in X$, there is an element $x_n \in X$ such that $\|T_n x\| \geq \gamma \|x_n\|$ and $T_n x = T_n x_n$. Then the range of T is closed in Y.

Acknowledgments. The authors would like to express thanks to the referee for useful and heart-warming suggestions. The present work of the first author was partially supported by National Board for Higher Mathematics (NBHM), Ministry of Atomic Energy, Government of India (Reference No.2/48(16)/2012/ NBHM (R.P.)/R&D 11/9133).

References

1. B.A. Barnes, *Restrictions of bounded linear operators: closed range*, Proc. Amer. Math. Soc. **135** (2007), no. 6, 1735–1740.
2. R. Bouldin, *The product of operators with closed range*, Tohoku Math. J. (2) **25** (1973) 359–363.
3. P. Brunovsky, M. Holecyova, *Optimal control, dichotomy, and closed range*, SIAM J. Control Optim. **56** (2018), no. 2, 633–648.
4. G. Corach, A. Maestripieri, M. Mbekhta, *Metric and homogeneous structure of closed range operators*, J. Operator Theory **61** (2009) 171–190.
5. J. Farokhi-Ostad, A.R. Janfada, *On closed range C^*-modular operators*, Aust. J. Math. Anal. Appl. **15** (2018), no. 2, 1–9.
6. S. Goldberg, *Unbounded Linear Operators: Theory and Applications*, McGraw-Hill, New York, 1966.
7. S. Izumino, *Inequalities on operators with closed range*, Math. Japon, **25** (1980), no. 4, 423–429.
8. S. Izumino, *The product of operators with closed range and an extension of the reverse order law*, Tohoku Math. J. (2) **34** (1982), no. 1, 43–52.
9. P.S. Johnson, C.G. Moorthy, *Composition of closed range operators*, J. Anal. **14** (2006) 79–80.
10. J.T. Joichi, *On closed operators with closed range*, Proc. Amer. Math. Soc. **11** (1960), no. 1, 80–83.
11. S.H. Kulkarni, M.T. Nair, *A characterization of closed range operators*, Indian J. Pure Appl. Math. **31** (2000), no. 4, 353–361.
12. W. Rudin, *Functional Analysis*, McGraw-Hill, New Delhi, 1991.
13. I. Serban, F. Turcu, *Compact perturbations and factorizations of closed range operators*, Acta Sci. Math. (Szeged) **72**(2006), no. 1-2, 345–351.
14. M. Vosough, M.S. Moslehian, Q. Xu, *Closed range and nonclosed range adjointable operators on Hilbert C^*-modules*, Positivity **22** (2018) no. 3, 701–710.
15. K. Yosida, *Functional Analysis*, Narosa Publishing House, New Delhi, 1979.

[1]Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal, Karnataka - 575 025, India.

E-mail address: nitksam@gmail.com

[2]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore, Tamilnadu - 632 014, India.

E-mail address: balajimath@gmail.com