Available online at www.sciencedirect.com

ScienceDirect

Composite Structures 80 (2007) 595–608

COMPOSITE STRUCTURES

www.elsevier.com/locate/compstruct

Higher order refined computational model with 12 degrees of freedom for the stress analysis of antisymmetric angle-ply plates – analytical solutions

K. Swaminathan *, S.S. Patil

Department of Civil Engineering, National Institute of Technology Karnataka, Srinivasnagar, Karnataka 575 025, India

Available online 11 September 2006

Abstract

Analytical formulations and solutions for the stress analysis of simply supported antisymmetric angle-ply composite and sandwich plates hitherto not reported in the literature based on a higher order refined computational model with twelve degrees of freedom already reported in the literature are presented. The theoretical model presented herein incorporates laminate deformations which account for the effects of transverse shear deformation, transverse normal strain/stress and a nonlinear variation of in-plane displacements with respect to the thickness coordinate thus modelling the warping of transverse cross sections more accurately and eliminating the need for shear correction coefficients. In addition, two higher order computational models, one with nine and the other with five degrees of freedom already available in the literature are also considered for comparison. The equations of equilibrium are obtained using Principle of Minimum Potential Energy (PMPE). Solutions are obtained in closed form using Navier's technique by solving the boundary value problem. Accuracy of the theoretical formulations and the solution method is first ascertained by comparing the results with that already available in the literature. After establishing the accuracy of the solutions, numerical results with real properties using all the computational models are presented for the stress analysis of multilayer antisymmetric angle-ply composite and sandwich plates, which will serve as a benchmark for future investigations.

Keywords: Stress analysis; Higher order theory; Shear deformation; Angle-ply plates; Analytical solutions

1. Introduction

Fibre reinforced composite, sandwich plates and shells are being increasingly used in aerospace, automobile and ship building industries due to their light weight and high stiffness and also due to their anisotropic material properties that can be tailored through variation of the fibre orientation and stacking sequence. Due to the special properties exhibited by the composite materials such as high degree of anisotropy and weak rigidities in transverse shear, the method of analysis based on Classical Laminate Plate Theory (CLPT) becomes inadequate. The First Order Shear Deformation Theory (FSDT) adequately describes the plate kinematics behaviour in most of the cases but requires a shear correction factor. Higher Order Shear Deformation Theories (HSDTs) can represent the kinematics better and can yield more accurate prediction of stress distributions. Owing to these reasons, an increasing number of higher order theories for the analysis of multilayered plates have been published over the past two decades. Results were reported in the literature using analytical and numerical methods. A complete review of various shear deformation theories for the analysis of single layer

* Corresponding author. Tel.: +91 824 2474340; fax: +91 824 2474033.
E-mail addresses: swami7192@yahoo.co.in, swami@vasnet.co.in (K. Swaminathan).

0263-8223/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruct.2006.07.006

isotropic, orthotropic and multilayer composite plates and shells is available in the review articles by Noor and Burton [1,2] and Noor et al. [3]. A selective review of the various analytical and numerical methods used for the stress analysis of laminated composite and sandwich plates was presented by Kant and Swaminathan [4]. Analytical formulations, solutions and comparison of numerical results for the buckling, free vibration, stress analyses of cross ply composite and sandwich plates based on the higher order refined theories already reported in the literature by Kant [5], Pandya and Kant [6–10] and Kant and Manjunatha [11] were presented recently by Kant and Swaminathan [12–15]. Recently the theoretical formulations and solutions for the static analysis of antisymmetric angle-ply laminated composite and sandwich plates using a nine degrees of freedom computational model were presented by Swaminathan and Ragounadin [16]. In this paper, analytical formulations developed and solutions obtained for the first time is presented for the stress analysis of antisymmetric angle-ply laminated composite and sandwich plates using a higher order refined computational model with twelve degrees of freedom. Solutions obtained using this model are also compared with the results of other two models considered in the present investigation. Correctness of the solutions is first established and then benchmark results with real properties using all the models are presented for the antisymmetric angle-ply composite and sandwich plates.

2. Theoretical formulation

2.1. Displacement models

In order to approximate the three-dimensional elasticity problem to a two-dimensional plate problem, the displacement components $u(x,y,z)$, $v(x,y,z)$ and $w(x,y,z)$ at any point in the plate space are expanded in Taylor's series in terms of the thickness coordinate. The elasticity solution indicates that the transverse shear stresses vary parabolically through the plate thickness. This requires the use of a displacement field in which the in-plane displacements are expanded as cubic functions of the thickness coordinate. In addition, the transverse normal strain may vary non-linearly through the plate thickness. The displacement field referred to as Model-1 in the present investigation which satisfies the above criteria may be assumed in the form [11]:

$$\begin{aligned}
u(x,y,z) &= u_o(x,y) + z\theta_x(x,y) + z^2u_o^*(x,y) + z^3\theta_x^*(x,y)\\
v(x,y,z) &= v_o(x,y) + z\theta_y(x,y) + z^2v_o^*(x,y) + z^3\theta_y^*(x,y)\\
w(x,y,z) &= w_o(x,y) + z\theta_z(x,y) + z^2w_o^*(x,y) + z^3\theta_z^*(x,y)
\end{aligned} \tag{1}$$

The parameters u_o, v_o are the in-plane displacements and w_o is the transverse displacement of a point (x,y) on the middle plane. The functions θ_x, θ_y are rotations of the normal to the middle plane about y and x axes respectively. The parameters $u_o^*, v_o^*, w_o^*, \theta_x^*, \theta_y^*, \theta_z^*$ and θ_z are the higher-order terms in the Taylor's series expansion and they represent higher-order transverse cross sectional deformation modes. Though the above theory was already reported earlier in the literature and numerical results were presented using finite element formulations, analytical formulations and solutions are obtained for the first time in this investigation and hence the results obtained using the above theory are referred to as *present* in all the tables and figures. In addition to the above, the following higher order theories already reported in the literature for the analysis of laminated composite and sandwich plates are also considered for the evaluation purpose. Results using these theories are generated independently and presented here with a view to have all the results on a common platform.

Model-2 [10]

$$\begin{aligned}
u(x,y,z) &= u_o(x,y) + z\theta_x(x,y) + z^2u_o^*(x,y) + z^3\theta_x^*(x,y)\\
v(x,y,z) &= v_o(x,y) + z\theta_y(x,y) + z^2v_o^*(x,y) + z^3\theta_y^*(x,y)\\
w(x,y,z) &= w_o(x,y)
\end{aligned} \tag{2}$$

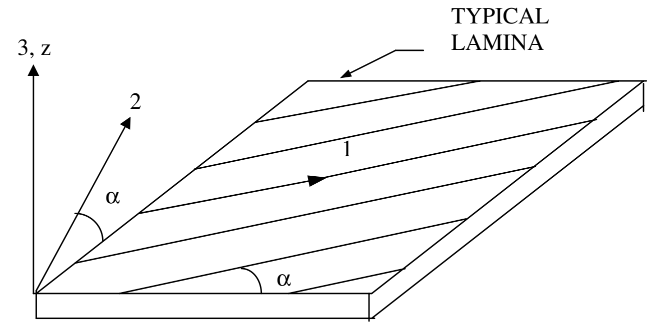
Model-3 [17]

$$\begin{aligned}
u(x,y,z) &= u_o(x,y) + z\left[\theta_x(x,y) - \frac{4}{3}\left(\frac{z}{h}\right)^2\left\{\theta_x(x,y) + \frac{\partial w_o}{\partial x}\right\}\right]\\
v(x,y,z) &= v_o(x,y) + z\left[\theta_y(x,y) - \frac{4}{3}\left(\frac{z}{h}\right)^2\left\{\theta_y(x,y) + \frac{\partial w_o}{\partial y}\right\}\right]\\
w(x,y,z) &= w_o(x,y)
\end{aligned} \tag{3}$$

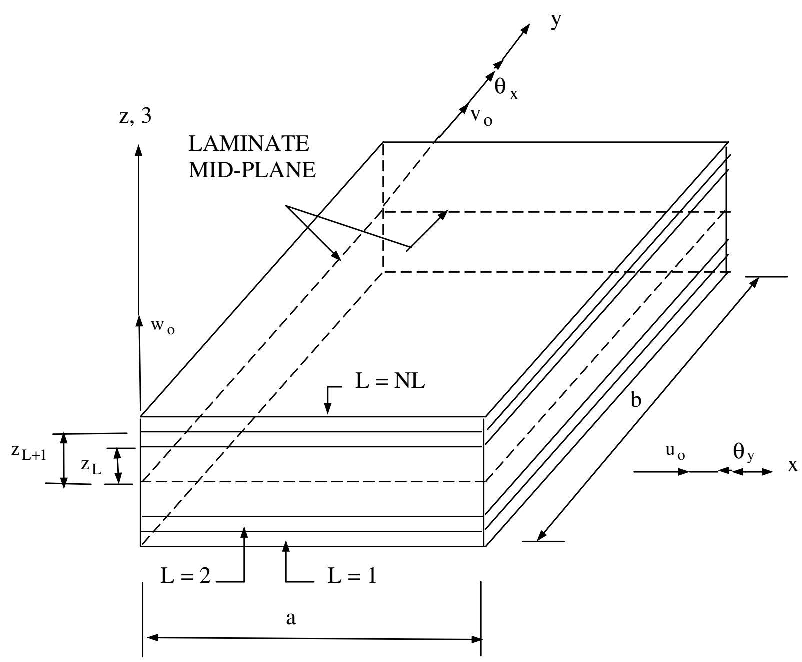
In this paper the analytical formulations and solution method followed using the higher order refined theory (Model-1) are only presented in detail and the same procedure is followed in obtaining the results using other models. The geometry of a two-dimensional laminated composite and sandwich plates with positive set of coordinate axes and the physical middle plane displacement terms are shown in Figs. 1 and 2 respectively. By substitution of the displacement relations given by Eq. (1) into the strain–displacement equations of the classical theory of elasticity, the following relations are obtained.

$$\begin{aligned}
\varepsilon_x &= \varepsilon_{xo} + z\kappa_x + z^2\varepsilon_{xo}^* + z^3\kappa_x^*\\
\varepsilon_y &= \varepsilon_{yo} + z\kappa_y + z^2\varepsilon_{yo}^* + z^3\kappa_y^*\\
\varepsilon_z &= \varepsilon_{zo} + z\kappa_z^* + z^2\varepsilon_{zo}^*\\
\gamma_{xy} &= \varepsilon_{xyo} + z\kappa_{xy} + z^2\varepsilon_{xyo}^* + z^3\kappa_{xy}^*\\
\gamma_{yz} &= \phi_y + z\kappa_{yz} + z^2\phi_y^* + z^3\kappa_{yz}^*\\
\gamma_{xz} &= \phi_x + z\kappa_{xz} + z^2\phi_x^* + z^3\kappa_{xz}^*
\end{aligned} \tag{4}$$

where



(1,2,3) - LAMINA REFERENCE AXES



(x,y,z) - LAMINATE REFERENCE AXES

Fig. 1. Laminate geometry with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

$$
\begin{aligned}
&(\varepsilon_{xo}, \varepsilon_{yo}, \varepsilon_{xyo}) = \left(\frac{\partial u_o}{\partial x}, \frac{\partial v_o}{\partial y}, \frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)\\
&\left(\varepsilon^*_{xo}, \varepsilon^*_{yo}, \varepsilon^*_{xyo}\right) = \left(\frac{\partial u^*_o}{\partial x}, \frac{\partial v^*_o}{\partial y}, \frac{\partial u^*_o}{\partial y} + \frac{\partial v^*_o}{\partial x}\right)\\
&\left(\varepsilon_{zo}, \varepsilon^*_{zo}\right) = \left(\theta_z, 3\theta^*_z\right)\\
&\left(\kappa_x, \kappa_y, \kappa^*_z, \kappa_{xy}\right) = \left(\frac{\partial \theta_x}{\partial x}, \frac{\partial \theta_y}{\partial y}, 2w^*_o, \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right)\\
&\left(\kappa^*_x, \kappa^*_y, \kappa^*_{xy}\right) = \left(\frac{\partial \theta^*_x}{\partial x}, \frac{\partial \theta^*_y}{\partial y}, \frac{\partial \theta^*_x}{\partial y} + \frac{\partial \theta^*_y}{\partial x}\right)\\
&(\kappa_{xz}, \kappa_{yz}) = \left(2u^*_o + \frac{\partial \theta_z}{\partial x}, 2v^*_o + \frac{\partial \theta_z}{\partial y}\right)\\
&\left(\kappa^*_{xz}, \kappa^*_{yz}\right) = \left(\frac{\partial \theta^*_z}{\partial x}, \frac{\partial \theta^*_z}{\partial y}\right)\\
&\left(\phi_x, \phi^*_x, \phi_y, \phi^*_y\right)\\
&\quad = \left(\theta_x + \frac{\partial w_o}{\partial x}, 3\theta^*_x + \frac{\partial w^*_o}{\partial x}, \theta_y + \frac{\partial w_o}{\partial y}, 3\theta^*_y + \frac{\partial w^*_o}{\partial y}\right)
\end{aligned}
\tag{5}
$$

2.2. Constitutive equations

Each lamina in the laminate is assumed to be in a three-dimensional stress state so that the constitutive relation for a typical lamina L with reference to the fibre-matrix coordinate axes (1–2–3) can be written as

$$
\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{12} \\ \tau_{23} \\ \tau_{13} \end{Bmatrix}^L = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{13} \end{Bmatrix}^L \tag{6}
$$

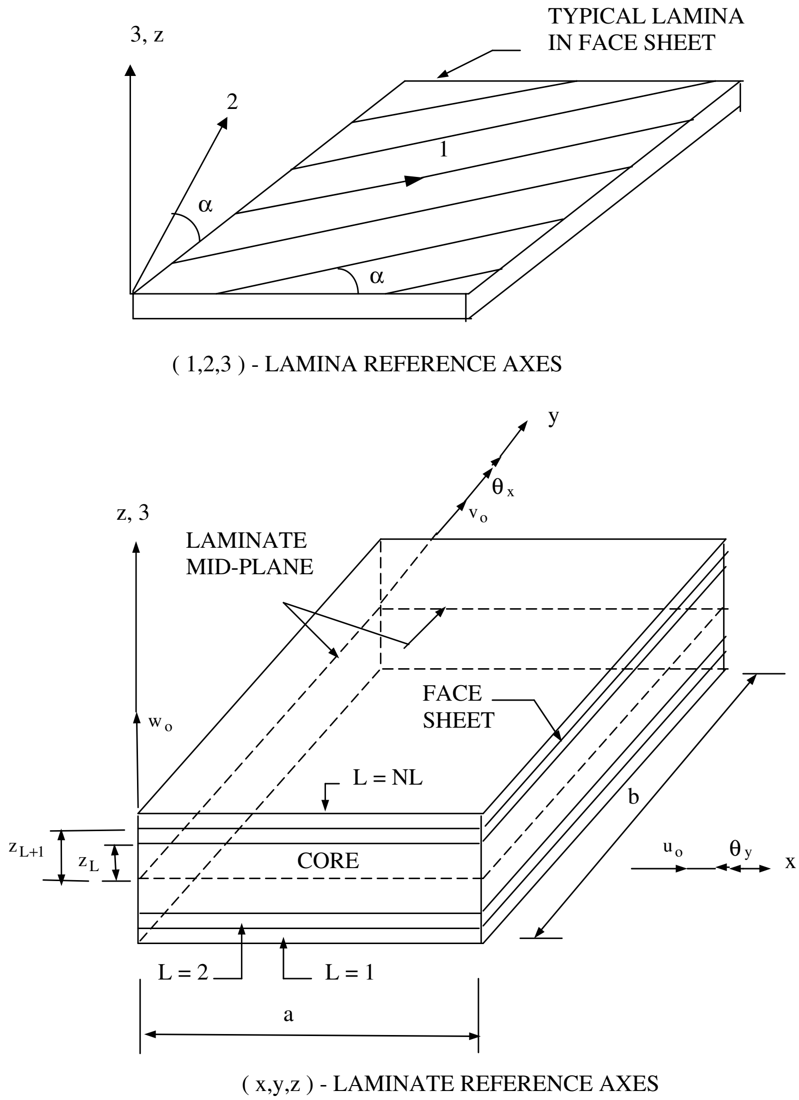


Fig. 2. Geometry of a sandwich plates with positive set of lamina/laminate reference axes, displacement components and fibre orientation.

where $(\sigma_1, \sigma_2, \sigma_3, \tau_{12}, \tau_{23}, \tau_{13})$ are the stresses and $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \gamma_{12}, \gamma_{23}, \gamma_{13})$ are the linear strain components referred to the lamina coordinates (1–2–3) and the C'_{ij}s are the elastic constants or the elements of stiffness matrix of the Lth lamina with reference to the fibre axes (1–2–3). In the laminate coordinate (x, y, z) the stress strain relations for the Lth lamina can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix}^L = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & Q_{14} & 0 & 0 \\ & Q_{22} & Q_{23} & Q_{24} & 0 & 0 \\ & & Q_{33} & Q_{34} & 0 & 0 \\ & & & Q_{44} & 0 & 0 \\ & \text{symmetric} & & & Q_{55} & Q_{56} \\ & & & & & Q_{66} \end{bmatrix}^L \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}^L \quad (7)$$

where $(\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz})$ are the stresses and $(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz})$ are the strains with respect to the laminate axes. Q'_{ij}s are the transformed elastic constants or the stiffness matrix with respect to the laminate axes x, y, z. The elements of matrices $[C]$ and $[Q]$ are defined in Appendices A and B.

2.3. Governing equations of equilibrium

The equations of equilibrium for the stress analysis are obtained using the principle of minimum potential energy (PMPE). In analytical form it can be written as follows [18]:

$$\delta(U + V) = 0 \quad (8)$$

where U is the total strain energy due to deformations, V is the potential of the external loads, and $U + V = \Pi$ is the total potential energy and δ denotes the variational symbol. Substituting the appropriate energy expression in the above equation, the final expression can thus be written as

$$\left[\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_A(\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\sigma_z\delta\varepsilon_z+\tau_{xy}\delta\gamma_{xy}+\tau_{yz}\delta\gamma_{yz}+\tau_{xz}\delta\gamma_{xz})\,\mathrm{d}A\,\mathrm{d}z - \int_A p_z^+\delta w^+\,\mathrm{d}A\right]=0 \tag{9}$$

where $w^+ = w_o + (h/2)\theta_z + (h^2/4)w_o^* + (h^3/8)\theta_z^*$ is the transverse displacement of any point on the top surface of the plate and p_z^+ is the transverse load applied at the top surface of the plate. Using Eqs. (1), (4) and (5) in Eq. (9) and integrating the resulting expression by parts, and collecting the coefficients of $\delta u_0, \delta v_0, \delta w_0, \delta\theta_x, \delta\theta_y, \delta\theta_z, \delta u_0^*, \delta v_0^*, \delta w_0^*, \delta\theta_x^*, \delta\theta_y^*, \delta\theta_z^*$ the following equations of equilibrium are obtained:

$$\begin{aligned}
\delta u_o &: \frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0\\
\delta v_o &: \frac{\partial N_y}{\partial y}+\frac{\partial N_{xy}}{\partial x}=0\\
\delta w_o &: \frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}+p_z^+=0\\
\delta\theta_x &: \frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-Q_x=0\\
\delta\theta_y &: \frac{\partial M_y}{\partial y}+\frac{\partial M_{xy}}{\partial x}-Q_y=0\\
\delta\theta_z &: \frac{\partial S_x}{\partial x}+\frac{\partial S_y}{\partial y}-N_z+\frac{h}{2}(p_z^+)=0\\
\delta u_o^* &: \frac{\partial N_x^*}{\partial x}+\frac{\partial N_{xy}^*}{\partial y}-2S_x=0\\
\delta v_o^* &: \frac{\partial N_y^*}{\partial y}+\frac{\partial N_{xy}^*}{\partial x}-2S_y=0\\
\delta w_o^* &: \frac{\partial Q_x^*}{\partial x}+\frac{\partial Q_y^*}{\partial y}-2M_z^*+\frac{h^2}{4}\left(p_z^+\right)=0\\
\delta\theta_x^* &: \frac{\partial M_x^*}{\partial x}+\frac{\partial M_{xy}^*}{\partial y}-3Q_x^*=0\\
\delta\theta_y^* &: \frac{\partial M_y^*}{\partial y}+\frac{\partial M_{xy}^*}{\partial x}-3Q_y^*=0\\
\delta\theta_z^* &: \frac{\partial S_x^*}{\partial x}+\frac{\partial S_y^*}{\partial y}-3N_z^*+\frac{h^3}{8}\left(p_z^+\right)=0
\end{aligned} \tag{10}$$

and boundary conditions are the form:

On the edge $x=$ constant

$$\begin{aligned}
&u_o=\overline{u}_o \text{ or } N_x=\overline{N}_x, \quad u_o^*=\overline{u}_o^* \text{ or } N_x^*=\overline{N}_x^*\\
&v_o=\overline{v}_o \text{ or } N_{xy}=\overline{N}_{xy}, \quad v_o^*=\overline{v}_o^* \text{ or } N_{xy}^*=\overline{N}_{xy}^*\\
&w_o=\overline{w}_o \text{ or } Q_x=\overline{Q}_x, \quad w_o^*=\overline{w}_o^* \text{ or } Q_x^*=\overline{Q}_x^*\\
&\theta_x=\overline{\theta}_x \text{ or } M_x=\overline{M}_x, \quad \theta_x^*=\overline{\theta}_x^* \text{ or } M_x^*=\overline{M}_x^*\\
&\theta_y=\overline{\theta}_y \text{ or } M_{xy}=\overline{M}_{xy}, \quad \theta_y^*=\overline{\theta}_y^* \text{ or } M_{xy}^*=\overline{M}_{xy}^*\\
&\theta_z=\overline{\theta}_z \text{ or } S_x=\overline{S}_x, \quad \theta_z^*=\overline{\theta}_z^* \text{ or } S_x^*=\overline{S}_x^*
\end{aligned} \tag{11}$$

On the edge $y=$ constant

$$\begin{aligned}
&u_o=\overline{u}_o \text{ or } N_{xy}=\overline{N}_{xy}, \quad u_o^*=\overline{u}_o^* \text{ or } N_{xy}^*=\overline{N}_{xy}^*\\
&v_o=\overline{v}_o \text{ or } N_y=\overline{N}_y, \quad v_o^*=\overline{v}_o^* \text{ or } N_y^*=\overline{N}_y^*\\
&w_o=\overline{w}_o \text{ or } Q_y=\overline{Q}_y, \quad w_o^*=\overline{w}_o^* \text{ or } Q_y^*=\overline{Q}_y^*\\
&\theta_x=\overline{\theta}_x \text{ or } M_{xy}=\overline{M}_{xy}, \quad \theta_x^*=\overline{\theta}_x^* \text{ or } M_{xy}^*=\overline{M}_{xy}^*\\
&\theta_y=\overline{\theta}_y \text{ or } M_y=\overline{M}_y, \quad \theta_y^*=\overline{\theta}_y^* \text{ or } M_y^*=\overline{M}_y^*\\
&\theta_z=\overline{\theta}_z \text{ or } S_y=\overline{S}_y, \quad \theta_z^*=\overline{\theta}_z^* \text{ or } S_y^*=\overline{S}_y^*
\end{aligned} \tag{12}$$

where the stress resultants are defined by

$$\begin{bmatrix} M_x & M_x^*\\ M_y & M_y^*\\ M_z^* & 0\\ M_{xy} & M_{xy}^*\end{bmatrix}=\sum_{L=1}^{NL}\int_{z_L}^{z_{L+1}}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\end{Bmatrix}\begin{bmatrix} z & z^3\end{bmatrix}\mathrm{d}z \tag{13}$$

$$\begin{bmatrix} Q_x & Q_x^*\\ Q_y & Q_y^*\end{bmatrix}=\sum_{L=1}^{NL}\int_{z_L}^{z_{L+1}}\begin{Bmatrix}\tau_{xz}\\ \tau_{yz}\end{Bmatrix}\begin{bmatrix} 1 & z^2\end{bmatrix}\mathrm{d}z \tag{14}$$

$$\begin{bmatrix} N_x & N_x^*\\ N_y & N_y^*\\ N_z & N_z^*\\ N_{xy} & N_{xy}^*\end{bmatrix}=\sum_{L=1}^{NL}\int_{z_L}^{z_{L+1}}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\end{Bmatrix}\begin{bmatrix} 1 & z^2\end{bmatrix}\mathrm{d}z \tag{15}$$

$$\begin{bmatrix} S_x & S_x^*\\ S_y & S_y^*\end{bmatrix}=\sum_{L=1}^{NL}\int_{z_L}^{z_{L+1}}\begin{Bmatrix}\tau_{xz}\\ \tau_{yz}\end{Bmatrix}\begin{bmatrix} z & z^3\end{bmatrix}\mathrm{d}z \tag{16}$$

The resultants in Eqs. (13)–(16) can be related to the total strains in Eq. (4) by the following equations:

$$\begin{Bmatrix} N_x\\ N_y\\ N_x^*\\ N_y^*\\ N_z\\ N_z^*\\ M_x\\ M_y\\ M_x^*\\ M_y^*\\ M_z^*\end{Bmatrix}=[A]\begin{Bmatrix}\frac{\partial u_o}{\partial x}\\ \frac{\partial v_o}{\partial y}\\ \frac{\partial u_o^*}{\partial x}\\ \frac{\partial v_o^*}{\partial y}\\ \theta_z\\ \theta_z^*\\ \frac{\partial\theta_x}{\partial x}\\ \frac{\partial\theta_y}{\partial y}\\ \frac{\partial\theta_x^*}{\partial x}\\ \frac{\partial\theta_y^*}{\partial y}\\ w_o^*\end{Bmatrix}+[A']\begin{Bmatrix}\frac{\partial u_o}{\partial y}\\ \frac{\partial v_o}{\partial x}\\ \frac{\partial u_o^*}{\partial y}\\ \frac{\partial v_o^*}{\partial x}\\ \frac{\partial\theta_x}{\partial y}\\ \frac{\partial\theta_y}{\partial x}\\ \frac{\partial\theta_x^*}{\partial y}\\ \frac{\partial\theta_y^*}{\partial x}\end{Bmatrix}$$

$$\begin{Bmatrix} N_{xy}\\ N_{xy}^*\\ M_{xy}\\ M_{xy}^*\end{Bmatrix}=[B']\begin{Bmatrix}\frac{\partial u_o}{\partial x}\\ \frac{\partial v_o}{\partial y}\\ \frac{\partial u_o^*}{\partial x}\\ \frac{\partial v_o^*}{\partial y}\\ \theta_z\\ \theta_z^*\\ \frac{\partial\theta_x}{\partial x}\\ \frac{\partial\theta_y}{\partial y}\\ \frac{\partial\theta_x^*}{\partial x}\\ \frac{\partial\theta_y^*}{\partial y}\\ w_o^*\end{Bmatrix}+[B]\begin{Bmatrix}\frac{\partial u_o}{\partial y}\\ \frac{\partial v_o}{\partial x}\\ \frac{\partial u_o^*}{\partial y}\\ \frac{\partial v_o^*}{\partial x}\\ \frac{\partial\theta_x}{\partial y}\\ \frac{\partial\theta_y}{\partial x}\\ \frac{\partial\theta_x^*}{\partial y}\\ \frac{\partial\theta_y^*}{\partial x}\end{Bmatrix} \tag{17}$$

$$\begin{Bmatrix} Q_x \\ Q_x^* \\ S_x \\ S_x^* \end{Bmatrix} = [D]\begin{Bmatrix} \theta_x \\ \frac{\partial w_o}{\partial x} \\ \theta_x^* \\ \frac{\partial w_o^*}{\partial x} \\ u_o^* \\ \frac{\partial \theta_z}{\partial x} \\ \frac{\partial \theta_z^*}{\partial x} \end{Bmatrix} + [D']\begin{Bmatrix} \theta_y \\ \frac{\partial w_o}{\partial y} \\ \theta_y^* \\ \frac{\partial w_o^*}{\partial y} \\ v_o^* \\ \frac{\partial \theta_z}{\partial y} \\ \frac{\partial \theta_z^*}{\partial y} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_y \\ Q_y^* \\ S_y \\ S_y^* \end{Bmatrix} = [E']\begin{Bmatrix} \theta_x \\ \frac{\partial w_o}{\partial x} \\ \theta_x^* \\ \frac{\partial w_o^*}{\partial x} \\ u_o^* \\ \frac{\partial \theta_z}{\partial x} \\ \frac{\partial \theta_z^*}{\partial x} \end{Bmatrix} + [E]\begin{Bmatrix} \theta_y \\ \frac{\partial w_o}{\partial y} \\ \theta_y^* \\ \frac{\partial w_o^*}{\partial y} \\ v_o^* \\ \frac{\partial \theta_z}{\partial y} \\ \frac{\partial \theta_z^*}{\partial y} \end{Bmatrix} \tag{18}$$

where the matrices $[A]$, $[A']$, $[B]$, $[B']$, $[D]$, $[D']$, $[E]$, $[E']$ are the matrices of plate stiffnesses whose elements are defined in Appendix C.

3. Analytical solutions

Here the exact solutions of Eqs. (10)–(18) for antisymmetric angle-ply plates are considered. Assuming that the plate is simply supported with SS-2 boundary conditions [19] in such a manner that tangential displacement is admissible, but the normal displacement is not, the following boundary conditions are appropriate:

At edges $x = 0$ and $x = a$;

$$\begin{aligned} &u_o = 0; \quad w_o = 0; \quad \theta_y = 0; \quad \theta_z = 0; \quad M_x = 0; \quad N_{xy} = 0; \\ &u_o^* = 0; \quad w_o^* = 0; \quad \theta_y^* = 0; \quad \theta_z^* = 0; \quad M_x^* = 0; \quad N_{xy}^* = 0. \end{aligned} \tag{19}$$

At edges $y = 0$ and $y = b$;

$$\begin{aligned} &v_o = 0; \quad w_o = 0; \quad \theta_x = 0; \quad \theta_z = 0; \quad M_y = 0; \quad N_{xy} = 0; \\ &v_o^* = 0; \quad w_o^* = 0; \quad \theta_x^* = 0; \quad \theta_z^* = 0; \quad M_y^* = 0; \quad N_{xy}^* = 0 \end{aligned} \tag{20}$$

following Navier's solution procedure [19–21], the solution to the displacement variables satisfying the above boundary conditions can be expressed in the following forms:

$$\begin{aligned}
u_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{o_{mn}} \sin\alpha x \cos\beta y, \quad u_o^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{o_{mn}}^* \sin\alpha x \cos\beta y \\
v_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{o_{mn}} \cos\alpha x \sin\beta y, \quad v_o^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{o_{mn}}^* \cos\alpha x \sin\beta y \\
w_o &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{o_{mn}} \sin\alpha x \sin\beta y, \quad w_o^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{o_{mn}}^* \sin\alpha x \sin\beta y \\
\theta_x &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{x_{mn}} \cos\alpha x \sin\beta y, \quad \theta_x^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{x_{mn}}^* \cos\alpha x \sin\beta y \\
\theta_y &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}} \sin\alpha x \cos\beta y, \quad \theta_y^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{y_{mn}}^* \sin\alpha x \cos\beta y \\
\theta_z &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{z_{mn}} \sin\alpha x \sin\beta y, \quad \theta_z^* = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \theta_{z_{mn}}^* \sin\alpha x \sin\beta y
\end{aligned}$$

and the loading term is expanded as

$$p_z^+ = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} p_{z_{mn}}^+ \sin\alpha x \sin\beta y \tag{21}$$

where

$$\alpha = \frac{m\pi}{a} \quad \text{and} \quad \beta = \frac{n\pi}{b}$$

Substituting Eqs. (19)–(21) in to Eq. (10) and collecting the coefficients one obtains

$$[X]_{12\times 12}\begin{Bmatrix} u_o \\ v_o \\ w_o \\ \theta_x \\ \theta_y \\ \theta_z \\ u_o^* \\ v_o^* \\ w_o^* \\ \theta_x^* \\ \theta_y^* \\ \theta_z^* \end{Bmatrix}_{12\times 1} = \begin{Bmatrix} 0 \\ 0 \\ p_z^+ \\ 0 \\ 0 \\ \frac{h}{2}(p_z^+) \\ 0 \\ 0 \\ \frac{h^2}{4}(p_z^+) \\ 0 \\ 0 \\ \frac{h^3}{8}(p_z^+) \end{Bmatrix}_{12\times 1} \tag{22}$$

for any fixed values of m and n. The elements of coefficient matrix $[X]$ are given in Appendix D.

4. Numerical results and discussion

In this section, various numerical examples solved are described and discussed for establishing the accuracy of the theory for the stress analysis of antisymmetric angle-ply laminated composite and sandwich plates. For all the problems a simply supported plate with SS-2 boundary conditions is considered for the analysis. The transverse loading considered is sinusoidal. Results are obtained in closed-form using *Navier's* solution technique for the above geometry and loading and the accuracy of the solution is established by comparing the results with the solutions wherever available in the literature.

The following sets of data are used in obtaining numerical results.

Material 1 [22]

$E_1 = 40 \times 10^6$ psi (276 GPa)
$E_2 = E_3 = 1 \times 10^6$ psi (6.895 Gpa)
$G_{12} = G_{13} = 0.5 \times 10^6$ psi (3.45 GPa)
$G_{23} = 0.6 \times 10^6$ psi (4.12 GPa)
$\upsilon_{12} = \upsilon_{23} = \upsilon_{13} = 0.25$

Material 2

Glass epoxy
$E_1 = 5.6 \times 10^6$ psi (38.61 GPa)
$E_2 = 1.2 \times 10^6$ psi (8.27 GPa)
$E_3 = 1.3 \times 10^6$ psi (8.96 GPa)
$G_{12} = 0.60 \times 10^6$ psi (4.14 GPa)
$G_{13} = 0.60 \times 10^6$ psi (4.14 GPa)
$G_{23} = 0.50 \times 10^6$ psi (3.45GPa)
$\upsilon_{12} = 0.26,\ \upsilon_{13} = 0.26,\ \upsilon_{23} = 0.34$

Material 3

Face sheets (Graphite epoxy T300/934)
$E_1 = 19 \times 10^6$ psi (131 GPa)
$E_2 = 1.5 \times 10^6$ psi (10.34 GPa)
$E_2 = E_3$
$G_{12} = 1 \times 10^6$ psi (6.895 GPa)
$G_{13} = 0.90 \times 10^6$ psi (6.205 GPa)
$G_{23} = 1 \times 10^6$ psi (6.895 GPa)
$\upsilon_{12} = 0.22,\ \upsilon_{13} = 0.22,\ \upsilon_{23} = 0.49$

Core (Isotropic)
$E_1 = E_2 = E_3 = 2G = 1000$ psi (6.90×10^{-3} GPa)
$G_{12} = G_{13} = G_{23} = 500$ psi (3.45×10^{-3} GPa)
$\upsilon_{12} = \upsilon_{13} = \upsilon_{23} = 0$

Results reported in tables and plots are using the following non-dimensional form:

$$\bar{u} = u\left(\frac{100h^3E_2}{p_oa^4}\right),\quad \bar{v} = v\left(\frac{100h^3E_2}{p_oa^4}\right),\quad \bar{w} = w\left(\frac{100h^3E_2}{p_oa^4}\right)$$

$$\bar{\sigma}_x = \sigma_x\left(\frac{h^2}{p_oa^2}\right),\quad \bar{\sigma}_y = \sigma_y\left(\frac{h^2}{p_oa^2}\right),\quad \bar{\tau}_{xy} = \tau_{xy}\left(\frac{h^2}{p_oa^2}\right)$$

Table 1
Non-dimensionalized transverse deflection in a simply supported antisymmetric angle-ply ($\theta/-\theta\cdots$) square laminate under sinusoidal transverse load

θ	a/h	Theory	$\bar{w}$	
			$n^a = 2$	$n^a = 4$
15°	4	Ren[b]	1.4989	1.3050
		Model-1 (present)	1.4258	1.2608
		Model-2	1.4596	1.2869
		Model-3	1.3307	1.1903
	10	Ren[b]	0.6476	0.4505
		Model-1 (present)	0.6296	0.4423
		Model-2	0.6374	0.4446
		Model-3	0.6213	0.4329
	100	Ren[b]	0.4680	0.2668
		Model-1 (present)	0.4621	0.2662
		Model-2	0.4679	0.2667
		Model-3	0.4678	0.2666
30°	4	Ren[b]	1.4865	1.0943
		Model-1 (present)	1.3439	1.0399
		Model-2	1.3775	1.0605
		Model-3	1.1082	0.9494
	10	Ren[b]	0.6731	0.3543
		Model-1 (present)	0.6367	0.3439
		Model-2	0.6432	0.3454
		Model-3	0.5985	0.3291
	100	Ren[b]	0.4975	0.2049
		Model-1 (present)	0.4931	0.2046
		Model-2	0.4972	0.2048
		Model-3	0.4967	0.2046
45°	4	Ren[b]	1.4471	1.0160
		Model-1 (present)	1.2852	0.9626
		Model-2	1.3175	0.9814
		Model-3	1.0203	0.8747
	10	Ren[b]	0.6427	0.3201
		Model-1 (present)	0.6028	0.3101
		Model-2	0.6084	0.3114
		Model-3	0.5581	0.2956
	100	Ren[b]	0.4685	0.1821
		Model-1 (present)	0.4649	0.1818
		Model-2	0.4682	0.1820
		Model-3	0.4676	0.1818

[a] Number of layers.
[b] See [22].

Table 2
Non-dimensionalized transverse deflection in a simply supported two layered antisymmetric angle-ply ($\theta/-\theta$) rectangular ($b = 3$a) laminate under sinusoidal transverse load

θ	a/h	Theory	$\bar{w}$
15°	4	Ren[a]	2.1922
		Model-1 (present)	2.0980
		Model-2	2.1245
		Model-3	2.0119
	10	Ren[a]	1.0272
		Model-1 (present)	1.0095
		Model-2	1.0146
		Model-3	1.0017
	100	Ren[a]	0.8020
		Model-1 (present)	0.7987
		Model-2	0.8019
		Model-3	0.8018
30°	4	Ren[a]	2.8881
		Model-1 (present)	2.6635
		Model-2	2.6980
		Model-3	2.3752
	10	Ren[a]	1.5787
		Model-1 (present)	1.5321
		Model-2	1.5388
		Model-3	1.4872
	100	Ren[a]	1.3163
		Model-1 (present)	1.3120
		Model-2	1.3158
		Model-3	1.3154
45°	4	Ren[a]	3.9653
		Model-1 (present)	3.6239
		Model-2	3.6716
		Model-3	3.1562
	10	Ren[a]	2.3953
		Model-1 (present)	2.3215
		Model-2	2.3323
		Model-3	2.2440
	100	Ren[a]	2.0686
		Model-1 (present)	2.0609
		Model-2	2.0679
		Model-3	2.0673

[a] See [22].

Table 3
Non-dimensionalized in-plane stresses in a simply supported two ($n=2$) and four ($n=4$) layered antisymmetric angle-ply ($15°/-15°\cdots$) rectangular ($b=3a$) laminate under sinusoidal transverse load

a/h	Theory	$\overline{\sigma}_x$		$\overline{\sigma}_y$		$\overline{\tau}_{xy}$	
		$n=2$	$n=4$	$n=2$	$n=4$	$n=2$	$n=4$
2	Model-1 (present)	1.3908	1.4803	0.1892	0.1955	−0.0571	−0.0607
	Model-2	1.6312	1.6427	0.1471	0.1452	−0.0544	−0.0567
	Model-3	1.9874	1.8616	0.1689	0.1561	−0.1576	−0.1587
4	Model-1 (present)	1.0324	0.8954	0.1030	0.0922	−0.0442	−0.0435
	Model-2	1.1110	0.9453	0.0934	0.0796	−0.0443	−0.0435
	Model-3	1.1722	0.9160	0.0974	0.0766	−0.0812	−0.0767
10	Model-1 (present)	0.8847	0.6141	0.0739	0.0522	−0.0443	−0.0333
	Model-2	0.9002	0.6223	0.0725	0.0502	−0.0443	−0.0333
	Model-3	0.9066	0.6096	0.0729	0.0491	−0.0511	−0.0385
20	Model-1 (present)	0.8603	0.5655	0.0695	0.0457	−0.0448	−0.0307
	Model-2	0.8655	0.5675	0.0692	0.0452	−0.0446	−0.0306
	Model-3	0.8670	0.5641	0.0693	0.0449	−0.0446	−0.0319
50	Model-1 (present)	0.8532	0.5515	0.0682	0.0438	−0.0449	−0.0298
	Model-2	0.8556	0.5518	0.0682	0.0438	−0.0448	−0.0298
	Model-3	0.8558	0.5512	0.0682	0.0438	−0.0450	−0.0299
100	Model-1 (present)	0.8522	0.5494	0.0681	0.0436	−0.0449	−0.0297
	Model-2	0.8542	0.5495	0.0681	0.0436	−0.0448	−0.0297
	Model-3	0.8543	0.5494	0.0681	0.0436	−0.0448	−0.0297

Unless otherwise specified within the table(s) the locations (i.e. x-, y-, and z-coordinates) for maximum values of displacements and stresses for the present evaluations are as follows:

In-plane displacement (u): $(0, b/2, \pm h/2)$
In-plane displacement (v): $(a/2, 0, \pm h/2)$
Transverse displacement (w): $(a/2, b/2, 0)$
In-plane normal stress (σ_x): $(a/2, b/2, \pm h/2)$
In-plane normal stress (σ_y): $(a/2, b/2, \pm h/2)$
In-plane shear stress (τ_{xy}): $(0, 0, \pm h/2)$

Example 1. A simply supported two and four layered square and two layered rectangular antisymmetric angle-ply ($\theta/-\theta/\cdots$) composite plates under sinusoidal transverse load are considered. The layers are of equal thickness. Material set 1 is used. The numerical values of non-dimensionalized maximum transverse deflection $\overline{w}$ for the square and rectangular plates are given in Tables 1 and 2 respectively. In the case of thick plates (a/h ratios 4 and 10) with different fibre orientations considered, there is a considerable difference exists between the results computed using the various models and the values reported by Ren [22]. For a/h ratio equal to 4 and fibre orientation equal to 15°, the transverse deflection $\overline{w}$ values predicted by Model-1, Model-2 and Model-3 are 4.88%, 2.62% and 11.22% lower for a two layered square plate and 3.39%, 1.39% and 8.79% lower for a four layered square plate as compared to the values obtained by Ren. Both for the square and rectangular thin laminates ($a/h=100$), for all the values of fibre orientation θ considered, the results computed using all the three models are in good agreement with those reported by Ren. The numerical values of non-dimensionalized in-plane stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$ and $\overline{\tau}_{xy}$ computed using all the models considered in the present study for a two and four layered rectangular plate with different a/h ratios and fibre orientations are given in Table 3. For a/h ratio equal to 2, Models 2 and 3 over predicts the $\overline{\sigma}_x$ values by

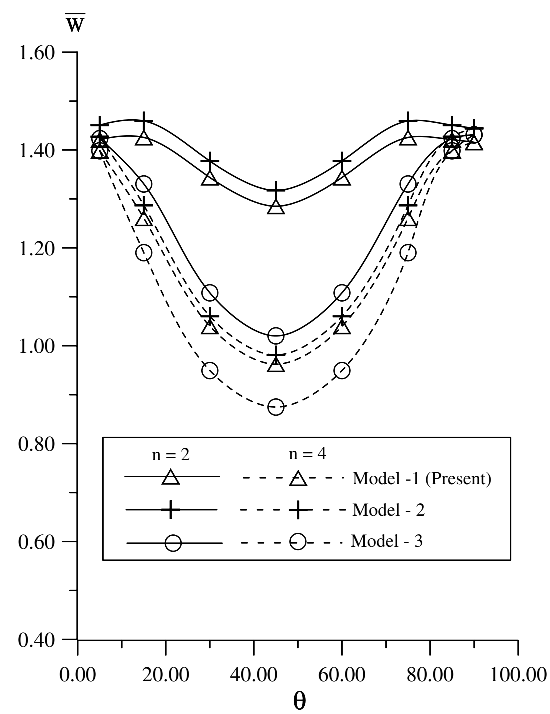


Fig. 3. Variation of non-dimensionalized transverse displacement ($\overline{w}$) for various angle of orientation (θ) in a two and four layered simply supported antisymmetric angle-ply square plate subjected to transverse sinusoidal load for a/h ratio 4.

17.28% and 42.90% compared to Model-1. For a/h value equal to 4, Fig. 3 gives the variation of maximum non-dimensionalized transverse displacement $\overline{w}$ with respect to different fibre orientations using all the models for a two and four layered square plate.

Example 2. A simply supported four layered antisymmetric angle-ply (15°/−15°/15°/−15°) rectangular composite plate with layers of equal thickness and with real material properties under sinusoidal transverse load is considered. Material set 2 is used. The non-dimensionalized maximum values of transverse displacement $\overline{w}$, in-plane stresses $\overline{\sigma}_x, \overline{\sigma}_y$ and $\overline{\tau}_{xy}$ for various values of side-to-thickness ratio and angle of orientation are given in Table 4. In the case of thick plate with a/h ratio equal to two and four, the values of $\overline{w}$, $\overline{\sigma}_x$ and $\overline{\sigma}_y$ predicted by Models 2 and 3 are almost similar whereas those predicted by Model-1 are very much different due to the effects of both transverse shear and normal deformations being considered in the displacement Model-1.

Example 3. In order to study the flexural behaviour of laminated sandwich plate, a five layered rectangular plate (15°/−15°/core/15°/−15°) with isotropic core and antisymmetric angle-ply face sheets are considered. Material set 3 is used. The ratio of the thickness of core to thickness of the face sheet t_c/t_f considered equal to 10. The non-dimensionalized maximum values of transverse displacement $\overline{w}$, in-plane stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$ and $\overline{\tau}_{xy}$ for various values of side-to-thickness ratio are given in Table 5. For plates with a/h ratio equal to 2, 4 and 10, the $\overline{w}$, $\overline{\sigma}_x$, $\overline{\sigma}_y$ and $\overline{\tau}_{xy}$ values predicted by Model-1 and Model-2 are very much closer whereas Model-3 very much underpredicts these values. For a thick plate with a/h ratio equal to 2, the values of $\overline{w}$ predicted by Model-2 and Model-3 are respectively 1.85% higher and 41.64% lower as compared to Model-1. The through the thickness variation of the non-dimensionalized in-plane stresses $\overline{\sigma}_x$, $\overline{\sigma}_y$ and $\overline{\tau}_{xy}$ and the non-dimen-

Table 4
Non-dimensionalized transverse deflection and in-plane stresses in a simply supported four layered antisymmetric angle-ply (15°/−15°/15°/−15°) rectangular ($b = 3a$) laminate under sinusoidal transverse load

a/h	Theory	$\overline{w}$	$\overline{\sigma}_x$	$\overline{\sigma}_y$	$\overline{\tau}_{xy}$
2	Model-1 (present)	8.0031	0.7807	0.2046	−0.0830
	Model-2	8.3750	0.7354	0.1082	−0.0889
	Model-3	8.3193	0.7388	0.1078	−0.0954
4	Model-1 (present)	4.0947	0.6107	0.1079	−0.0663
	Model-2	4.1587	0.6029	0.0838	−0.0676
	Model-3	4.1474	0.5995	0.0833	−0.0693
10	Model-1 (present)	2.9353	0.5622	0.0796	−0.0602
	Model-2	2.9469	0.5610	0.0757	−0.0602
	Model-3	2.9451	0.5603	0.0756	−0.0604
20	Model-1 (present)	2.7671	0.5552	0.0755	−0.0593
	Model-2	2.7724	0.5549	0.0745	−0.0590
	Model-3	2.7720	0.5547	0.0745	−0.0591
50	Model-1 (present)	2.7198	0.5532	0.0743	−0.0590
	Model-2	2.7234	0.5531	0.0742	−0.0586
	Model-3	2.7134	0.5531	0.0742	−0.0587
100	Model-1 (present)	2.7128	0.5529	0.0742	−0.0589
	Model-2	2.7162	0.5528	0.0741	−0.0586
	Model-3	2.7164	0.5529	0.0741	−0.0586

Table 5
Non-dimensionalized transverse deflection and in-plane stresses in a simply supported five layered antisymmetric angle-ply (15°/−15°/core/15°/−15°) rectangular ($b = 3a$) sandwich plate under sinusoidal transverse load

a/h	Theory	$\overline{w}$	$\overline{\sigma}_x$	$\overline{\sigma}_y$	$\overline{\tau}_{xy}$
2	Model-1 (present)	501.9803	20.8372	2.1631	−1.6997
	Model-2	511.2636	20.6804	1.7298	−1.6993
	Model-3	292.9478	10.2735	0.9041	−0.9903
4	Model-1 (present)	177.3093	8.0323	0.7504	−0.8227
	Model-2	178.5455	7.9837	0.7142	−0.8159
	Model-3	83.8321	3.7227	0.3712	−0.5052
10	Model-1 (present)	34.4140	2.4447	0.2170	−0.3629
	Model-2	34.4698	2.4602	0.2544	−0.3604
	Model-3	16.1482	1.6842	0.1775	−0.2482
20	Model-1 (present)	10.7284	1.5966	0.1462	−0.2086
	Model-2	10.7337	1.6040	0.1616	−0.2074
	Model-3	5.9387	1.4174	0.1359	−0.1531
50	Model-1 (present)	3.7792	1.3763	0.1234	−0.1253
	Model-2	3.7800	1.3771	0.1259	−0.1247
	Model-3	2.9926	1.3489	0.1209	−0.1126
100	Model-1 (present)	2.7658	1.3465	0.1195	−0.1096
	Model-2	2.7662	1.3462	0.1199	−0.1092
	Model-3	2.5682	1.3392	0.1186	−0.1059

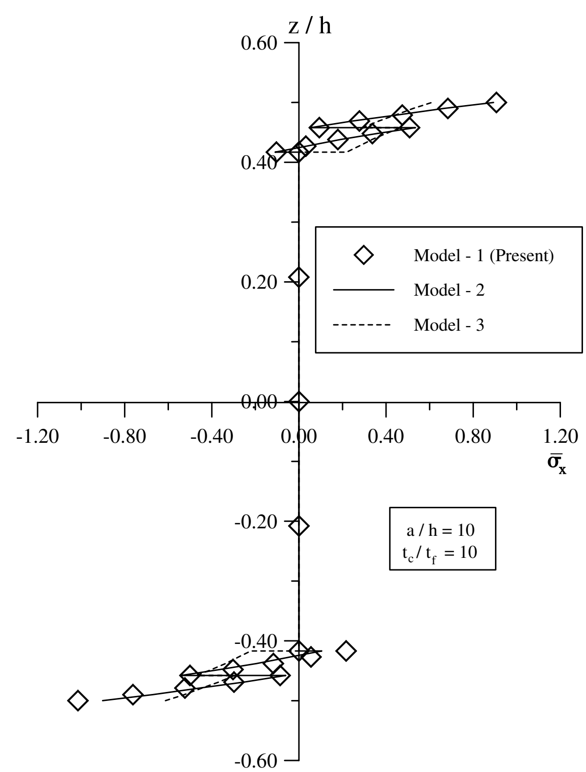


Fig. 4. Variation of non-dimensionalized in-plane normal stress ($\overline{\sigma}_x$) through the thickness (z/h) of a five layered (30°/−30°/core/30°/−30°) simply supported antisymmetric angle-ply square sandwich plate under sinusoidal transverse load.

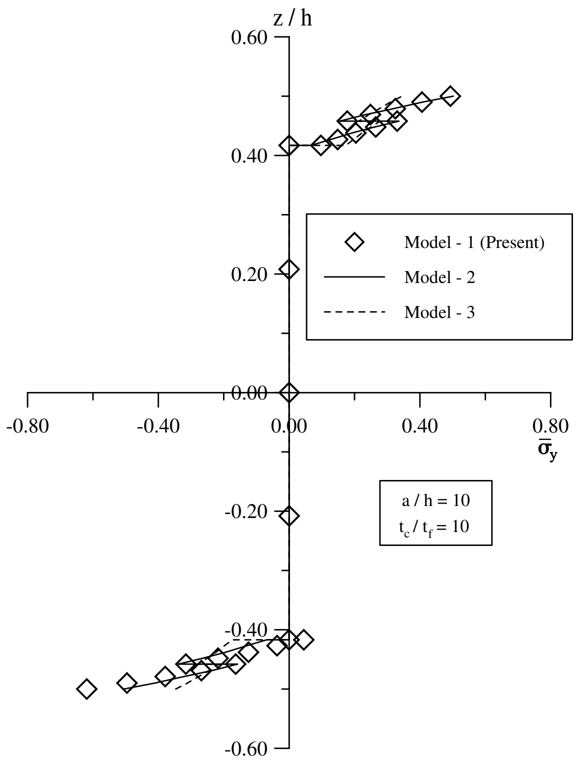


Fig. 5. Variation of non-dimensionalized in-plane normal stress ($\bar{\sigma}_y$) through the thickness (z/h) of a five layered (30°/−30°/core/30°/−30°) simply supported antisymmetric angle-ply square sandwich plate under sinusoidal transverse load.

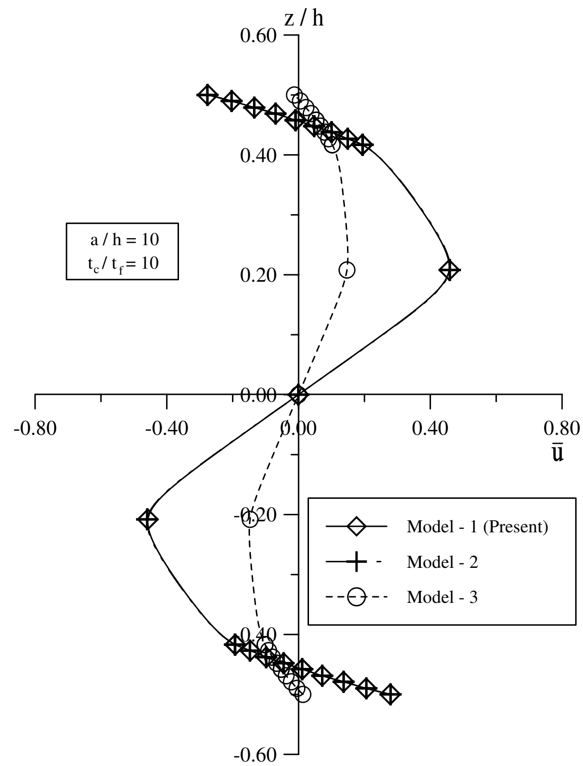


Fig. 7. Variation of non-dimensionalized in-plane displacement ($\bar{u}$) through the thickness (z/h) of a five layered (30°/−30°/core/30°/−30°) simply supported antisymmetric angle-ply square sandwich plate under sinusoidal transverse load.

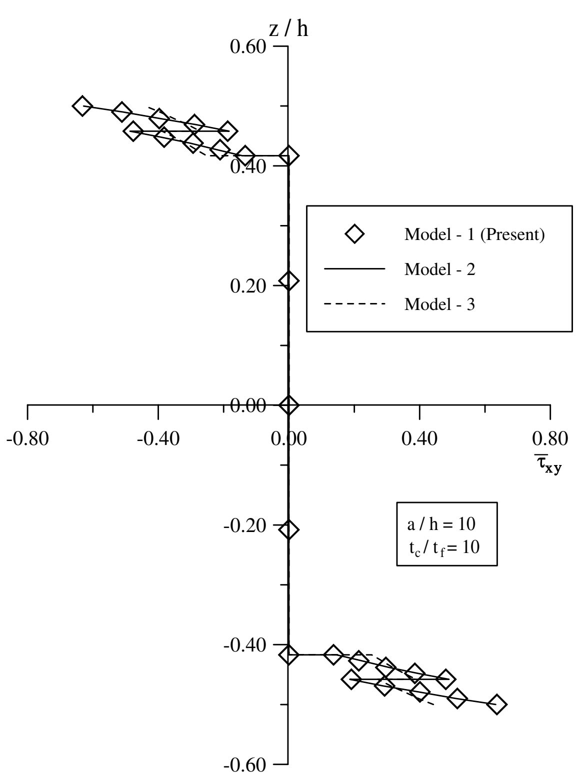


Fig. 6. Variation of non-dimensionalized in-plane shear stress ($\bar{\tau}_{xy}$) through the thickness (z/h) of a five layered (30°/−30°/core/30°/−30°) simply supported antisymmetric angle-ply square sandwich plate under sinusoidal transverse load.

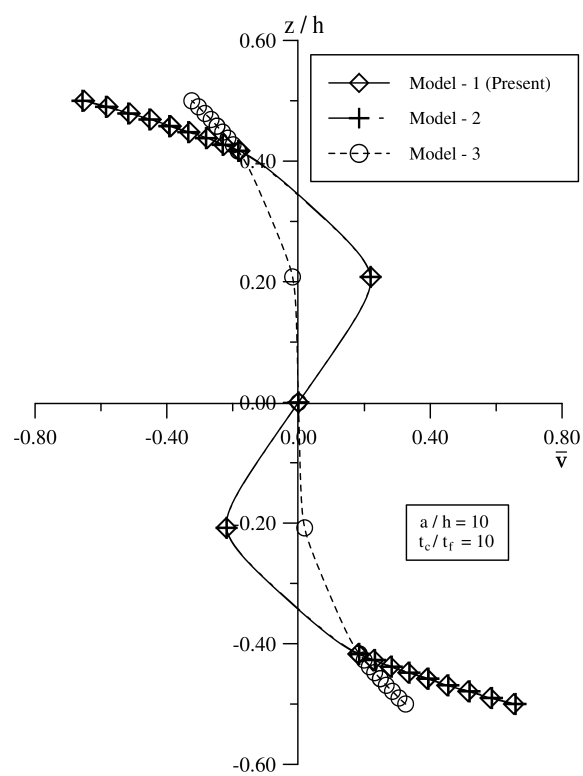


Fig. 8. Variation of non-dimensionalized in-plane displacement ($\bar{v}$) through the thickness (z/h) of a five layered (30°/−30°/core/30°/−30°) simply supported antisymmetric angle-ply square sandwich plate under sinusoidal transverse load.

sionalized in-plane displacements $\overline{u}$ and $\overline{v}$ for a simply supported five layered square sandwich plate (30°/−30°/core/30°/−30°) with isotropic core and antisymmetric angle-ply face sheets are shown in Figs. 4–8. Material set 3 is used. The through the thickness variation of in-plane displacements $\overline{u}$ and $\overline{v}$ for a plate with a/h ratio equal to 10 and ratio of the thickness of core to thickness of the face sheet t_c/t_f equal to 10 are shown in Figs. 7 and 8. It clearly indicates that the Models 1 and 2 predict the realistic through the thickness variation of displacements more accurately than Model-3.

5. Conclusion

Analytical formulations and solutions to the static analysis of simply supported antisymmetric angle-ply composite and sandwich plates hitherto not reported in the literature based on a higher order refined theory which takes in to account the effects of both transverse shear and transverse normal deformations are presented. The accuracy of the present computational model with twelve degrees of freedom in comparison to other higher order models with nine and five degrees of freedom considered in the present investigation in predicting the in-plane stresses, in-plane and transverse displacements has been established. After ascertaining the accuracy, new results for multilayered sandwich plates with antisymmetric angle-ply face sheets are presented which will serve as a benchmark for future investigations.

Appendix A

Coefficients of $[C]$ matrix

$$C_{11}=\frac{E_1(1-\nu_{23}\nu_{32})}{\Delta};\quad C_{12}=\frac{E_1(\nu_{21}+\nu_{31}\nu_{23})}{\Delta}$$
$$C_{13}=\frac{E_1(\nu_{31}+\nu_{21}\nu_{32})}{\Delta};\quad C_{22}=\frac{E_2(1-\nu_{13}\nu_{31})}{\Delta}$$
$$C_{23}=\frac{E_2(\nu_{32}+\nu_{12}\nu_{31})}{\Delta};\quad C_{33}=\frac{E_3(1-\nu_{12}\nu_{21})}{\Delta}$$
$$C_{44}=G_{12};\quad C_{55}=G_{23};\quad C_{66}=G_{13}$$

where

$$\Delta=(1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{12}\nu_{23}\nu_{31})$$

and

$$\varepsilon_1=\frac{\sigma_1}{E_1}-\nu_{21}\frac{\sigma_2}{E_2}-\nu_{31}\frac{\sigma_3}{E_3}$$
$$\varepsilon_2=\frac{\sigma_2}{E_2}-\nu_{32}\frac{\sigma_3}{E_3}-\nu_{12}\frac{\sigma_1}{E_1}$$
$$\varepsilon_3=\frac{\sigma_3}{E_3}-\nu_{13}\frac{\sigma_1}{E_1}-\nu_{23}\frac{\sigma_2}{E_2}$$
$$\frac{\nu_{12}}{E_1}=\frac{\nu_{21}}{E_2},\quad \frac{\nu_{31}}{E_3}=\frac{\nu_{13}}{E_1},\quad \frac{\nu_{32}}{E_3}=\frac{\nu_{23}}{E_2}$$

Appendix B

Coefficients of $[Q]$ matrix

$$Q_{11}=C_{11}c^4+2(C_{12}+2C_{44})s^2c^2+C_{22}s^4$$
$$Q_{12}=C_{12}(c^4+s^4)+(C_{11}+C_{22}-4C_{44})s^2c^2$$
$$Q_{13}=C_{13}c^2+C_{23}s^2$$
$$Q_{14}=(C_{11}-C_{12}-2C_{44})sc^3+(C_{12}-C_{22}+2C_{44})cs^3$$
$$Q_{22}=C_{11}s^4+C_{22}c^4+(2C_{12}+4C_{44})s^2c^2$$
$$Q_{23}=C_{13}s^2+C_{23}c^2$$
$$Q_{24}=(C_{11}-C_{12}-2C_{44})s^3c+(C_{12}-C_{22}+2C_{44})c^3s$$
$$Q_{33}=C_{33}$$
$$Q_{34}=(C_{31}-C_{32})sc$$
$$Q_{44}=(C_{11}-2C_{12}+C_{22}-2C_{44})c^2s^2+C_{44}(c^4+s^4)$$
$$Q_{55}=C_{55}c^2+C_{66}s^2$$
$$Q_{56}=(C_{66}-C_{55})cs$$
$$Q_{66}=C_{55}s^2+C_{66}c^2$$

and $Q_{ij}=Q_{ji}$, $i, j=$ 1–6, where $c=\cos\alpha$ and $s=\sin\alpha$

Appendix C

Elements of $[A]$, $[A']$, $[B]$, $[B']$, $[D]$, $[D']$, $[E]$, $[E']$ matrices

$$[A]=\sum_{L=1}^{NL}\begin{bmatrix}
Q_{11}H_1 & Q_{12}H_1 & Q_{11}H_3 & Q_{12}H_3 & Q_{13}H_1 & 3Q_{13}H_3 & Q_{11}H_2 & Q_{12}H_2 & Q_{11}H_4 & Q_{12}H_4 & 2Q_{13}H_2\\
Q_{12}H_1 & Q_{22}H_1 & Q_{12}H_3 & Q_{22}H_3 & Q_{23}H_1 & 3Q_{23}H_3 & Q_{12}H_2 & Q_{22}H_2 & Q_{12}H_4 & Q_{22}H_4 & 2Q_{23}H_2\\
Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_5 & Q_{12}H_5 & Q_{13}H_3 & 3Q_{13}H_5 & Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_6 & Q_{12}H_6 & 2Q_{13}H_4\\
Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_5 & Q_{22}H_5 & Q_{23}H_3 & 3Q_{23}H_5 & Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_6 & Q_{22}H_6 & 2Q_{23}H_4\\
Q_{13}H_1 & Q_{23}H_1 & Q_{13}H_3 & Q_{23}H_3 & Q_{33}H_1 & 3Q_{33}H_3 & Q_{13}H_2 & Q_{23}H_2 & Q_{13}H_4 & Q_{23}H_4 & 2Q_{33}H_2\\
Q_{13}H_3 & Q_{23}H_3 & Q_{13}H_5 & Q_{23}H_5 & Q_{33}H_3 & 3Q_{33}H_5 & Q_{13}H_4 & Q_{23}H_4 & Q_{13}H_6 & Q_{23}H_6 & 2Q_{33}H_4\\
Q_{11}H_2 & Q_{12}H_2 & Q_{11}H_4 & Q_{12}H_4 & Q_{13}H_2 & 3Q_{13}H_4 & Q_{11}H_3 & Q_{12}H_3 & Q_{11}H_5 & Q_{12}H_5 & 2Q_{13}H_3\\
Q_{12}H_2 & Q_{22}H_2 & Q_{12}H_4 & Q_{22}H_4 & Q_{23}H_2 & 3Q_{23}H_4 & Q_{12}H_3 & Q_{22}H_3 & Q_{12}H_5 & Q_{22}H_5 & 2Q_{23}H_3\\
Q_{11}H_4 & Q_{12}H_4 & Q_{11}H_6 & Q_{12}H_6 & Q_{13}H_4 & 3Q_{13}H_6 & Q_{11}H_5 & Q_{12}H_5 & Q_{11}H_7 & Q_{12}H_7 & 2Q_{13}H_5\\
Q_{12}H_4 & Q_{22}H_4 & Q_{12}H_6 & Q_{22}H_6 & Q_{23}H_4 & 3Q_{23}H_6 & Q_{12}H_5 & Q_{22}H_5 & Q_{12}H_7 & Q_{22}H_7 & 2Q_{23}H_5\\
Q_{13}H_2 & Q_{23}H_2 & Q_{13}H_4 & Q_{23}H_4 & Q_{33}H_2 & 3Q_{33}H_4 & Q_{13}H_3 & Q_{23}H_3 & Q_{13}H_5 & Q_{23}H_5 & 2Q_{33}H_3
\end{bmatrix}$$

$$[A'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{14}H_1 & Q_{14}H_1 & Q_{14}H_3 & Q_{14}H_3 & Q_{14}H_2 & Q_{14}H_2 & Q_{14}H_4 & Q_{14}H_4 \\ Q_{24}H_1 & Q_{24}H_1 & Q_{24}H_3 & Q_{24}H_3 & Q_{24}H_2 & Q_{24}H_2 & Q_{24}H_4 & Q_{24}H_4 \\ Q_{14}H_3 & Q_{14}H_3 & Q_{14}H_5 & Q_{14}H_5 & Q_{14}H_4 & Q_{14}H_4 & Q_{14}H_6 & Q_{14}H_6 \\ Q_{24}H_3 & Q_{24}H_3 & Q_{24}H_5 & Q_{24}H_5 & Q_{24}H_4 & Q_{24}H_4 & Q_{24}H_6 & Q_{24}H_6 \\ Q_{34}H_1 & Q_{34}H_1 & Q_{34}H_3 & Q_{34}H_3 & Q_{34}H_2 & Q_{34}H_2 & Q_{34}H_4 & Q_{34}H_4 \\ Q_{34}H_3 & Q_{34}H_3 & Q_{34}H_5 & Q_{34}H_5 & Q_{34}H_4 & Q_{34}H_4 & Q_{34}H_6 & Q_{34}H_6 \\ Q_{14}H_2 & Q_{14}H_2 & Q_{14}H_4 & Q_{14}H_4 & Q_{14}H_3 & Q_{14}H_3 & Q_{14}H_5 & Q_{14}H_5 \\ Q_{24}H_2 & Q_{24}H_2 & Q_{24}H_4 & Q_{24}H_4 & Q_{24}H_3 & Q_{24}H_3 & Q_{24}H_5 & Q_{24}H_5 \\ Q_{14}H_4 & Q_{14}H_4 & Q_{14}H_6 & Q_{14}H_6 & Q_{14}H_5 & Q_{14}H_5 & Q_{14}H_7 & Q_{14}H_7 \\ Q_{24}H_4 & Q_{24}H_4 & Q_{24}H_6 & Q_{24}H_6 & Q_{24}H_5 & Q_{24}H_5 & Q_{24}H_7 & Q_{24}H_7 \\ Q_{34}H_2 & Q_{34}H_2 & Q_{34}H_4 & Q_{34}H_4 & Q_{34}H_3 & Q_{34}H_3 & Q_{34}H_5 & Q_{34}H_5 \end{bmatrix}$$

$$[B] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{44}H_1 & Q_{44}H_1 & Q_{44}H_3 & Q_{44}H_3 & Q_{44}H_2 & Q_{44}H_2 & Q_{44}H_4 & Q_{44}H_4 \\ Q_{44}H_3 & Q_{44}H_3 & Q_{44}H_5 & Q_{44}H_5 & Q_{44}H_4 & Q_{44}H_4 & Q_{44}H_6 & Q_{44}H_6 \\ Q_{44}H_2 & Q_{44}H_2 & Q_{44}H_4 & Q_{44}H_4 & Q_{44}H_3 & Q_{44}H_3 & Q_{44}H_5 & Q_{44}H_5 \\ Q_{44}H_4 & Q_{44}H_4 & Q_{44}H_6 & Q_{44}H_6 & Q_{44}H_5 & Q_{44}H_5 & Q_{44}H_7 & Q_{44}H_7 \end{bmatrix}$$

$$[B'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{14}H_1 & Q_{24}H_1 & Q_{14}H_3 & Q_{24}H_3 & Q_{34}H_1 & 3Q_{34}H_3 & Q_{14}H_2 & Q_{24}H_2 & Q_{14}H_4 & Q_{24}H_4 & 2Q_{34}H_2 \\ Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_5 & Q_{24}H_5 & Q_{34}H_3 & 3Q_{34}H_5 & Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_6 & Q_{24}H_6 & 2Q_{34}H_4 \\ Q_{14}H_2 & Q_{24}H_2 & Q_{14}H_4 & Q_{24}H_4 & Q_{34}H_2 & 3Q_{34}H_4 & Q_{14}H_3 & Q_{24}H_3 & Q_{14}H_5 & Q_{24}H_5 & 2Q_{34}H_3 \\ Q_{14}H_4 & Q_{24}H_4 & Q_{14}H_6 & Q_{24}H_6 & Q_{34}H_4 & 3Q_{34}H_6 & Q_{14}H_5 & Q_{24}H_5 & Q_{14}H_7 & Q_{24}H_7 & 2Q_{34}H_5 \end{bmatrix}$$

$$[D] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{66}H_1 & Q_{66}H_1 & 3Q_{66}H_3 & Q_{66}H_3 & 2Q_{66}H_2 & Q_{66}H_2 & Q_{66}H_4 \\ Q_{66}H_3 & Q_{66}H_3 & 3Q_{66}H_5 & Q_{66}H_5 & 2Q_{66}H_4 & Q_{66}H_4 & Q_{66}H_6 \\ Q_{66}H_2 & Q_{66}H_2 & 3Q_{66}H_4 & Q_{66}H_4 & 2Q_{66}H_3 & Q_{66}H_3 & Q_{66}H_5 \\ Q_{66}H_4 & Q_{66}H_4 & 3Q_{66}H_6 & Q_{66}H_6 & 2Q_{66}H_5 & Q_{66}H_5 & Q_{66}H_7 \end{bmatrix}$$

$$[D'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{56}H_1 & Q_{56}H_1 & 3Q_{56}H_3 & Q_{56}H_3 & 2Q_{56}H_2 & Q_{56}H_2 & Q_{56}H_4 \\ Q_{56}H_3 & Q_{56}H_3 & 3Q_{56}H_5 & Q_{56}H_5 & 2Q_{56}H_4 & Q_{56}H_4 & Q_{56}H_6 \\ Q_{56}H_2 & Q_{66}H_2 & 3Q_{56}H_4 & Q_{56}H_4 & 2Q_{56}H_3 & Q_{56}H_3 & Q_{56}H_5 \\ Q_{56}H_4 & Q_{56}H_4 & 3Q_{56}H_6 & Q_{56}H_6 & 2Q_{56}H_5 & Q_{56}H_5 & Q_{56}H_7 \end{bmatrix}$$

$$[E] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{55}H_1 & Q_{55}H_1 & 3Q_{55}H_3 & Q_{55}H_3 & 2Q_{55}H_2 & Q_{55}H_2 & Q_{55}H_4 \\ Q_{55}H_3 & Q_{55}H_3 & 3Q_{55}H_5 & Q_{55}H_5 & 2Q_{55}H_4 & Q_{55}H_4 & Q_{55}H_6 \\ Q_{55}H_2 & Q_{55}H_2 & 3Q_{55}H_4 & Q_{55}H_4 & 2Q_{55}H_3 & Q_{55}H_3 & Q_{55}H_5 \\ Q_{55}H_4 & Q_{55}H_4 & 3Q_{55}H_6 & Q_{55}H_6 & 2Q_{55}H_5 & Q_{55}H_5 & Q_{55}H_7 \end{bmatrix}$$

$$[E'] = \sum_{L=1}^{NL} \begin{bmatrix} Q_{56}H_1 & Q_{56}H_1 & 3Q_{56}H_3 & Q_{56}H_3 & 2Q_{56}H_2 & Q_{56}H_2 & Q_{56}H_4 \\ Q_{56}H_3 & Q_{56}H_3 & 3Q_{56}H_5 & Q_{56}H_5 & 2Q_{56}H_4 & Q_{56}H_4 & Q_{56}H_6 \\ Q_{56}H_2 & Q_{56}H_2 & 3Q_{56}H_4 & Q_{56}H_4 & 2Q_{56}H_3 & Q_{56}H_3 & Q_{56}H_5 \\ Q_{56}H_4 & Q_{56}H_4 & 3Q_{56}H_6 & Q_{56}H_6 & 2Q_{56}H_5 & Q_{56}H_5 & Q_{56}H_7 \end{bmatrix}$$

Appendix D

Coefficients of matrix $[X]$

$X_{1,1} = A_{1,1}\alpha^2 + B_{1,1}\beta^2, \quad X_{1,2} = A_{1,2}\alpha\beta + B_{1,2}\alpha\beta$

$X_{1,3} = 0, \quad X_{1,4} = A'_{1,5}\alpha\beta + B'_{1,7}\alpha\beta$

$X_{1,5} = A'_{1,6}\alpha^2 + B'_{1,8}\beta^2, \quad X_{1,6} = -B'_{1,5}\beta$

$X_{1,7} = A_{1,3}\alpha^2 + B_{1,3}\beta^2, \quad X_{1,8} = A_{1,4}\alpha\beta + B_{1,4}\alpha\beta$

$X_{1,9} = -B'_{1,11}\beta, \quad X_{1,10} = A'_{1,7}\alpha\beta + B'_{1,9}\alpha\beta$

$X_{1,11} = A'_{1,8}\alpha^2 + B'_{1,10}\beta^2, \quad X_{1,12} = -B'_{1,6}\beta$

$X_{2,1} = A_{2,1}\alpha\beta + B_{1,1}\alpha\beta, \quad X_{2,2} = A_{2,2}\beta^2 + B_{1,2}\alpha^2$

$X_{2,3} = 0, \quad X_{2,4} = A'_{2,5}\beta^2 + B'_{1,7}\alpha^2$

$X_{2,5} = A'_{2,6}\alpha\beta + B'_{1,8}\alpha\beta, \quad X_{2,6} = -B'_{1,5}\alpha$

$X_{2,7} = A_{2,3}\alpha\beta + B_{1,3}\alpha\beta, \quad X_{2,8} = A_{2,4}\beta^2 + B_{1,4}\alpha^2$

$X_{2,9} = -B'_{1,11}\alpha, \quad X_{2,10} = A'_{2,7}\beta^2 + B'_{1,9}\alpha^2$

$X_{2,11} = A'_{2,8}\alpha\beta + B'_{1,10}\alpha\beta, \quad X_{2,12} = -B'_{1,6}\alpha$

$X_{3,1} = 0, \quad X_{3,2} = 0, \quad X_{3,3} = D_{1,2}\alpha^2 + E_{1,2}\beta^2$

$X_{3,4} = D_{1,1}\alpha, \quad X_{3,5} = E_{1,1}\beta, \quad X_{3,6} = D_{1,6}\alpha^2 + E_{1,6}\beta^2$

$X_{3,7} = E'_{1,5}\beta, \quad X_{3,8} = D'_{1,5}\alpha, \quad X_{3,9} = D_{1,4}\alpha^2 + E_{1,4}\beta^2$

$X_{3,10} = D_{1,3}\alpha, \quad X_{3,11} = E_{1,3}\beta, \quad X_{3,12} = D_{1,7}\alpha^2 + E_{1,7}\beta^2$

$X_{4,1} = A'_{7,1}\alpha\beta + B'_{3,1}\alpha\beta, \quad X_{4,2} = A'_{7,2}\alpha^2 + B'_{3,2}\beta^2$

$X_{4,3} = D_{1,2}\alpha, \quad X_{4,4} = A_{7,7}\alpha^2 + B_{3,5}\beta^2 + D_{1,1}$

$X_{4,5} = A_{7,8}\alpha\beta + B_{3,6}\alpha\beta, \quad X_{4,6} = D_{1,6}\alpha - A_{7,5}\alpha$

$X_{4,7} = A'_{7,3}\alpha\beta + B'_{3,3}\alpha\beta, \quad X_{4,8} = A'_{7,4}\alpha^2 + B'_{3,4}\beta^2 + D'_{1,5}$

$X_{4,9} = D_{1,4}\alpha - A_{7,11}\alpha, \quad X_{4,10} = A_{7,9}\alpha^2 + B_{3,7}\beta^2 + D_{1,3}$

$X_{4,11} = A_{7,10}\alpha\beta + B_{3,8}\alpha\beta, \quad X_{4,12} = D_{1,7}\alpha - A_{7,6}\alpha$

$X_{5,1} = A'_{8,1}\beta^2 + B'_{3,1}\alpha^2, \quad X_{5,2} = A'_{8,2}\alpha\beta + B'_{3,2}\alpha\beta$

$X_{5,3} = E_{1,2}\beta, \quad X_{5,4} = A_{8,7}\alpha\beta + B_{3,5}\alpha\beta$

$X_{5,5} = A_{8,8}\beta^2 + B_{3,6}\alpha^2 + E_{1,1}, \quad X_{5,6} = E_{1,6}\beta - A_{8,5}\beta$

$X_{5,7} = A'_{8,3}\beta^2 + B'_{3,3}\alpha^2 + E'_{1,5}, \quad X_{5,8} = A'_{8,4}\alpha\beta + B'_{3,4}\alpha\beta$

$X_{5,9} = E_{1,4}\beta - A_{8,11}\beta, \quad X_{5,10} = A_{8,9}\alpha\beta + B_{3,7}\alpha\beta$

$X_{5,11} = A_{8,10}\beta^2 + B_{3,8}\alpha^2 + E_{1,3}, \quad X_{5,12} = E_{1,7}\beta - A_{8,6}\beta$

$X_{6,1} = -A'_{5,1}\beta, \quad X_{6,2} = -A'_{5,2}\alpha$

$X_{6,3} = D_{3,2}\alpha^2 + E_{3,2}\beta^2, \quad X_{6,4} = D_{3,1}\alpha - A_{5,7}\alpha$

$X_{6,5} = E_{3,1}\beta - A_{5,8}\beta, \quad X_{6,6} = D_{3,6}\alpha^2 + E_{3,6}\beta^2 + A_{5,5}$

$X_{6,7} = E'_{3,5}\beta - A'_{5,3}\beta, \quad X_{6,8} = D'_{3,5}\alpha - A'_{5,4}\alpha$

$X_{6,9} = D_{3,4}\alpha^2 + E_{3,4}\beta^2 + A_{5,11}, \quad X_{6,10} = D_{3,3}\alpha - A_{5,9}\alpha$

$X_{6,11} = E_{3,3}\beta - A_{5,10}\beta, \quad X_{6,12} = D_{3,7}\alpha^2 + E_{3,7}\beta^2 + A_{5,6}$

$X_{7,1} = A_{3,1}\alpha^2 + B_{2,1}\beta^2, \quad X_{7,2} = A_{3,2}\alpha\beta + B_{2,2}\alpha\beta$

$X_{7,3} = 2D'_{3,2}\beta, \quad X_{7,4} = A'_{3,5}\alpha\beta + B'_{2,7}\alpha\beta$

$X_{7,5} = A'_{3,6}\alpha^2 + B'_{2,8}\beta^2 + 2D'_{3,1}, \quad X_{7,6} = 2D'_{3,6}\beta - B'_{2,5}\beta$

$X_{7,7} = A_{3,3}\alpha^2 + B_{2,3}\beta^2 + 2D_{3,5}, \quad X_{7,8} = A_{3,4}\alpha\beta + B_{2,4}\alpha\beta$

$X_{7,9} = 2D'_{3,4}\beta - B'_{2,11}\beta, \quad X_{7,10} = A'_{3,7}\alpha\beta + B'_{2,9}\alpha\beta$

$X_{7,11} = A'_{3,8}\alpha^2 + B'_{2,10}\beta^2 + 2D'_{3,3}, \quad X_{7,12} = 2D'_{3,7}\beta - B'_{2,6}\beta$

$X_{8,1} = A_{4,1}\alpha\beta + B_{2,1}\alpha\beta, \quad X_{8,2} = A_{4,2}\beta^2 + B_{2,2}\alpha^2$

$X_{8,3} = 2E'_{3,2}\alpha, \quad X_{8,4} = A'_{4,5}\beta^2 + B'_{2,7}\alpha^2 + 2E'_{3,1}$

$X_{8,5} = A'_{4,6}\alpha\beta + B'_{2,8}\alpha\beta, \quad X_{8,6} = 2E'_{3,6}\alpha - B'_{2,5}\alpha$

$X_{8,7} = A_{4,3}\alpha\beta + B_{2,3}\alpha\beta, \quad X_{8,8} = A_{4,4}\beta^2 + B_{2,4}\alpha^2 + 2E_{3,5}$

$X_{8,9} = 2E'_{3,4}\alpha - B'_{2,11}\alpha, \quad X_{8,10} = A'_{4,7}\beta^2 + B'_{2,9}\alpha^2 + 2E'_{3,3}$

$X_{8,11} = A'_{4,8}\alpha\beta + B'_{2,10}\alpha\beta, \quad X_{8,12} = 2E'_{3,7}\alpha - B'_{2,6}\alpha$

$X_{9,1} = -2A'_{11,1}\beta, \quad X_{9,2} = -2A'_{11,2}\alpha$

$X_{9,3} = D_{2,2}\alpha^2 + E_{2,2}\beta^2, \quad X_{9,4} = D_{2,1}\alpha - 2A_{11,7}\alpha$

$X_{9,5} = E_{2,1}\beta - 2A_{11,8}\beta, \quad X_{9,6} = D_{2,6}\alpha^2 + E_{2,6}\beta^2 + 2A_{11,5}$

$X_{9,7} = E'_{2,5}\beta - 2A'_{11,3}\beta, \quad X_{9,8} = D'_{2,5}\alpha - 2A'_{11,4}\alpha$

$X_{9,9} = D_{2,4}\alpha^2 + E_{2,4}\beta^2 + 2A_{11,11}, \quad X_{9,10} = D_{2,3}\alpha - 2A_{11,9}\alpha$

$X_{9,11} = E_{2,3}\beta - 2A_{11,10}\beta, \quad X_{9,12} = D_{2,7}\alpha^2 + E_{2,7}\beta^2 + 2A_{11,6}$

$X_{10,1} = A'_{9,1}\alpha\beta + B'_{4,1}\alpha\beta, \quad X_{10,2} = A'_{9,2}\alpha^2 + B'_{4,2}\beta^2$

$X_{10,3} = 3D_{2,2}\alpha, \quad X_{10,4} = A_{9,7}\alpha^2 + B_{4,5}\beta^2 + 3D_{2,1}$

$X_{10,5} = A_{9,8}\alpha\beta + B_{4,6}\alpha\beta, \quad X_{10,6} = 3D_{2,6}\alpha - A_{9,5}\alpha$

$X_{10,7} = A'_{9,3}\alpha\beta + B'_{4,3}\alpha\beta, \quad X_{10,8} = A'_{9,4}\alpha^2 + B'_{4,4}\beta^2 + 3D'_{2,5}$

$X_{10,9} = 3D_{2,4}\alpha - A_{9,11}\alpha, \quad X_{10,10} = A_{9,9}\alpha^2 + B_{4,7}\beta^2 + 3D_{2,3}$

$X_{10,11} = A_{9,10}\alpha\beta + B_{4,8}\alpha\beta, \quad X_{10,12} = 3D_{2,7}\alpha - A_{9,6}\alpha$

$X_{11,1} = A'_{10,1}\beta^2 + B'_{4,1}\alpha^2, \quad X_{11,2} = A'_{10,2}\alpha\beta + B'_{4,2}\alpha\beta$

$X_{11,3} = 3E_{2,2}\beta, \quad X_{11,4} = A_{10,7}\alpha\beta + B_{4,5}\alpha\beta$

$X_{11,5} = A_{10,8}\beta^2 + B_{4,6}\alpha^2 + 3E_{2,1}, \quad X_{11,6} = 3E_{2,6}\beta - A_{10,5}\beta$

$X_{11,7} = A'_{10,3}\beta^2 + B'_{4,3}\alpha^2 + 3E'_{2,5}, \quad X_{11,8} = A'_{10,4}\alpha\beta + B'_{4,4}\alpha\beta$

$X_{11,9} = 3E_{2,4}\beta - A_{10,11}\beta, \quad X_{11,10} = A_{10,9}\alpha\beta + B_{4,7}\alpha\beta$

$X_{11,11} = A_{10,10}\beta^2 + B_{4,8}\alpha^2 + 3E_{2,3}, \quad X_{11,12} = 3E_{2,7}\beta - A_{10,6}\beta$

$X_{12,1} = -3A'_{6,1}\beta, \quad X_{12,2} = -3A'_{6,2}\alpha$

$X_{12,3} = D_{4,2}\alpha^2 + E_{4,2}\beta^2, \quad X_{12,4} = D_{4,1}\alpha - 3A_{6,7}\alpha$

$X_{12,5} = E_{4,1}\beta - 3A_{6,8}\beta, \quad X_{12,6} = D_{4,6}\alpha^2 + E_{4,6}\beta^2 + 3A_{6,5}$

$X_{12,7} = E'_{4,5}\beta - 3A'_{6,3}\beta, \quad X_{12,8} = D'_{4,5}\alpha - 3A'_{6,4}\alpha$

$X_{12,9} = D_{4,4}\alpha^2 + E_{4,4}\beta^2 + 3A_{6,11}, \quad X_{12,10} = D_{4,3}\alpha - 3A_{6,9}\alpha$

$X_{12,11} = E_{4,3}\beta - 3A_{6,10}\beta, \quad X_{12,12} = D_{4,7}\alpha^2 + E_{4,7}\beta^2 + 3A_{6,6}$

References

[1] Noor AK, Burton WS. Assessment of shear deformation theories for multilayered composite plates. Appl Mech Rev 1989;42(1):1–13.

[2] Noor AK, Burton WS. Assessment of computational model for multilayered composite shells. Appl Mech Rev 1990;43(4):67–97.

[3] Noor AK, Burton WS, Bert CW. Computational models for sandwich panels and shells. Appl Mech Rev 1996;49(3):155–99.

[4] Kant T, Swaminathan K. Estimation of transverse/interlaminar stresses in laminated composites – a selective review and survey of current developments. Compos Struct 2001;49(1):65–75.

[5] Kant T. Numerical analysis of thick plates. Comput Methods Appl Mech Eng 1982;31:1–18.

[6] Pandya BN, Kant T. A consistent refined theory for flexure of a symmetric laminate. Mech Res Commun 1987;14:107–13.
[7] Pandya BN, Kant T. Higher order shear deformable theories for flexure of sandwich plates – finite element evaluations. Int J Solids Struct 1988;24(12):1267–86.
[8] Pandya BN, Kant T. Flexure analysis of laminated composites using refined higher order C° plate bending elements. Comput Methods Appl Mech Eng 1988;66:173–98.
[9] Pandya BN, Kant T. A refined higher order generally orthotropic C° plate bending element. Comput Struct 1988;28:119–33.
[10] Pandya BN, Kant T. Finite element stress analysis of laminated composites using higher order displacement model. Compos Sci Technol 1988;32:137–55.
[11] Kant T, Manjunatha BS. An unsymmetric FRC laminate C° finite element model with 12 degrees of freedom per node. Eng Comput 1988;5(3):300–8.
[12] Kant T, Swaminathan K. Analytical solutions using a higher order refined theory for the stability analysis of laminated composite and sandwich plates. Struct Eng Mech Int J 2000;10(4):337–57.
[13] Kant T, Swaminathan K. Free vibration of isotropic, orthotropic and multilayer plates based on higher order refined theories. J Sound Vib 2001;241(2):319–27.
[14] Kant T, Swaminathan K. Analytical solutions for free vibration of laminated composite and sandwich plates based on a higher order refined theory. Compos Struct 2001;53(1):73–85.
[15] Kant T, Swaminathan K. Analytical solutions for static analysis of laminated composite and sandwich plates based on a higher order refined theory. Compos Struct 2002;56(4):329–44.
[16] Swaminathan K, Ragounadin D. Analytical solutions using a higher order refined theory for the static analysis of antisymmetric angle ply composite and sandwich plates. Compos Struct 2004;64(3–4): 405–17.
[17] Reddy JN. A simple higher-order theory for laminated composite plates. J Appl Mech 1984;51:745–52.
[18] Reddy JN. Energy and variational methods in applied mechanics. New York: John Wiley and Sons; 1984.
[19] Reddy JN. Mechanics of laminated composite plates, theory and analysis. Boca Raton, FL, USA: CRC Press; 1996.
[20] Timoshenko SP, Woinowsky-Krieger S. Theory of plates and shells. New York: McGraw-Hill; 1959.
[21] Szilard R. Theory and analysis of plates (classical and numerical methods). New Jersy: Prentice-Hall; 1974.
[22] Ren JG. Bending, vibration and buckling of laminated plates. In: Cheremisinoff NP, editor. Handbook of ceramics and composites, vol. 1. New York: Marcel Dekker; 1990. p. 413–50.