Improved Secant-Updates of Rank 1 in Hilbert Space

Ioannis K. Argyros
Cameron University
Department of Mathematical Sciences
Lawton, OK 73505, USA
iargyros@cameron.edu

Santhosh George
NIT Karnataka
Department of Math. & Computational Sci.
India-575 025
sgeorge@nitk.ac.in

Abstract

Galperin(2013) considered a class of secant-update iterative methods for solving nonlinear operator equations in Hilbert spaces and characterize optimal inverse secant-update of rank 1. In this study we improve the results in Galperin (2013).

1 Introduction

Let $F : D \subseteq \mathbb{X} \longrightarrow \mathbb{X}$ be a nonlinear operator defined on a Hilbert space $\mathbb{X}$ with innerproduct $\langle .,. \rangle$. In this study we consider the equation

$$F(x) = 0 \qquad (1.1)$$

and secant-update methods of rank 1 for solving (1.1).

Recall [6, 7] that in this method, for a given initial pair $(x_0, A_0) \in D \times \mathcal{L}(\mathbb{X})$ ($\mathcal{L}(\mathbb{X})$ is the space of all bounded linear operators on $\mathbb{X}$) with invertible A_0 generate a sequence $(x_n, A_n) \in D \times \mathcal{L}(\mathbb{X})$ according to the rule:

$$x_+ := x - AF(x), \ A_+ := A + B$$

where the update B is a linear operator of rank 1, such that A_+ is invertible and satisfies the equation

$$A_+^{-1}(x_+ - x) = F(x_+) - F(x).$$

The most widely used representation of the secant-update of rank 1 is the Broyden's update [3, 6, 7, 12];

$$B = \frac{AF(x_+)}{\langle y, y \rangle} \langle y, . \rangle, \; y := F(x_+) - F(x).$$

The secant-update method is free of inversion and no linear subproblem has to be solved at each iteration [7] are the advantage of secant-method. The method alos exibits superlinear convergence [7].

Due to its wide applications, finding solutions to (1.1) is an important problem in applied mathematics. Most of the solution methods for (1.1) are iterative, so optimality of iterative methods for solving nonlinear equations is an issue. In [6, 7], Galperin introduced another concept of optimality of an iterative method using the concept of entropy of solution's position within a set of its guaranteed existence and uniqueness. In this concept the merit of a method is measured by how much this entropy is reduced by one iteration. Using this concept, Galperin designed a new iterative method for nonlinear equations with regularly smooth functions in [7]. He further extended the ideas in [11] to study the Secant-type method defined by

$$x_{n+1} = x_n - [x_n, x_{n-1}; F]^{-1} F(x_n), \;\; n = 0, 1, 2, \ldots, \tag{1.2}$$

where x_0, x_{-1} are some initial guess. Here $[., .; F]$ is the divided difference of order one defined by

$$F(x) - F(y) = [x, y; F](x - y).$$

Galperin in [7], proved an existence and uniqueness theorem for method (1.2), when the divided difference of F are Lipschitz continuous on D. Using the idea of restricted convergence domain, we extend the applicability of secant-update method considered in [7].

The paper is structured as follows. In Section 2 we present the existence and uniqueness of solutions . In Section 3 improved optimal Secant updates of rank 1 and a numerical example is given in Section 4.

2   Existence and Uniqueness of solutions

Suppose that the divided difference $[x, x_{-1}; F]^{-1} \in \mathcal{L}(\mathbb{Y}, \mathbb{X})$. Set $F_0 = [x, x_{-1}; F]^{-1} F$ to normalize F. Without loss of generality we assume that F is already normalized, i.e. $F : \Omega \longrightarrow \mathbb{X}, [x, x_{-1}; F] = I$. Let $\gamma_{-1}, \gamma_0, \gamma, \delta_{-1}, \delta_0$ and δ be positive parameters. It is helpfull for the analysis that follows to define functions $\alpha, \beta, \varphi, \psi, \psi_1$ and h on the interval $[0, +\infty)$ by

$$\alpha(t) \;\; := \;\; \|F(x)\| t, \; \beta(t) := \|x - x_{-1}\| t, \tag{2.1}$$

$$\varphi(t) \;\; := \;\; \frac{2\alpha(t)}{\sqrt{(1 + \beta(t))^2 + 4\alpha(t)} + 1 + \beta(t)}, \tag{2.2}$$

$$\psi(t) \;\; := \;\; \frac{2\alpha(t)}{1 - \beta(t) + \sqrt{(1 - \beta(t))^2 - 4\alpha(t)}}, \tag{2.3}$$

$$\psi_1(t) : \;\; = \;\; \frac{2\alpha(t)}{1 - \beta(t) - \sqrt{(1 - \beta(t))^2 - 4\alpha(t)}},$$

$$h(t) \quad := \quad \beta(t) + 2\sqrt{\alpha(t)}, \tag{2.4}$$

parameter r_0 by

$$r_0 := \frac{1 - \gamma \|x_0 - x_{-1}\|}{\gamma_0 + \gamma}, \tag{2.5}$$

and the open ball $U(x_0, r)$ by $U(x, r) := \{x' \in \Omega : \|x' - x\| < r\}$. Denote by $\bar{U}(x, r)$ the closure of $U(x, r)$. Suppose that

$$\gamma \|x - x_{-1}\| < 1. \tag{2.6}$$

Set

$$D_0 = D \cap U(x, r_0) \tag{2.7}$$

and

$$\delta = \max\{\delta_{-1}, \delta_0\}. \tag{2.8}$$

The set D_0 exists by (2.5) and (2.6). The existence and uniqueness result was shown in [?] using the preceding notation.

THEOREM 2.1 *Suppose that there exist* $x^* \in D, \gamma_{-1} \geq 0$ *such that* $F(x^*) = 0$ *and*

$$\|[x, x^*; F] - [x, x_{-1}; F]\| \leq \gamma_{-1} \|x_{-1} - x^*\|. \tag{2.9}$$

Then, the following assertions hold

(i)

$$\|x - x^*\| \geq \varphi(\gamma_{-1}). \tag{2.10}$$

(ii) If $h(\gamma_{-1}) \leq 1$, *then*

$$\varphi(\gamma_{-1}) \leq \|x - x^*\| \leq \psi(\gamma_{-1}). \tag{2.11}$$

  and x^* *is the unique solution of equation* $F(x) = 0$ *in* $U(x, \psi_1(\gamma_{-1}))$.

(iii) The bounds in (2.11) are sharp.

(iv) Moreover, suppose that

$$\|[z, y; F] - [x, x_{-1}; F]\| \leq \gamma_0 \|z - x\| + \gamma \|y - x_{-1}\| \text{ for each } z, y \in D \tag{2.12}$$

$$\|[x, y; F] - [u, z; F]\| \leq \delta_{-1} \|x - u\| + \delta_0 \|y - z\| \text{ for each } x, y, u, z \in D_0 \tag{2.13}$$

$$h(\delta) \leq 1 \tag{2.14}$$

  and (2.6) hold. Then, the equation (1.1) has a solution in the set

$$A_0(x, x_{-1}) := \{x/ \in D_0 / \varphi(\gamma_{-1}) \leq \|x' - x\| \leq \psi(\gamma_{-1})\}. \tag{2.15}$$

REMARK 2.2 *The advantages of our Theorem 2.1 over the corresponding one in [6] or [7] follow, since* $\gamma_0, \gamma \leq \max\{\gamma_0, \gamma\} = \bar{\gamma}$ *and* $\delta_{-1}, \bar{\delta} \leq \max\{\delta_{-1}, \delta_0\} = \bar{\delta}$, *where* $\bar{\gamma}, \bar{\delta}$ *were used in [6, 7].*

3   Improved Optimal Secant updates of rank 1

Consider optimal secant update

$$x_+ := x - AF(x),\; y := F(x_+) - F(x)$$

$$A_+ := A(I - \frac{F(x_+)}{\langle v, y \rangle} \langle v, . \rangle).$$

Let

$$\omega := \frac{2\sqrt{1 + \beta + \beta^2} - 1 - 2\beta}{3}, \tag{3.1}$$

$$s := 4\alpha_+, t := \beta_+, \alpha := \delta \|F(x_+)\|, \beta := \delta \|x_+ - x\| \tag{3.2}$$

and

$$\alpha_+ := \delta \|F(x_{++})\|,\; \beta_+ := \delta \|x_{++} - x_+\| = \delta \|AF(x_+)\| \left| \frac{\langle v, F(x) \rangle}{\langle u, y \rangle} \right| \tag{3.3}$$

As in [7] by simply replacing $a, b, \alpha, \beta, \gamma, g(b), \rho, v, g(t)$ by $\alpha, \beta, \delta\varphi, \delta\psi, g(\beta), \bar{\rho}, \bar{v}, \bar{g}(t)$, respectively, we obtain the corresponding improvements:

LEMMA 3.1 *Suppose that $0 \le \beta \le 1$ and $0 \le \alpha \le 0.25(1-\beta)^2$. Then, the following assertions hold*

(i) $\delta\varphi(\delta) < \omega$

(ii) $w(1 - \beta - \omega) < 0.25(1-\beta)^2 \Leftrightarrow w < 0.25(1-\beta) \Leftrightarrow \beta < 0.6$

(iii) $\omega < \delta\psi(\delta) \Leftrightarrow b < 0.6$ and $w(1 - \beta - \omega) < \alpha < 0.25(1-\beta)^2$.

LEMMA 3.2 *The function*

$$\bar{g}(t) := \sqrt{(1+t)^2 + 4t(t+\beta)} + \sqrt{(1-t)^2 - 4t(t+\beta)} \tag{3.4}$$

is decreasing in the interval $[\delta\varphi(\delta), \delta\psi(\delta)]$.

PROPOSITION 3.3 *The optimal values for the parameter v in (3.2) satisfy*

$$\left| \frac{\langle v, F(x) \rangle}{\langle u, y \rangle} \right| = \frac{\bar{\rho}}{\|AF(x_+)\|}, \tag{3.5}$$

where

$$\bar{\rho} := \begin{cases} \varphi(\delta), & \text{if } \omega \ge \delta\psi(\delta) \vee \omega < \delta\psi(\delta) \text{ and } \bar{g}(\delta\varphi(\delta) \ge \sqrt{2(1 + (\delta\psi(\delta))^2)} \\ \psi(\delta), & \text{if } \omega < \delta\psi(\delta) \text{ and } \bar{g}(\delta\varphi(\delta) \le \sqrt{2(1 + (\delta\psi(\delta))^2)}. \end{cases} \tag{3.6}$$

It is convenient to let

$$\alpha : F(x_+),\; y := F(x_+) - F(x),\; det := \|\alpha\|^2 \|y\|^2 - \langle \alpha, y \rangle^2 \tag{3.7}$$

PROPOSITION 3.4 *Suppose:*

(i) $det > 0$, *then the condition number of the operator $I - F(x_+)\langle \bar{v}, y \rangle^{-1}$ in (their 2.1) is minimized for all optimal v given by Proposition 3.3 by the vectors*

$$\bar{v} = \pm \left(\frac{\omega \|y\|^2 - \langle \alpha, y \rangle}{\|\alpha - \omega y\|\sqrt{det}} \alpha + \frac{\|\alpha\|^2 - \omega \langle \alpha, y \rangle}{\|\alpha - \omega y\|\sqrt{det}} \right) \tag{3.8}$$

$$\omega = \begin{cases} 1 - \bar{\rho}\|A\alpha\|^{-1}, & \text{if } \bar{g}(1 - \bar{\rho}\|A\alpha\|^{-1}) \le \bar{g}(1 + \bar{\rho}\|A\alpha\|^{-1}) \\ 1 + \bar{\rho}\|A\alpha\|^{-1}, & \text{if } \bar{g}(1 - \bar{\rho}\|A\alpha\|^{-1}) \ge \bar{g}(1 + \bar{\rho}\|A\alpha\|^{-1}) \end{cases} \tag{3.9}$$

and

$$\bar{g}(t) := \frac{\|\alpha\|^2 \|\alpha - ty\|^2}{det} + \frac{\|\alpha\|\|\alpha - ty\|}{\sqrt{det}} \sqrt{\frac{\|\alpha\|^2 \|\alpha - ty\|^2}{det} + 4(1 - t)} - 2t \tag{3.10}$$

(ii) *If $det = 0$, then the condition number is minimized by $\bar{v} = \pm \frac{y}{\|y\|}$. According to the preceding results, if $x, x_{-1} \in D$ with $\alpha := \delta\|F(x)\|$ and $\beta := \delta\|x - x_{-1}\|$ satisfy $\beta + 2\sqrt{\alpha} \le 1$ and $A \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$ is invertible, then the next pair (x_+, A_+) is given by the following scheme:*

Algorithm.

1: $x_+ := x - AF(x)$

2: *Compute* $F(x_+), y := F(x_+) - F(x), AF(x_+);$

3: *Compute* $\|F(x_+)\|, \|y\|, \|AF(x_+)\|, \langle F(x_+), y \rangle;$

4:

$$\delta\varphi(\delta) := \frac{\sqrt{(1 + \delta\psi(\delta))^2 + 4\alpha} - 1 - \beta}{2};$$

$$\delta\psi(\delta) := \frac{1 - \beta - \sqrt{(1 - \beta)^2 - 4\alpha}}{2}$$

$$\omega := \frac{2\sqrt{1 + \beta + \beta^2} - 1 - 2\beta}{3};$$

5: *Compute $\bar{\rho}$ as defined in (3.6);*

6: *Compute ω as defined in (3.9);*

7: *Compute $\bar{v}$ as defined in (3.8);*

8: $A_+ := A(I - \frac{F(x_+)}{\langle \bar{v}, y \rangle} \langle \bar{v}, . \rangle).$

REMARK 3.5 *The results of this section reduce to the corresponding ones in [7], if $\delta = c$ and $D_0 = D$. However, if $\delta < c$ the new results improve the old ones with the advantages as already stated in the introduction of this study.*

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