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LAVRENTIEV'S REGULARIZATION METHOD FOR NONLINEAR ILL-POSED EQUATIONS IN BANACH SPACES*

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Abstract In this paper, we deal with nonlinear ill-posed problems involving m-accretive mappings in Banach spaces. We consider a derivative and inverse free method for the implementation of Lavrentiev regularization method. Using general Hölder type source condition we obtain an optimal order error estimate. Also we consider the adaptive parameter choice strategy proposed by Pereverzev and Schock (2005) for choosing the regularization parameter.

Key words nonlinear ill-posed problem; Banach space; Lavrentiev regularization; m-accretive mappings; adaptive parameter choice strategy

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1 Introduction

Let E be a real Banach space with its dual space E^*. The norm of E and E^* are denoted by $\| \, . \, \|$ and we write $\langle x, j\rangle$ instead of $j(x)$ for $j \in E^*$ and $x \in E$. In this paper we consider the problem of approximately solving the non linear ill-posed equation

$$F(u) = f, \quad f \in E. \tag{1.1}$$

Here $F : E \to E$ is an m-accretive (see [1, 2, 4]), Fréchet differentiable and single valued non-linear mapping. The Fréchet derivative of F at x is denoted by $F'(x)$.

Note that F is an m-accretive and single valued in E means, F has the following properties (see [6, 9, 12]):

1) $\langle F(x) - F(y), J(x - y)\rangle \geq 0$, where J is the dual mapping on E;

2) $R(F + \lambda I) = E$ for each $\lambda \geq 0$ where $R(F)$ and I denote the range of F and the identity mapping on E respectively.

In other words, if F is m-accretive, then the equation

$$F(u) + \alpha(u - u_0) = f_\delta, \quad \| f_\delta - f \| \leq \delta \longrightarrow 0 \tag{1.2}$$

has a unique solution u_α^δ for $\alpha \geq 0$ and $f^\delta \in E$ (see [6, 9, 16]). Here and below u_0 is the initial guess of the exact solution $\hat{u}$ (which is assumed to exist) of (1.1).

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A typical example of (1.1) is the parameter identification problem in an elliptic PDE [11]; i.e., to find the source term q in the elliptic boundary value problem

$$\begin{aligned} -\Delta u + \xi(u) &= q \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega \end{aligned}$$

from measurement of u in Ω. Here $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuously differentiable monotonically increasing function and $\Omega \subseteq \mathbb{R}^3$ is a smooth domain. The corresponding forward operator in this case is $F : H^2(\Omega) \longrightarrow H^2(\Omega)$ defined by

$$F(q) = u,$$

which is monotone. This can be seen as follows:

$$\begin{aligned} \langle F(q_1) - F(q_2), q_1 - q_2\rangle &= \int_\Omega (u_1 - u_2)(q_1 - q_2)\mathrm{d}x \\ &= \int_\Omega (u_1 - u_2)(-\Delta(u_1 - u_2) + \xi(u_1) - \xi(u_2))\mathrm{d}x \\ &= \int_\Omega (|\Delta(u_1 - u_2)|^2 + (\xi(u_1) - \xi(u_2))(u_1 - u_2))\mathrm{d}x \\ &\geq \| \bigtriangledown (u_1 - u_2)\|^2_{L^2(\Omega)} \geq 0. \end{aligned}$$

In the earlier studies such as [3, 6, 9, 13, 18], the optimal order convergence rate for $\|u_\alpha^\delta - \hat{u}\|$ is obtained under the Hölder type assumption

$$u_0 - \hat{u} = F'(\hat{u})v. \tag{1.3}$$

To our knowledge, for ill posed operator equation (1.1) in the setting of Banach space, no error estimate is known for $\| u_\alpha^\delta - \hat{u} \|$ under the general Hölder type condition

$$u_0 - \hat{u} = F'(\hat{u})^\nu v, \quad 0 < \nu \leq 1. \tag{1.4}$$

Our goal is to bridge this gap. We also provide a derivative and inverse free iterative method for obtaining an approximation for u_α^δ, although for the purpose of analysis of our method we assume that F possesses uniformly bounded Fréchet derivatives. Precisely, we consider the Hölder type source condition

$$u_0 - \hat{u} = F'(u_0)^\nu v, \quad 0 < \nu \leq 1 \tag{1.5}$$

and obtain the optimal order error estimate for $\|u_\alpha^\delta - \hat{u}\|$ in the Banach space setting. Note that (1.3) is depending on the unknown solution $\hat{u}$ but (1.5) is depending on the known u_0. This is one of the advantages of our approach. Using our idea one can obtain the optimal order error estimate for $\|u_\alpha^\delta - \hat{u}\|$ under the assumption (1.4) (see Corollary 2.5).

The rest of the paper is organized as follows. In Section 2, we consider Hölder type source condition for obtaining error estimate for $\|u_\alpha^\delta - \hat{u}\|$. In Section 3 we consider an iterative method and its convergence analysis. A priori choice of the Parameter and adaptive choice of the parameter are considered in Section 4. The implementation of the adaptive method and the algorithm are given in Section 5. Finally, the paper ends with a conclusion in Section 6.

2 Error Estimates Using Hölder Type Source Condition

We briefly introduce some results from [6, 18] to make the study self-contained. Let u_α^δ be the unique solution of (1.2) and u_α is the unique solution of

$$F(u) + \alpha(u - u_0) = f. \tag{2.1}$$

Then

$$\|u_\alpha^\delta - u_\alpha\| \le \frac{\delta}{\alpha} \tag{2.2}$$

and

$$\|u_\alpha - \hat{u}\| \le \|\hat{u} - u_0\|. \tag{2.3}$$

The following lemma from [18] is used for proving our results in this paper.

Lemma 2.1 (see [18]) Let $F : E \to E$ be accretive and Fréchet differentiable on E. Then for any real number $\alpha > 0$ and $x \in E$, $F'(x) + \alpha I$ is invertible,

$$\|(F'(x) + \alpha I)^{-1}\| \le \frac{1}{\alpha} \tag{2.4}$$

and

$$\|(F'(x) + \alpha I)^{-1} F'(x)\| \le 2. \tag{2.5}$$

Note that by (2.4) we have,

$$\|\alpha(F'(x) + \alpha I)^{-1}\| \le 1.$$

So for $0 < \nu \le 1$, we have (see [12, page 287]),

$$F'(x)^\nu w = \frac{\sin \pi\nu}{\pi\nu} \int_0^\infty t^\nu (F'(x) + tI)^{-2} F'(x) w \mathrm{d}t. \tag{2.6}$$

One of the crucial result for proving error estimate is the following lemma, proof of which is analogous to the proof of Lemma 14.1 in [12], but for us to make this paper as self-contained as possible we give the proof.

Lemma 2.2 Let $F : E \to E$ be a Fréchet differentiable and monotone operator. Then for $x \in E$ and $0 < \nu < 1$,

$$\|\alpha(F' + \alpha I)^{-1} F'(x)^\nu\| \le 4 \frac{\sin(\pi\nu)}{\pi\nu^2} \left(\frac{\nu}{1-\nu}\right)^\nu \alpha^\nu. \tag{2.7}$$

Proof By (2.6) we have

$$\begin{aligned}
(F' + \alpha I)^{-1} F'(x)^\nu w &= \frac{\sin \pi\nu}{\pi\nu} \int_0^\infty t^\nu (F' + \alpha I)^{-1} (F'(x) + tI)^{-2} F'(x) w \mathrm{d}t \\
&= \frac{\sin \pi\nu}{\pi\nu} \Big[\int_0^\rho t^\nu (F' + \alpha I)^{-1} (F'(x) + tI)^{-2} F'(x) w \mathrm{d}t \\
&\quad + \int_\rho^\infty t^\nu (F' + \alpha I)^{-1} (F'(x) + tI)^{-2} F'(x) w \mathrm{d}t \Big] \\
&= \frac{\sin \pi\nu}{\pi\nu} [H_1 + H_2],
\end{aligned} \tag{2.8}$$

where $H_1 = \int_0^\rho t^\nu (F' + \alpha I)^{-1} (F'(x) + tI)^{-2} F'(x) w \mathrm{d}t$ and $H_2 = \int_\rho^\infty t^\nu (F' + \alpha I)^{-1} (F'(x) + tI)^{-2} F'(x) w \mathrm{d}t$. So, by (2.4) and (2.5) we have

$$\|H_1\| = \left\| \int_0^\rho t^\nu F'(x) (F'(x) + tI)^{-2} (F' + \alpha I)^{-1} w \mathrm{d}t \right\|$$

$$\begin{aligned} &\leq 2\int_0^\rho \frac{t^{\nu-1}}{\alpha}\|w\|\mathrm{d}t \\ &= 2\frac{\rho^\nu}{\nu.\alpha}\|w\| \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} \|H_2\| &= \left\| \int_\rho^\infty t^\nu F'(x)(F'(x)+tI)^{-2}(F'+\alpha I)^{-1}w\mathrm{d}t \right\| \\ &\leq 2\int_\rho^\infty t^{\nu-2}\|w\|\mathrm{d}t \\ &= 2\frac{\rho^{\nu-1}}{1-\nu}\|w\|. \end{aligned} \tag{2.10}$$

Thus by (2.8), (2.9) and (2.10), we have

$$\|(F'+\alpha I)^{-1}F'(x)^\nu w\| \leq 2\frac{\sin(\pi\nu)}{\pi\nu}\left[\frac{\rho^\nu}{\nu\alpha}+\frac{\rho^{\nu-1}}{1-\nu}\right]\|w\|.$$

Now the result follows by taking minimum of the right side of the above expression (i.e., $\rho = \frac{\nu\alpha}{1-\nu}$). □

Assumption 2.3 (see [3, 14, 15]) There exists a constant $k_0 \geq 0$ such that for every $u \in B(u_0, r)$ and $v \in E$ there exists an element $\Phi(u, u_0, v) \in X$ such that $[F'(u) - F'(u_0)]v = F'(u_0)\Phi(u, u_0, v), \|\Phi(u, u_0, v)\| \leq k_0\|v\|\|u - u_0\|$.

Theorem 2.4 Let Assumption 2.3 and (1.5) hold. If $3k_0r < 1$, then

$$\|u_\alpha - \hat{u}\| \leq 4\frac{\frac{\sin\pi\nu}{\pi\nu^2}(\frac{\nu}{1-\nu})^\nu\|v\|}{1-3k_0r}\alpha^\nu,$$

where v is as in (1.5).

Proof We have

$$F(u_\alpha) - F(\hat{u}) + \alpha(u_\alpha - u_0) = 0.$$

Thus by mean value theorem of integral calculus, we have

$$(F'(u_0)+\alpha I)(u_\alpha - \hat{u}) = \alpha(u_0 - \hat{u}) - \int_0^1 [F'(\hat{u}+t(u_\alpha - \hat{u})) - F'(u_0)](u_\alpha - \hat{u})\mathrm{d}t.$$

Therefore by (1.5), Lemma 2.1, Lemma 2.2, Assumption 2.3, (2.3), we have in turn

$$\begin{aligned} \|u_\alpha - \hat{u}\| &\leq \|\alpha(F'(u_0)+\alpha I)^{-1}F'(u_0)^\nu v\| \\ &\quad + \left\|(F'(u_0)+\alpha I)^{-1}\int_0^1 [F'(\hat{u}+t(u_\alpha - \hat{u})) - F'(u_0)](u_\alpha - \hat{u})\mathrm{d}t\right\| \\ &\leq 4\frac{\sin\pi\nu}{\pi\nu^2}\Big(\frac{\nu}{1-\nu}\Big)^\nu\|v\|\alpha^\nu + 2\int_0^1 \|\Phi(\hat{u}+t(u_\alpha - \hat{u}), u_0, u_\alpha - \hat{u})\mathrm{d}t\| \\ &\leq 4\frac{\sin\pi\nu}{\pi\nu^2}\Big(\frac{\nu}{1-\nu}\Big)^\nu\|v\|\alpha^\nu + 2k_0(\|\hat{u} - u_0\| + \frac{1}{2}\|u_\alpha - \hat{u}\|)\|u_\alpha - \hat{u}\| \\ &\leq 4\frac{\sin\pi\nu}{\pi\nu^2}\Big(\frac{\nu}{1-\nu}\Big)^\nu\|v\|\alpha^\nu + 2k_0(\|\hat{u} - u_0\| + \frac{1}{2}\|u_0 - \hat{u}\|)\|u_\alpha - \hat{u}\| \\ &\leq 4\frac{\sin\pi\nu}{\pi\nu^2}\Big(\frac{\nu}{1-\nu}\Big)^\nu\|v\|\alpha^\nu + 3k_0\|\hat{u} - u_0\|\|u_\alpha - \hat{u}\| \\ &\leq 4\frac{\sin\pi\nu}{\pi\nu^2}\Big(\frac{\nu}{1-\nu}\Big)^\nu\|v\|\alpha^\nu + 3k_0r\|u_\alpha - \hat{u}\|. \end{aligned}$$

This completes the proof of the theorem. □

Corollary 2.5 Let Assumption 2.3 and (1.4) hold. If $k_0r < 1$, then

$$\|u_\alpha - \hat{u}\| \leq 4\frac{\frac{\sin \pi\nu}{\pi\nu^2}(\frac{\nu}{1-\nu})^\nu \|v\|}{1-k_0r}\alpha^\nu,$$

where v is as in (1.4).

Proof Since

$$F(u_\alpha) - F(\hat{u}) + \alpha(u_\alpha - u_0) = 0,$$

we have

$$(F'(\hat{x}) + \alpha I)(u_\alpha - \hat{u}) = \alpha(u_0 - \hat{u}) - \int_0^1 [F'(\hat{u} + t(u_\alpha - \hat{u})) - F'(\hat{u})](u_\alpha - \hat{u})\mathrm{d}t.$$

Therefore by (1.4), Lemma 2.1, Lemma 2.2, Assumption 2.3, (2.3), we have in turn

$$\begin{aligned}
\|u_\alpha - \hat{u}\| &\leq \|\alpha(F'(\hat{u}) + \alpha I)^{-1}F'(\hat{u})^\nu v\| \\
&\quad + \left\|(F'(\hat{u}) + \alpha I)^{-1}\int_0^1 [F'(\hat{u} + t(u_\alpha - \hat{u})) - F'(\hat{u})](u_\alpha - \hat{u})\mathrm{d}t\right\| \\
&\leq 4\frac{\sin \pi\nu}{\pi\nu^2}\Big(\frac{\nu}{1-\nu}\Big)^\nu \|v\|\alpha^\nu + 2\int_0^1 \|\varphi(\hat{u} + t(u_\alpha - \hat{u}), \hat{u}, u_\alpha - \hat{u})\mathrm{d}t\| \\
&\leq 4\frac{\sin \pi\nu}{\pi\nu^2}\Big(\frac{\nu}{1-\nu}\Big)^\nu \|v\|\alpha^\nu + 2k_0\frac{1}{2}\|u_\alpha - \hat{u}\|\|u_\alpha - \hat{u}\| \\
&\leq 4\frac{\sin \pi\nu}{\pi\nu^2}\Big(\frac{\nu}{1-\nu}\Big)^\nu \|v\|\alpha^\nu + k_0r\|u_\alpha - \hat{u}\|.
\end{aligned}$$

The rest of the proof is analogous to the proof of Theorem 2.4. □

3 Iterative Method and Convergence Analysis

In this section, we assume that X is a real Banach algebra and $F : X \longrightarrow X$ is twice Fréchet differentiable accretive operator. In order for us to introduce the method, it is convenient to introduce some notations. For $\alpha > 0$, let

$$R_\alpha(u) := F(u) + \alpha(u - u_0) - f^\delta \tag{3.1}$$

and

$$R'_\alpha(\cdot)h := F'(\cdot)h + \alpha h. \tag{3.2}$$

We consider the sequence defined iteratively by

$$u^\delta_{n+1,\alpha} = u^\delta_{n,\alpha} - \frac{2[R_\alpha(u^\delta_{n,\alpha})]^2}{R_\alpha(u^\delta_{n,\alpha} + R_\alpha(u^\delta_{n,\alpha})) - R_\alpha(u^\delta_{n,\alpha} - R_\alpha(u^\delta_{n,\alpha}))}, \tag{3.3}$$

where $u^\delta_{0,\alpha} = u_0$ is an initial guess. As in earlier papers such as [5–10, 16] etc., we choose the parameter $\alpha = \alpha_i$ from some finite set

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\},$$

using the adaptive method considered by Perverzev and Schock [14]. For convenience we use the notation

$$e_n = u^\delta_{n,\alpha} - u^\delta_\alpha \ \text{ for each } \ n = 0, 1, 2, \cdots, \tag{3.4}$$

where u^δ_α is the solution of $R_\alpha(x) = 0$.

Let

$$C_\beta := \min\left\{\frac{\|F(u_0)-f^\delta\|}{(2+\beta_1/\alpha_0)(\beta_1+\alpha_N)}, 2\right\}, \quad \delta < \frac{C_\beta}{2}\alpha_0 \tag{3.5}$$

and

$$\|\hat{u}-u_0\| \le r \text{ with } r < \min\left\{\frac{1}{3k_0}, \frac{1}{2}\Big(\frac{C_\beta}{2}-\frac{\delta}{\alpha_0}\Big)\right\}. \tag{3.6}$$

Further, we assume that

$$\|F'(\cdot)\| \le \beta_1 \text{ and } \|F''(\cdot)\| \le \beta_2.$$

We begin proving a series of lemmas to prove our main result (Theorem 3.5).

Lemma 3.1 Let e_n be as in (2.4). Then

$$\|e_0\| \le 2r + \frac{\delta}{\alpha_0}.$$

Proof Note that, by (2.2) and (2.3) we have

$$\|u_\alpha^\delta - \hat{u}\| \le \frac{\delta}{\alpha} + \|u_0-\hat{u}\|. \tag{3.7}$$

The result now follows from (3.7) and the following triangle inequality

$$\|u_\alpha^\delta - u_0\| \le \|u_\alpha^\delta - \hat{u}\| + \|\hat{u}-u_0\|.$$

□

Let us first define the operators $M(u), M_1(u)$ and $M_2(u)$:

$$M(u) = \int_0^1 R''_\alpha(u_\alpha^\delta + t(u-u_\alpha^\delta))(1-t)\mathrm{d}t \text{ for each } u\in D(F), \tag{3.8}$$

$$M_1(u) = \int_0^1 R''_\alpha(u_\alpha^\delta + t(u+R_\alpha(u)-u_\alpha^\delta))(1-t)\mathrm{d}t, \text{ for each } u\in D(F) \tag{3.9}$$

and

$$M_2(u) = \int_0^1 R''_\alpha(u_\alpha^\delta + t(u-R_\alpha(u)-u_\alpha^\delta))(1-t)\mathrm{d}t, \text{ for each } u\in D(F). \tag{3.10}$$

Let

$$\Gamma_1 := \frac{[M_1(u_{n,\alpha}^\delta)-M_2(u_{n,\alpha}^\delta)][(e_n)^2+(R_\alpha(u_{n,\alpha}^\delta))^2]}{2R'_\alpha(u_\alpha^\delta)R_\alpha(u_{n,\alpha}^\delta)}, \tag{3.11}$$

and

$$\Gamma_2 := \frac{[M_1(u_{n,\alpha}^\delta)+M_2(u_{n,\alpha}^\delta)]e_nR_\alpha(u_{n,\alpha}^\delta)}{R'_\alpha(u_\alpha^\delta)R_\alpha(u_{n,\alpha}^\delta)}. \tag{3.12}$$

□

Lemma 3.2 Let R'_α be as in (3.2), Γ_1 and Γ_2 be as above. Then

$$R_\alpha(u_{n,\alpha}^\delta + R_\alpha(u_{n,\alpha}^\delta)) - R_\alpha(u_{n,\alpha}^\delta - R_\alpha(u_{n,\alpha}^\delta)) = 2R'_\alpha(u_\alpha^\delta)R_\alpha(u_{n,\alpha}^\delta)[1+\Gamma_1+\Gamma_2].$$

Proof Using the Taylor expansion of the operator $R_\alpha(u)$ around the solution u_α^δ of $R_\alpha(u)=0$, we get

$$R_\alpha(u_{n,\alpha}^\delta) = R'_\alpha(u_\alpha^\delta)(u_{n,\alpha}^\delta - u_\alpha^\delta) + M(u_{n,\alpha}^\delta)(u_{n,\alpha}^\delta - u_\alpha^\delta)^2. \tag{3.13}$$

Similarly the Taylor expansion of $R_\alpha(u^\delta_{n,\alpha}+R_\alpha(u^\delta_\alpha))$ and $R_\alpha(u^\delta_{n,\alpha}-R_\alpha(u^\delta_\alpha))$ around the solution u^δ_α of $R_\alpha(u)=0$ we get

$$\begin{aligned}
&R_\alpha(u^\delta_{n,\alpha}+R_\alpha(u^\delta_{n,\alpha}))\\
=\;& R'_\alpha(u^\delta_\alpha)(u^\delta_{n,\alpha}-u^\delta_\alpha+R_\alpha(u^\delta_{n,\alpha}))+M_1(u^\delta_{n,\alpha})(u^\delta_{n,\alpha}-u^\delta_\alpha+R_\alpha(u^\delta_{n,\alpha}))^2\\
=\;& R'_\alpha(u^\delta_\alpha)[(u^\delta_{n,\alpha}-u^\delta_\alpha)+R_\alpha(u^\delta_{n,\alpha})]\\
&+M_1(u^\delta_{n,\alpha})[(u^\delta_{n,\alpha}-u^\delta_\alpha)^2+(R_\alpha(u^\delta_{n,\alpha}))^2+2(u^\delta_{n,\alpha}-u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})]\\
=\;& R'_\alpha(u^\delta_\alpha)[e_n+R_\alpha(u^\delta_{n,\alpha})]+M_1(u^\delta_{n,\alpha})[(e_n)^2+(R_\alpha(u^\delta_{n,\alpha}))^2+2e_nR_\alpha(u^\delta_{n,\alpha})] \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
&R_\alpha(u^\delta_{n,\alpha}-R_\alpha(u^\delta_{n,\alpha}))\\
=\;& R'_\alpha(u^\delta_\alpha)(u^\delta_{n,\alpha}-u^\delta_\alpha-R_\alpha(u^\delta_{n,\alpha}))+M_2(u^\delta_{n,\alpha})(u^\delta_{n,\alpha}-u^\delta_\alpha-R_\alpha(u^\delta_{n,\alpha}))^2\\
=\;& R'_\alpha(u^\delta_\alpha)[(u^\delta_{n,\alpha}-u^\delta_\alpha)-R_\alpha(u^\delta_{n,\alpha})]\\
&+M_2(u^\delta_{n,\alpha})[(u^\delta_{n,\alpha}-u^\delta_\alpha)^2+(R_\alpha(u^\delta_{n,\alpha}))^2-2(u^\delta_{n,\alpha}-u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})]\\
=\;& R'_\alpha(u^\delta_\alpha)[e_n-R_\alpha(u^\delta_{n,\alpha})]+M_2(u^\delta_{n,\alpha})[(e_n)^2+(R_\alpha(u^\delta_{n,\alpha}))^2-2e_nR_\alpha(u^\delta_{n,\alpha})]. \qquad (3.15)
\end{aligned}$$

From (3.14) and (3.15), we have

$$\begin{aligned}
&R_\alpha(u^\delta_{n,\alpha}+R_\alpha(u^\delta_{n,\alpha}))-R_\alpha(u^\delta_{n,\alpha}-R_\alpha(u^\delta_{n,\alpha}))\\
=\;& 2R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})+[M_1(u^\delta_{n,\alpha})-M_2(u^\delta_{n,\alpha})]((e_n)^2+(R_\alpha(u^\delta_{n,\alpha}))^2)\\
&+2[M_1(u^\delta_{n,\alpha})+M_2(u^\delta_{n,\alpha})]e_nR_\alpha(u^\delta_{n,\alpha})\\
=\;& 2R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})[1+\Gamma_1+\Gamma_2]. \qquad (3.16)
\end{aligned}$$

□

Lemma 3.3 Let R_α, R'_α, Γ_1 and Γ_2 be as in (3.1), (3.2), (3.11) and (3.12) respectively. Then

(i)

$$\|R_\alpha(u^\delta_{n,\alpha})\|\le(\beta_1+\alpha)\|e_n\|+\frac{\beta_2+\alpha}{2}\|e_n\|^2;$$

(ii)

$$\|(R_\alpha(u^\delta_{n,\alpha}))^2(\Gamma_1+\Gamma_2)\|=O(\|e_n\|^3).$$

Proof Note that (i) follows from (3.13) and the inequalities

$$\|R'_\alpha(u^\delta_\alpha)\|\le\beta_1+\alpha \;\text{ and }\; \|M(u)\|\le\frac{\beta_2+\alpha}{2}. \qquad (3.17)$$

To prove (ii), we observe that

$$\|R_\alpha(u^\delta_{n,\alpha})\|=\|R'_\alpha(u^\delta_\alpha)^{-1}R'_\alpha(u^\delta_\alpha)(R_\alpha(u^\delta_{n,\alpha}))\|\le\frac{1}{\alpha}\|R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})\| \qquad (3.18)$$

and hence

$$\begin{aligned}
\|(R_\alpha(u^\delta_{n,\alpha}))^2(\Gamma_1+\Gamma_2)\| &\le \Big\|\frac{1}{\alpha}R_\alpha(u^\delta_{n,\alpha})([M_1(u^\delta_{n,\alpha})-M_2(u^\delta_{n,\alpha})][(e_n)^2+(R_\alpha(u^\delta_{n,\alpha}))^2]\\
&\quad+[M_1(u^\delta_{n,\alpha})+M_2(u^\delta_{n,\alpha})]e_nR_\alpha(u^\delta_{n,\alpha}))\Big\|\\
&=O(\|e_n\|^3). \qquad (3.19)
\end{aligned}$$

The last step follows from (i), (3.17) and the inequality $\|M_i(u)\|\le\frac{\beta_2+\alpha}{2}$, for $i=1,2$. □

Lemma 3.4 Let R_α and R'_α be as in (2.2) and (2.3) respectively. Suppose $\|u^\delta_{n,\alpha}-u_0\|<\frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$ for each $n=1,2,\cdots$. Then

$$\frac{1}{\|R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})\|}\leq\frac{1}{\alpha(\|F(u_0)-f^\delta\|-(\beta_1+\alpha)\|u^\delta_{n,\alpha}-u_0\|)}\ \text{ for each }\ n=1,2,\cdots.$$

Proof Observe that

$$\begin{aligned}R_\alpha(u^\delta_{n,\alpha})&=F(u^\delta_{n,\alpha})-f^\delta+\alpha(u^\delta_{n,\alpha}-u_0)\\&=F(u_0)-f^\delta+F(u^\delta_{n,\alpha})-F(u_0)+\alpha(u^\delta_{n,\alpha}-u_0)\\&=F(u_0)-f^\delta+\left[\int_0^1F'(u_0+t(u^\delta_{n,\alpha}-u_0)\mathrm{d}t+\alpha I\right](u^\delta_{n,\alpha}-u_0).\end{aligned}$$

So

$$\begin{aligned}\|R_\alpha(u^\delta_{n,\alpha})\|&\geq\|F(u_0)-f^\delta\|-\left\|\left[\int_0^1F'(u_0+t(u^\delta_{n,\alpha}-u_0)\mathrm{d}t+\alpha I\right](u^\delta_{n,\alpha}-u_0)\right\|\\&\geq\|F(u_0)-f^\delta\|-(\beta_1+\alpha)\|u^\delta_{n,\alpha}-u_0\|\end{aligned}\tag{3.20}$$

for each $n=1,2,\cdots$. The result now follows from (3.18) and (3.20). □

We state our main theorem of this section below.

Theorem 3.5 Let R_α be as in (3.1) and u^δ_α be the solution of $R_\alpha(u)=0$. Further the first and second Fréchet derivative of F exists at all $u\in D(F)$. Then the iteration defined in (3.3) converges quadratically to u^δ_α. Moreover

$$\|u^\delta_{n+1,\alpha}-u^\delta_\alpha\|=\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0)-f^\delta\|-(\beta_1+\alpha)2\|e_0\|)}\|e_n\|^2+O(\|e_n\|^3).$$

Proof $\Theta=\Gamma_1+\Gamma_2$. Then by (3.3), (3.16) and (3.18), we have

$$\begin{aligned}e_{n+1}&=e_n-\frac{(R_\alpha(u^\delta_{n,\alpha}))^2}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})(1+\Theta)}\\&=e_n-\frac{(R_\alpha(u^\delta_{n,\alpha}))^2}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}[I-\Theta+\Theta^2-\cdots]\\&=e_n-\frac{(R_\alpha(u^\delta_{n,\alpha}))^2}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}(I-\Theta)\\&\quad-\frac{(R_\alpha(u^\delta_{n,\alpha}))^2}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}\times\text{higher order terms in }\ \Theta\\&=\frac{1}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}[R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})e_n-R_\alpha(u^\delta_{n,\alpha})^2(I-\Theta)\\&\quad-(R_\alpha(u^\delta_{n,\alpha}))^2\times\text{higher order terms in }\ \Theta].\end{aligned}\tag{3.21}$$

Therefore, we have

$$\begin{aligned}\|e_{n+1}\|\leq&\left\|\frac{1}{R'_\alpha(u^\delta_\alpha)R_\alpha(u^\delta_{n,\alpha})}\right\|[\|R'_\alpha(u^\delta_\alpha)\|\|R_\alpha(u^\delta_{n,\alpha})\|\|e_n\|+\|(R_\alpha(u^\delta_{n,\alpha}))^2\|\\&+\|(R_\alpha(u^\delta_{n,\alpha}))^2\|\|\Theta\|+\text{higher order terms in }\ \|\Theta\|].\end{aligned}$$

If $\|u^\delta_{n,\alpha}-u_0\|<\frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$, then using Lemmas 3.1–3.4 one can prove that

$$\|e_{n+1}\|\leq\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0)-f^\delta\|-(\beta_1+\alpha)\|u^\delta_{n,\alpha}-u_0\|)}\|e_n\|^2+O(\|e_n\|^3).\tag{3.22}$$

Now it remains to show that $\|u^\delta_{n,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$. This can be shown as follows. Since $\frac{2(\beta_1/\alpha+1)(\beta_1+\alpha)}{\|F(u_0)-f^\delta\|}\|e_0\| \le \frac{2(\beta_1/\alpha_0+1)(\beta_1+\alpha_N)}{\|F(u_0)-f^\delta\|}\|e_0\| \le 1$, by (3.5) and (3.6),

$$\begin{aligned}\|u^\delta_{1,\alpha} - u_0\| &\le \|u^\delta_{1,\alpha} - u^\delta_\alpha\| + \|u^\delta_\alpha - u_0\| \\ &\le \frac{2(\beta_1/\alpha+1)(\beta_1+\alpha)}{\|F(u_0)-f^\delta\|}\|e_0\|^2 + O(\|e_0\|^3) + \|u^\delta_\alpha - u_0\| \\ &\le 2\|e_0\| \le C_\beta < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}\end{aligned}$$

(by ignoring higher order terms in $\|e_0\|$). Again by (3.22) and (3.6), we have,

$$\begin{aligned}\|u^\delta_{2,\alpha} - u_0\| &\le \|u^\delta_{2,\alpha} - u^\delta_\alpha\| + \|u^\delta_\alpha - u_0\| \\ &\le \frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0)-f^\delta\| - (\beta_1+\alpha)\|u^\delta_{1,\alpha} - u_0\|)}\|e_1\|^2 + O(\|e_1\|^3) + \|u^\delta_\alpha - u_0\| \\ &\le 2\|u^\delta_\alpha - u_0\| = 2\|e_0\| \le C_\beta < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}.\end{aligned}$$

By ignoring higher order terms in $\|e_0\|$ and observing that by (3.6)

$$\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0)-f^\delta\| - (\beta_1+\alpha)\|u^\delta_{1,\alpha} - u_0\|)}\|e_0\| < 1,$$

which shows $\|u^\delta_{n,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$ for $n = 2$. By simply replacing $u^\delta_{2,\alpha}$ by $u^\delta_{k+1,\alpha}$ in the preceding estimates we arrive at $\|u^\delta_{k+1,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$. Thus by induction $\|u^\delta_{n,\alpha} - u_0\| < \frac{\|F(u_0)-f^\delta\|}{\beta_1+\alpha}$ for $n > 0$. From the above relation it follows that

$$\|u^\delta_{n+1,\alpha} - u^\delta_\alpha\| \le \frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0)-f^\delta\| - (\beta_1+\alpha)2\|e_0\|)}\|e_n\|^2 + O(\|e_n\|^3). \tag{3.23}$$

This completes the proof of the theorem. □

Remark 3.6 Note that, repeated applications of (3.23) lead to the following estimate

$$\|e_{n+1}\| \le \left(\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(u_0)-f^\delta\| - 2(\beta_1+\alpha)\|e_0\|)}\right)^{2^{n+1}-1}\|e_0\|^{2^{n+1}} + O(\|e_n\|^{2^n+3}).$$

Since $\|e_0\| < 1$, we ignore the terms of order $\|e_0\|^{2^{n+1}+3}$ and take

$$\|e_{n+1}\| \le C_\alpha \mathrm{e}^{-\gamma 2^{n+1}}, \tag{3.24}$$

where $C_\alpha := \left(\frac{2(\beta_1+\alpha_N)^2}{\alpha_0(\|F(u_0)-f^\delta\| - 2(\beta_1+\alpha)\|e_0\|)}\right)^{2^{n+1}-1}$, $\gamma = -\log(\|e_0\|)$. Note that $C_\alpha \mathrm{e}^{-\gamma 2^{n+1}} = [C_\alpha \mathrm{e}^{-\gamma 2^n}]\mathrm{e}^{-\gamma 2^n}$, and for large n, $C_\alpha \mathrm{e}^{-\gamma 2^n} \le C$ for any $C > 0$. Therefore for large n, from (3.24), (2.2) and Theorem 2.4, we have

$$\|u^\delta_{n+1,\alpha} - \hat{u}\| \le C\mathrm{e}^{-\gamma 2^n} + \frac{\delta}{\alpha} + 4\frac{\frac{\sin \pi\nu}{\pi\nu^2}(\frac{\nu}{(1-\nu)})^\nu \|v\|}{1 - 3k_0 r}\alpha^\nu.$$

Let

$$n_\delta := \min\left\{n : \mathrm{e}^{-\gamma 2^n} \le \frac{\delta}{\alpha} \ \& \ C_\alpha \mathrm{e}^{-\gamma 2^n} \le C\right\} \tag{3.25}$$

for some constant C. In view of the above remark, we have the following theorem.

Theorem 3.7 Let $u^\delta_{n_\delta+1,\alpha}$ be as in (2.1) and let the assumptions in Theorem 2.4 and Theorem 3.5 be satisfied, where n_δ be as in (3.25). Then we have the following

$$\|u^\delta_{n_\delta+1,\alpha} - \hat{u}\| \leq \bar{C}\Big(\alpha^\nu + \frac{\delta}{\alpha}\Big), \tag{3.26}$$

where $\bar{C} = \max\big\{C+1, \frac{4\frac{\sin\pi\nu}{\pi\nu^2}(\frac{\nu}{1-\nu})^\nu\|v\|}{1-3k_0r}\big\}$.

4 A Priori Choice of the Parameter

Note that the error $\alpha^\nu + \frac{\delta}{\alpha}$ in (3.26) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies, $\alpha_\delta^{1+\nu} = \delta$. That is $\alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Hence by (3.26) we have the following theorem.

Theorem 4.1 Let the assumptions in Theorem 3.7 hold. For $\delta > 0$, let $\alpha := \alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Let n_δ be as in (3.25). Then

$$\|u^\delta_{n_\delta,\alpha} - \hat{u}\| = O(\delta^{\frac{\nu}{1+\nu}}).$$

4.1 Adaptive Scheme and Stopping Rule

We use the adaptive selection of the parameter strategy considered by Pereverzev and Schock [14], modified suitably for the situation for choosing the parameter α. For convenience, take $u^\delta_i := u^\delta_{n_i,\alpha_i}$. Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i\alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := \max\Big\{i : \alpha_i^\nu \leq \frac{\delta}{\alpha_i}\Big\} < N, \tag{4.1}$$

$$k := \max\Big\{i : \|u^\delta_i - u^\delta_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}, j = 0, 1, 2, \cdots, i-1\Big\}, \tag{4.2}$$

where $\bar{C}$ is as in Theorem 3.7. Now we have the following theorem.

Theorem 4.2 Assume that there exists $i \in \{0, 1, \cdots, N\}$ such that $\alpha_i^\nu \leq \frac{\delta}{\alpha_i}$. Let the assumptions of Theorem 3.7 be fulfilled, and l and k be as in (4.1) and (4.2) respectively. Then $l \leq k$ and

$$\|\hat{u} - u^\delta_k\| \leq 6\bar{C}\mu\delta^{\frac{\nu}{1+\nu}}.$$

Proof To prove $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \cdots, N\}$, $\alpha_i^\nu \leq \frac{\delta}{\alpha_i} \Longrightarrow \|u^\delta_i - u^\delta_j\| \leq 4\bar{C}\frac{\delta}{\alpha_j}$, $\forall j = 0, 1, 2, \cdots, i-1$. For $j < i$, we have

$$\begin{aligned}\| u^\delta_i - u^\delta_j \| &\leq \| u^\delta_i - \hat{u} \| + \| \hat{u} - u^\delta_j \| \\ &\leq \bar{C}\Big(\alpha_i^v + \frac{\delta}{\alpha_i}\Big) + \bar{C}\Big(\alpha_j^v + \frac{\delta}{\alpha_j}\Big) \\ &\leq 2\bar{C}\frac{\delta}{\alpha_i} + 2\bar{C}\frac{\delta}{\alpha_j} \\ &\leq 4\bar{C}\frac{\delta}{\alpha_j}.\end{aligned}$$

Thus the relation $l \leq k$ is proved. Observe that

$$\| \hat{u} - u^\delta_k \| \leq \| \hat{u} - u^\delta_l \| + \| u^\delta_k - u^\delta_l \|,$$

where

$$\| \hat{u} - u^\delta_l \| \leq \bar{C}\Big(\alpha_l^v + \frac{\delta}{\alpha_l}\Big) \leq 2\bar{C}\frac{\delta}{\alpha_l}.$$

Now since $l \leq k$, we have

$$\| u_k^\delta - u_l^\delta \| \leq 4\bar{C}\frac{\delta}{\alpha_l}.$$

Hence

$$\| \hat{u} - u_k^\delta \| \leq 6\bar{C}\frac{\delta}{\alpha_l}.$$

Now, since $\alpha_\delta = \delta^{\frac{1}{1+\nu}} \leq \alpha_{l+1} \leq \mu\alpha_l$, it follows that

$$\frac{\delta}{\alpha_l} \leq \frac{\mu\delta}{\alpha_\delta} = \mu\delta^{\frac{\nu}{1+\nu}}.$$

This completes the proof. □

5 Implementation of Adaptive Choice Rule

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 4.2 involves the following steps.

- Choose $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.

5.1 Algorithm

1. Set $i = 0$.
2. Choose $n_i := \min\left\{n : \mathrm{e}^{-\gamma 2^n} \leq \frac{\delta}{\alpha_i} \ \& \ C_\alpha \mathrm{e}^{-\gamma 2^n} \leq C\right\}$.
3. Solve $u_i := u_{n_i,\alpha_i}^\delta$ by using the iteration (3.3).
4. If $\|u_i - u_j\| > 4\bar{C}\frac{\delta}{\alpha_j}, j < i$, then take $k = i - 1$ and return u_k.
5. Else set $i = i + 1$ and go to 2.

6 Conclusion

In this paper we considered a derivative free iterative method for approximately solving ill-posed equation involving m-accretive mappings in a real reflexive Banach space. We obtained optimal order error estimate under a general Hölder type source condition. Also we considered the adaptive parameter choice strategy considered by Pereverzev and Schock [14], for choosing the regularization parameter.

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