Khayyam J. Math. 5 (2019), no. 2, 96–107
DOI: 10.22034/kjm.2019.88082

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

LOCAL CONVERGENCE OF A NOVEL EIGHTH ORDER METHOD UNDER HYPOTHESES ONLY ON THE FIRST DERIVATIVE

IOANNIS K. ARGYROS[1], SANTHOSH GEORGE[2] AND SHOBHA M. ERAPPA[3*]

Communicated by T. Riedel

Abstract. We expand the applicability of eighth order-iterative method studied by Jaiswal in order to approximate a locally unique solution of an equation in Banach space setting. We provide a local convergence analysis using only hypotheses on the first Frechet-derivative. Moreover, we provide computable convergence radii, error bounds, and uniqueness results. Numerical examples computing the radii of the convergence balls as well as examples where earlier results cannot apply to solve equations but our results can apply are also given in this study.

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Frechet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y. Many problems in computational sciences and also in engineering, mathematical biology, mathematical economics, and other disciplines can be written in the form of an equation like (1.1) by using mathematical modeling [1–28]. The solutions of such equations can rarely be found in a closed form. That is why most solution methods for such equations are usually iterative.

Date: Received: 18 June 2018; Revised : 8 January 2019; Accepted; 15 February 2019.
* Corresponding author.
2010 *Mathematics Subject Classification.* Primary 65D10, 65D99; Secondary 65G99, 65H10.
Key words and phrases. Eighth order of convergence, ball convergence, Banach space, Frechet-derivative.

Higher order convergence methods such as the Chebyshev–Halley-type methods [2, 5, 13] require the computation of derivatives of order higher than one, which are very expensive in general. However, these methods are important for faster convergence, especially in cases of stiff systems of equations. Recently many researchers have tried to find fast convergence methods using only the first derivative or divided differences of order one [13]. In particular, Jaiswal studied the convergence of the multistep method [17] defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} r_n &= x_n - F'(x_n)^{-1}F(x_n), \\ y_n &= \frac{1}{2}(r_n - x_n), \\ z_n &= \frac{1}{3}(4y_n - x_n), \\ u_n &= y_n + (F'(x_n) - F'(z_n))^{-1}F(x_n) \\ v_n &= u_n + 2(F'(x_n) - 3F'(z_n))^{-1}F(u_n) \end{aligned} \tag{1.2}$$

and

$$x_{n+1} = v_n + 2(F'(x_n) - 3F'(z_n))^{-1}F(v_n),$$

where $x_0 \in D$ is an initial point. The eighth order of convergence was shown by using a hypothesis reaching up to the Lipschitz continuity

$$\|F'''(x) - F'''(y)\| \le \alpha\|x - y\| \tag{1.3}$$

for some $\alpha > 0$ and each $x, y \in D$, although only the first derivative appears in method (1.2). These hypotheses limit the applicability of method (1.2). As a motivational example, define function F on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We have that $x^* = 1$,

$$F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x,$$

and

$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

The function $F'''(x)$ is unbounded on D. Hence, (1.3) and consequently the results in [17] cannot be applied to solve equation (1.1). We provide a local convergence analysis using only hypotheses on the first Frechet-derivative. This way we expand the applicability of these methods. Moreover, we provide computable convergence radii, error bounds on the distances $\|x_n - x^*\|$, and uniqueness results. Furthermore, we use the computational order of convergence (COC) and the approximate computational order of convergence (ACOC) (which do not depend on higher than one Frechet-derivative) to determine the order of convergence of method (1.2). Local results are important, because they provide the degree of difficulty for choosing initial points. Our idea can be used on other iterative methods.

This paper is organized as follows: Section 2 contains the local convergence analysis of method (1.2). The numerical examples are presented in the concluding Section 3.

2. Local convergence

The local convergence analysis of method (1.2) is based on some scalar functions and parameters. Let q_0 be a continuous, nondecreasing function defined on the interval $[0,+\infty)$ with values in $[0,+\infty)$ and satisfying $q_0(0) = 0$. Define the parameter ρ_0 by

$$\rho_0 := \sup\{t \geq 0 : q_0(t) < 1\}. \tag{2.1}$$

Let also q and q_1 be continuous, nondecreasing functions defined on the interval $[0,\rho_0)$ with values in $[0,+\infty)$ and satisfying $q(0) = 0$. Moreover, define functions $g_i, h_i, i = 1, 2, 3,\ p$, and h_p on the interval $[0,\rho_0)$ by

$$g_1(t) = \frac{\int_0^1 q((1-\theta)t)d\theta}{1-q_0(t)},$$

$$g_2(t) = \frac{1}{2}(1+g_1(t)),$$

$$g_3(t) = \frac{1}{3}(1+4g_1(t)),$$

$$h_i(t) = g_i(t) - 1,$$

$$p(t) = \frac{1}{2}(q_0(t) + 3q_0(g_3(t)t)),$$

and

$$h_p(t) = p(t) - 1.$$

We have that $h_1(0) = -1 < 0$, $h_2(0) = -\frac{1}{2} < 0$, $h_3(0) = -\frac{2}{3} < 0$, $h_p(t) = -1 < 0$ and $h_i(t) \to +\infty$, $h_p(t) \to +\infty$ as $t \to r_p^-$. It then follows from the intermediate value theorem that functions h_i and h_p have zeros in the interval $(0,\rho_0)$. Denote by ρ_i and ρ_p the smallest such zeros for functions h_i and h_p, respectively.

Furthermore, define functions $g_j, h_j, j = 4, 5, 6$ on the interval $[0,\rho_p)$ by

$$g_4(t) = g_1(t) + \frac{3(q_0(t) + q_0(g_3(t)t)\int_0^1 q_1(\theta t)d\theta}{4(1-p(t))(1-q_0(t))}, \tag{2.2}$$

$$g_5(t) = (1 + \frac{\int_0^1 q_1(\theta g_4(t)t)d\theta}{1-p(t)})g_4(t),$$

$$g_6(t) = (1 + \frac{\int_0^1 q_1(\theta g_5(t)t)d\theta}{1-p(t)})g_5(t),$$

and $h_j(t) = g_j(t) - 1$. Then, again we have that $h_j(0) = -1 < 0$ and $h_j(t) \to +\infty$ as $t \to \rho_p^-$. Denote by ρ_j the smallest zeros of functions h_j on the interval $(0,\rho_p)$.

Define the radius of convergence r by

$$\rho := \min\{\rho_i\}, \quad i = 1, 2, \ldots, 6 \tag{2.3}$$

Then, we have that for each $t \in [0, r)$

$$0 \leq g_i(t) < 1, \quad i = 1, 2, \ldots, 6, \tag{2.4}$$

and

$$0 \leq p(t) < 1. \tag{2.5}$$

Let $U(\gamma, \rho)$ and $\bar{U}(\gamma, \rho)$ stand, respectively, for the open and closed balls in X with center $\gamma \in X$ and of radius $\rho > 0$. Next, we present the local convergence analysis of method (1.2) using the preceding notation.

Theorem 2.1. *Let $F : D \subset X \to Y$ be a continuously Fréchet differentiable operator. Suppose that there exist $x^* \in D$ and continuous and nondecreasing function $q_0 : [0, +\infty) \to [0, +\infty)$, with $q_0(0) = 0$ such that for each $x \in D$*

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X) \tag{2.6}$$

and

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq q_0(\|x - x^*\|); \tag{2.7}$$

there exist $q, q_1 : [0, +\infty) \to [0, +\infty)$ continuous, nondecreasing functions satisfying $q(0) = 0$ such that for each $x, y \in D_0 = D \cap U(x^*, \rho_0)$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq q(\|x - y\|), \tag{2.8}$$

and

$$\|F'(x^*)^{-1}F'(x)\| \leq q_1(\|x - x^*\|), \tag{2.9}$$

and

$$\bar{U}(x^*, r) \subseteq D, \tag{2.10}$$

where ρ_0 and r are given by (2.1) and (2.3), respectively. Then, the sequence $\{x_n\}$ generated for $x_0 \in U(x^*, \rho) - \{x^*\}$ by method (1.2) is well defined in $U(x^*, \rho)$, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold

$$\|r_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho, \tag{2.11}$$

$$\|y_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.12}$$

$$\|z_n - x^*\| \leq g_3(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.13}$$

$$\|u_n - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.14}$$

$$\|v_n - x^*\| \leq g_5(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.15}$$

and

$$\|x_{n+1} - x^*\| \leq g_6(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.16}$$

where the functions g_i, $i = 1, 2, 3, 4, 5, 6$ are defined previously. Furthermore, if there exists $R > \rho$ such that

$$\int_0^1 q_0(\theta R) d\theta < 1, \tag{2.17}$$

then the limit point x^* is the only solution of the equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x^*, R)$.

Proof. Using mathematical induction, we show that the sequence $\{x_n\}$ is well defined in $U(x^*,\rho)$, remains in $U(x^*,\rho)$ for each $n=0,1,2,\dots$ and converges to x^*, so that the estimates (2.11)–(2.16) are satisfied. Using (2.1), (2.3), (2.7), and the hypothesis $x_0\in U(x^*,\rho)-\{x^*\}$, we have that

$$\|F'(x^*)^{-1}(F'(x)-F'(x^*)\|\le q_0(\|x-x^*\|)\le q_0(\rho)<1. \tag{2.18}$$

It follows from (2.18) and the Banach lemma on invertible operators [2–6, 18, 25], that $F'(x)^{-1}\in L(Y,X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\|\le\frac{1}{1-q_0(\|x-x^*\|)}. \tag{2.19}$$

The points r_0,y_0 and z_0 are also well defined by the first three substeps of method (1.2) for $n=0$, respectively. We can write by the first substeps of method (1.2) for $n=0$,

$$r_0-x^*=x_0-x^*-F'(x_0)^{-1}F(x_0). \tag{2.20}$$

By (2.1), (2.3), (2.4) (for i=1), (2.6), (2.8), (2.19) and (2.20), we get in turn that

$$\begin{aligned}\|r_0-x^*\| &\le \|F'(x_0)^{-1}F'(x^*)\|\|\textstyle\int_0^1 F'(x^*)^{-1}(F'(x^*+\theta(x_0-x^*))\\ &\quad -F'(x_0))(x_0-x^*)d\theta\\ &\le \frac{\int_0^1 q((1-\theta)\|x_0-x^*\|)\|x_0-x^*\|d\theta}{1-q_0(\|x_0-x^*\|)}\\ &= g_1(\|x_0-x^*\|)\|x_0-x^*\|\\ &\le \|x_0-x^*\|<r,\end{aligned} \tag{2.21}$$

which shows (2.11) for $n=0$ and $r_0\in U(x^*,\rho)$. Notice also that we used that $x^*+\theta(x_0-x^*)\in U(x^*,\rho)$, for each $\theta\in[0,1]$ $\|x^*+\theta(x_0-x^*)-x^*\|=\theta\|x_0-x^*\|<\rho$, for each $\theta\in[0,1]$, so $x^*+\theta(x_0-x^*)\in U(x^*,\rho)$. By (2.24)(for i=2 and i=3),(2.21) and the second substep and third substep of method (1.2), we get in turn that

$$\begin{aligned}\|y_0-x^*\| &= \|\tfrac12(r_0+x_0)-x^*\|\\ &= \tfrac12\|(r_0-x_0)+(x_0-x^*)\|\\ &\le \tfrac12[\|r_0-x_0\|+\|x_0-x^*\|]\\ &\le \tfrac12[g_1(\|x_0-x^*\|)+1]\|x_0-x^*\|\\ &= g_2(\|x_0-x^*\|)\|x_0-x^*\|\le\|x_0-x^*\|<\rho\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}\|z_0 - x^*\| &= \|\tfrac{1}{3}[(4y_0 - x_0) - x^*\| \\ &\leq \tfrac{4}{3}\|(y_0 - x^*)\| + \tfrac{1}{3}\|x_0 - x^*\| \\ &\leq \tfrac{1}{3}[4g_1(\|x_0 - x^*\|) + 1]\|x_0 - x^*\| \\ &= g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \rho,\end{aligned} \tag{2.23}$$

which shows (2.12), (2.13), $y_0 \in U(x^*, \rho)$, and $z_0 \in U(x^*, \rho)$.

Next, we must show that $(F'(x_0) - 3F'(z_0))^{-1} \in L(Y, X)$. In view of (2.3), (2.5), (2.7), and (2.23), we obtain in turn that

$$\begin{aligned}&\|(2F'(x^*))^{-1}[F'(x_0) - 3F'(z_0)]\| \\ &\leq \tfrac{1}{2}\|F'(x^*)^{-1}[(F'(x_0) - F'(x^*)) - 3(F'(z_0) - F'(x^*))]\| \\ &\leq \tfrac{1}{2}[\|F'(x^*)^{-1}[F'(x_0) - F'(x^*)]\| + 3\|F'(x^*)^{-1}(F'(z_0) - F'(x^*))\|] \\ &\leq \tfrac{1}{2}[q_0(\|x^* - x_0\|) + 3q_0(\|z_0 - x^*\|) \\ &\leq \tfrac{1}{2}[q_0(\|x^* - x_0\|) + 3q_0(g_3(\|x^* - x_0\|)\|x^* - x_0\|)] \\ &= p(\|x^* - x_0\|) \leq p(\rho) < 1,\end{aligned} \tag{2.24}$$

so, $(F'(x_0) - 3F'(z_0))^{-1} \in L(Y, X)$ and

$$\|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\| \leq \frac{1}{2(1 - p(\|x_0 - x^*\|))}. \tag{2.25}$$

It also follows that u_0, v_0, and x_1 are well defined by the last three substeps of method (1.2), respectively. We can write by (2.6) that

$$F(x_0) = F(x_0) - F(x^*) = \int_0^1 F'(x^* + t(x_0 - x^*))(x_0 - x^*)d\theta. \tag{2.26}$$

Then, by (2.9) and (2.26), we have that

$$\begin{aligned}\|F'(x^*)^{-1}F(x_0)\| &= \|\textstyle\int_0^1 F'(x^*)^{-1}F'(x^* + t(x_0 - x^*))(x_0 - x^*)d\theta\| \\ &\leq \textstyle\int_0^1 q_1(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|.\end{aligned} \tag{2.27}$$

Using the fourth substep of method (1.2) for $n=0$, we can write

$$\begin{aligned} u_0 - x^* &= (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + \tfrac{1}{2}F'(x_0)^{-1}F(x_0) \\ &\quad +(F'(x_0) - 3F'(z_0))^{-1}F(x_0) \\ &= (x_0 - x^* - F'(x_0)^{-1}F(x_0)) + \tfrac{3}{2}F'(x_0)^{-1}[(F'(x_0) - F'(x^*)) \\ &\quad +(F'(x^*) - F'(z_0))](F'(x_0) - 3F'(z_0))^{-1}F(x_0). \end{aligned} \tag{2.28}$$

Then, by (2.3), (2.4) (for i=4), (2.19), (2.21), (2.25), (2.27), and (2.28), we have in turn that

$$\begin{aligned} \|u_0 - x^*\| &\leq \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\ &\quad +\tfrac{3}{2}\|F'(x_0)^{-1}F'(x^*)\|[\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \\ &\quad +\|F'(x^*)^{-1}(F'(x^*) - F'(z_0))\|]\|(F'(x_0) - 3F'(z_0))^{-1}F'(x^*)\| \\ &\quad \times\|F'(x^*)^{-1}F(x_0)\| \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &\quad +\tfrac{3}{4}\tfrac{[q_0(\|x_0-x^*\|)+q_0(\|z_0-x^*\|)]\int_0^1 q_1(\theta\|x_0-x^*\|)d\theta\|x_0-x^*\|}{(1-q_0(\|x_0-x^*\|))(1-p\|x_0-x^*\|)} \\ &= g_4(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \rho, \end{aligned} \tag{2.29}$$

which shows (2.14) for $n=0$ and $u_0 \in U(x^*, \rho)$. As in (2.27) for $x_0 = u_0$, using (2.29), we get also that

$$\begin{aligned} \|F'(x^*)^{-1}F(u_0)\| &\leq \|\int_0^1 q_1(\theta\|u_0 - x^*\|)d\theta\|u_0 - x^*\| \\ &\leq \int_0^1 q_1(\theta g_4(\|x_0 - x^*\|)\|x_0 - x^*\|)d\theta g_4(\|x_0 - x^*\|)\|x_0 - x^*\|. \end{aligned} \tag{2.30}$$

Then, it follows from (2.3), (2.4) (for i=5), (2.19), (2.21), (2.25), (2.29), (2.30), and the identity obtained from the fifth substep of method (1.2) for $n=0$

$$v_0 - x^* = u_0 - x^* + 2(F'(x_0) - 3F'(z_0))^{-1}F(u_0) \tag{2.31}$$

that

$$\begin{aligned}\|v_0-x^*\| &\le \|u_0-x^*\|+2\|(F'(x_0)-3F'(z_0))^{-1}F(x^*)\|\|F'(x^*)^{-1}F(u_0)\|\\ &\le \|u_0-x^*\|+\tfrac{\int_0^1 q_1(\theta\|u_0-x^*\|)d\theta\|u_0-x^*\|}{1-p\|x_0-x^*\|}\\ &\le (1+\tfrac{\int_0^1 q_1(\theta\|u_0-x^*\|)d\theta}{1-p\|x_0-x^*\|})\|u_0-x^*\|\\ &\le g_5(\|x_0-x^*\|)\|x_0-x^*\|\le\|x_0-x^*\|<\rho,\end{aligned} \tag{2.32}$$

which shows (2.15) for $n=0$ and $v_0\in U(x^*,\rho)$. From (2.27)(for $x_0=u_0$) and (2.32), we also have that

$$\begin{aligned}\|x_1-x^*\| &= \|v_0-x^*+2(F'(x_0)-3F'(z_0))^{-1}F(v_0)\|\\ &\le \|v_0-x^*\|+2\|(F'(x_0)-3F'(z_0))^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(v_0)\|\\ &\le \|v_0-x^*\|+\tfrac{\int_0^1 q_1(\theta\|v_0-x^*\|)d\theta\|v_0-x^*\|}{1-p\|x_0-x^*\|}\\ &\le (1+\tfrac{\int_0^1 q_1(\theta\|v_0-x^*\|)d\theta}{1-p\|x_0-x^*\|})\|v_0-x^*\|\\ &= g_6(\|x_0-x^*\|)\|x_0-x^*\|\le\|x_0-x^*\|<\rho,\end{aligned} \tag{2.33}$$

which shows (2.16) for $n=0$ and $x_1\in U(x^*,\rho)$. By simply replacing $x_0, r_0, y_0, z_0, u_0, v_0, x_1$ by $x_k, r_k, y_k, z_k, u_k, v_k, x_{k+1}$ in the preceding estimates, we arrive at (2.11)–(2.16). Using the estimate

$$\|x_{k+1}-x^*\|\le c\|x_k-x^*\|<\rho, \tag{2.34}$$

where $c=g_6(\|x_0-x^*\|)\in[0,1)$, we deduce that $\lim_{k\to\infty} x_k=x^*$ and $x_{k+1}\in U(x^*,\rho)$.

Finally to show the uniqueness part, let $y^*\in D_1$ be such that $F(y^*)=0$. Define the linear operator $T=\int_0^1 F'(x^*+\theta(y^*-x^*))d\theta$. Then, using (2.7) and (2.17), we get that

$$\begin{aligned}\|F'(x^*)^{-1}(T-F'(x^*))\| &\le \int_0^1 q_0(\theta\|x^*-y^*\|)d\theta\\ &\le \int_0^1 q_0(\theta R)d\theta<1,\end{aligned} \tag{2.35}$$

so, $T^{-1}\in L(Y,X)$. Then, from the identity $0=F(y^*)-F(x^*)=T(y^*-x^*)$, we conclude that $x^*=y^*$. □

Remark 2.2. (a) The radius ρ_1 was obtained by Argyros in [2–5] as the convergence radius for Newton's method under condition (2.6)–(2.8). Notice that the convergence radius for Newton's method given independently by Rheinboldt [25] and Traub [28] is given by

$$\rho=\frac{2}{3L}<\rho_1.$$

As an example, let us consider the function $f(x) = e^x - 1$. Then $x^* = 0$. Set $\Omega = U(0,1)$. Then, we have that $L_0 = e - 1 < l = e$, so $\rho = 0.24252961 < \rho_1 = 0.324947231$.

Moreover, the new error bounds [2–5] are

$$\|x_{n+1} - x^*\| \leq \frac{q}{1 - q_0\|x_n - x^*\|}\|x_n - x^*\|^2,$$

whereas the old ones [25, 28]

$$\|x_{n+1} - x^*\| \leq \frac{q}{1 - q\|x_n - x^*\|}\|x_n - x^*\|^2.$$

Clearly, the new error bounds are more precise, if $q_0 < q$. Also, the radius of convergence of method (1.2) given by ρ is larger than ρ_1 (see (2.2)).

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method(GMREM), the generalized conjugate method(GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2–5].

(c) The results can be also be used to solve equations, where the operator F' satisfies the autonomous differential equation [2–5, 18, 23]:

$$F'(x) = P(F(x)),$$

where P is a known continuous operator. Since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing the solution x^*. Set as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that method (1.2) is not changing if we use the new instead of the old conditions [17]. Moreover, for the error bounds in practice, we can use the computational order of convergence (COC)

$$\xi = \frac{\ln \frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{\ln \frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \qquad \text{for each } n = 1, 2, \ldots,$$

or the approximate computational order of convergence (ACOC)

$$\xi^* = \frac{\ln \frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{\ln \frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \qquad \text{for each } \ n = 0, 1, 2, \ldots,$$

instead of the error bounds obtained in Theorem 2.1.

(e) In view of (2.7) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + q_0(\|x - x^*\|)\end{aligned}$$

condition (2.9) can be dropped and q can be replaced by

$$q(t) = 1 + q_0(t)$$

or

$$q = 1 + q_0(\rho_0) \quad \text{or} \quad q(t) = 2,$$

since $q_0(\rho_0) < 1$.

3. Numerical examples

The numerical examples are presented in this section.

Example 3.1. Let $X = Y = \mathbb{R}^3, D = \bar{U}(0,1), x^* = (0,0,0)^T$. Define a function F on D by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T,$$

where $w = (x,y,z)^T$. Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using condition (2.8), we get $q_0(t) = (e-1)t, q(t) = e^{\frac{1}{e-1}}t$, and $q_1(t) = e^{\frac{1}{e-1}}t$.

The parameters are $\rho_0 = 0.581976, \rho_1 = 0.38269, \rho_2 = 0.38269, \rho_3 = 0.2850, \rho_4 = 0.186668, \rho_5 = 0.17217, \rho_6 = 0.161725$, and $\rho_p = 0.288031$.

Example 3.2. Let $X = Y = C[0,1]$, be the space of continuous functions defined on $[0,1]$ equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta \quad \text{for each } \xi \in D.$$

Then, we get that $x^* = 0$, $q_0(t) = 7.5t, q(t) = 15t, q_1(t) = 15t$. The parameters are $\rho_0 = 0.13333, \rho_1 = 0.06666, \rho_2 = 0.0666, \rho_3 = 0.044$, $\rho_4 = 0.0358, \rho_5 = 0.0318636, \rho_6 = 0.0292235$, and $\rho_p = 0.0552285$.

Example 3.3. Let $X = Y = C[0,1]$, be the space of continuous functions defined on $[0,1]$ equipped with the max norm. Let us define f on $D = [-\frac{1}{2}, \frac{5}{2})$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{3.2}$$

Choose $x^* = 1$. We also have that

$$f'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2,$$
$$f''(x) = 6x \ln x^2 + 20x^3 + 12x^2 + 10x,$$

and

$$f'''(x) = 6\ln x^2 + 60x^2 - 24x + 22.$$

Notice that $f'''(x)$ is unbounded on D. Hence, the results in [11], cannot apply to show the convergence of method (1.2). Then, we get that $x^* = 0$, $q_0(t) = q(t) =$

$147t, q_1(t) = 1 + q_0(\rho_0)$. The parameters are $\rho_0 = 0.006802, \rho_1 = 0.004535, \rho_2 = 0.004535, \rho_3 = 0.00340$, $\rho_4 = 0.00173487, \rho_5 = 0.000843696, \rho_6 = 0.000353272$, and $\rho_p = 0.00340136$.

References

1. M.F. Abad, A. Cordero, J.R. Torregrosa, *Fourth and fifth-order methods for Solving nonlinear systems of equations: An application to the global positioning system*, Abstr. Appl. Anal. **2013** (2013), Art. ID 586708, 10 pp.
2. I.K. Argyros, *Computational Theory of Iterative Methods, Vol. 15,* Elsevier, New York, 2007.
3. I.K. Argyros, *A semilocal convergence analysis for directional Newton methods*, Math. Comput. **80** (2011) 327–343.
4. I.K. Argyros, S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity **28** (2012) 364–387.
5. I.K. Argyros, S. Hilout, *Computational Methods in Nnonlinear Analysis. Efficient Algorithms, Fixed Point Theory and Applications*, World Scientific, 2013.
6. I.K. Argyros, H. Ren, *Improved local analysis for certain class of iterative methods with cubic convergence*, Numer. Algorithms, **59** (2012) 505–521.
7. C. Chun, P. Stănică, B. Neta, *Third-order family of methods in Banach spaces*, Comput. Math. Appl. **61** (2011) 1665–1675.
8. A. Cordero, J. Hueso, E. Martinez, J.R. Torregrosa, *A modified Newton-Tarratt's composition*, Numer. Algorithms, **55** (2010) 87–99.
9. A. Cordero, J.R. Torregrosa, *Variants of Newton's method using fifth order quadrature formulas*, Appl. Math. Comput. **190** (2007) 686–698.
10. A. Cordero, J.R. Torregrosa, M.P. Vasileva, *Increasing the order of convergence of iterative schemes for solving nonlinear systems*, J. Comput. Appl. Math. **252** (2013) 86–94.
11. R. Ezzati, E. Azandegan, *A simple iterative method with fifth order convergence by using Potra and Ptak's method*, Math. Sci. **3** (2009), no. 2, 191–200.
12. G.M. Grau-Sanchez, A. Grau, M. Noguera, *On the computational efficiency index and some iterative methods for solving systems of non-linear equations*, J. Comput. Appl Math. **236** (2011) 1259–1266.
13. J.M. Gutiérrez, A.A. Magreñán, N. Romero, *On the semi-local convergence of Newton-Kantorovich method under center-Lipschitz conditions*, Appl. Math. Comput. **221** (2013) 79–88.
14. V.I. Hasanov, I.G. Ivanov, F. Nebzhibov, *A new modification of Newton's method*, Appl. Math. Eng. **27** (2002) 278–286.
15. M.A. Hernández, M.A. Salanova, *Modification of the Kantorovich assumptions for semilocal convergence of the Chebyshev method*, J. Comput. Appl. Math. **126** (2000) 131–143.
16. H.H.H. Homeier, *On Newton type methods with cubic convergence*, J. Comput. Appl. Math. **176** (2005) 425–432.
17. J.P. Jaiswal, *Semilocal convergence of an eighth-order method in Banach spaces and its computational efficiency*, Numer. Algorithms, **71** (2016) 933–951.
18. L.V. Kantorovich, G.P. Akilov, *Functional Analysis*, Pergamon Press, Oxford, 1982.
19. J.S. Kou, Y.T. Li, X.H. Wang, *A modification of Newton method with fifth-order convergence*, J. Comput. Appl. Math. **209** (2007) 146–152.
20. A.A. Magrenan, *Different anomalies in a Jarratt family of iterative root finding methods*, Appl. Math. Comput. **233** (2014) 29–38.
21. A.A. Magrenan, *A new tool to study real dynamics: The convergence plane*, Appl. Math. Comput. **248** (2014), 29–38.
22. M.S. Petkovic, B. Neta, L. Petkovic, J. Džunič, *Multipoint Methods for Solving Nonlinear Equations*, Elsevier, 2013

23. F.A. Potra, V. Pták, *Nondiscrete Induction and Iterative Processes*, Research Notes in Mathematics 103, Pitman, Boston, 1984.
24. A.N .Romero, J.A. Ezquerro, M.A. Hernandez, *Approximacion de soluciones de algunas equacuaciones integrals de Hammerstein mediante metodos iterativos tipo. Newton*, in: XXI Congresode Ecuaciones Diferenciales y Aplicaciones, Universidad de Castilla-La Mancha, 2009.
25. W.C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations*, in: Mathematical Models and Numerical Methods (A.N. Tikhonov et al. eds.) pp. 129–142, Banach Center Publ. 3, Warsaw, 1978.
26. J.R. Sharma, P.K. Guha, R. Sharma, *An efficient fourth order weighted-Newton method for systems of nonlinear equations*, Numer. Algorithms **62** (2013), no. 2, 307–323.
27. J.R. Sharma, P.K. Guha, *An efficient fifth order method for systems of nonlinear equations*, Comput. Math. Appl **67** (2014) 591–601.
28. J.F. Traub, *Iterative Methods for the Solution of Equations*, AMS Chelsea Publishing, 1982.

[1] Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA
E-mail address: iargyros@cameron.edu

[2] Department of Mathematical and Computational Sciences, NIT Karnataka, 575 025, India
E-mail address: sgeorge@nitk.ac.in

[3] Department of Mathematics, Manipal Institute of Technology, Manipal, Karnataka, 576104, India
E-mail address: shobha.me@gmail.com