Journal of Computational and Applied Mathematics 330 (2018) 488–498

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Numerical approximation of a Tikhonov type regularizer by a discretized frozen steepest descent method

Santhosh George *, M. Sabari

Department of Mathematical and Computational Sciences, NIT Karnataka, Mangaluru-575 025, India

ARTICLE INFO

Article history:
Received 18 December 2015
Received in revised form 29 December 2016

MSC:
47A52
65J15

Keywords:
Nonlinear ill-posed problem
Steepest descent method
Balancing principle

ABSTRACT

We present a frozen regularized steepest descent method and its finite dimensional realization for obtaining an approximate solution for the nonlinear ill-posed operator equation $F(x) = y$. The proposed method is a modified form of the method considered by Argyros et al. (2014). The balancing principle considered by Pereverzev and Schock (2005) is used for choosing the regularization parameter. The error estimate is derived under a general source condition and is of optimal order. The provided numerical example proves the efficiency of the proposed method.

1. Introduction

Inverse problems arise in many practical applications, such as inverse scattering problem and tomographic, parameter identification in partial differential equations (see [1–3]). They can be modeled as an operator equation

$$F(x) = y, \tag{1.1}$$

where $F : D(F) \subseteq X \to Y$ is a nonlinear Fréchet differentiable operator between the Hilbert spaces X and Y. Throughout this study, $D(F)$, $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively, stand for the domain of F, innerproduct and norm which can always be identified from the context in which they appear. Fréchet derivative of F is denoted by $F'(\cdot)$ and its adjoint by $F'(\cdot)^*$. Further we assume that Eq. (1.1) has a solution $\hat{x}$, which is not depending continuously on the right-hand side data y. The problems in which the solution $\hat{x}$ is not depending continuously on the right hand data are called ill-posed problems. It is a common practice to use iterative methods or iterative regularization methods for approximating $\hat{x}$. For example, Landweber method [4,5], Levenberg–Marquardt method [6], Gauss–Newton [7,8], Conjugate Gradient [9], Newton-like methods [10,11], TIGRA (Tikhonov-gradient method) [12].

It is assumed further that we have only approximate data $y^\delta \in Y$ with

$$\|y - y^\delta\| \le \delta.$$

The steepest descent method was considered by Scherzer [13], Neubauer and Scherzer [14] for approximately solving (1.1). In general, the steepest descent method for (1.1) with y^δ in place of y can be written as

$$x_{k+1} = x_k + \alpha_k s_k, \tag{1.2}$$

* Corresponding author.
E-mail addresses: sgeorge@nitk.ac.in (S. George), msabarijothi16@gmail.com (M. Sabari).

http://dx.doi.org/10.1016/j.cam.2017.09.022

where s_k is the search direction taken as the negative gradient of the minimization functional involved and α_k is the descent. For solving Eq. (1.1) with y^δ in place of y, method (1.2) was studied by Scherzer [13] when $s_k = -F'(x_k)^*(F(x_k) - y^\delta)$ and $\alpha_k = \frac{\|s_k\|}{\|F'(x_k)^* s_k\|}$, and Neubauer and Scherzer [14] studied (the minimal error method) method (1.2) when $s_k = -F'(x_k)^*(F(x_k) - y^\delta)$ and $\alpha_k = \frac{\|F(x_k)-y^\delta\|}{\|F'(x_k)s_k\|}$. For linear operator F, Gilyazov [15] studied (α-process) method (1.2) when $s_k = -F'(x_k)^*(F(x_k) - y^\delta)$ and $\alpha_k = \frac{\langle (F^*F)^\alpha, s_k\rangle}{\langle (F^*F)^\alpha s_k, F^*Fs_k\rangle}$. Vasin [16] considered a regularized version of the steepest descent method in which $s_k = -F'(x_k)^*(F(x_k) - y^\delta) + \alpha(x_k - x_0)$ and $\alpha_k = \frac{\|s_k\|^2}{\|F'(x_k)s_k\|^2 + \alpha\|s_k\|^2}$. Here and below x_0 is the initial guess. Also, observe that the TIGRA-method of Ramlau [12] is of the form (1.2) with $s_k = -[F'(x_k)^*(F(x_k) - y^\delta) + \alpha_k(x_0 - x_k)]$ and $\alpha_k = \beta_k$. Note that, in all these methods, one has to compute the Fréchet derivative of F at each iterate x_k and α_k in each iteration step which is in general very expensive.

In the present study, we consider a modified form of (1.2), namely the frozen regularized steepest method defined for each $k = 0, 1, 2, \ldots$ by

$$x_{k+1} = x_k - \beta[F'(x_0)^*(F(x_k) - y^\delta) + \alpha(x_k - x_0)], \tag{1.3}$$

where x_0 is the initial point, $\beta > 0$ is a fixed parameter and $\alpha > 0$ is the regularization parameter. Further, note that in method (1.3), we have frozen the Fréchet derivative at x_0 throughout the iteration. That is why, we call method (1.3) the frozen regularized steepest method. This is one of the advantage of the proposed method. Observe that (1.3) is of the form (1.2) with $s_k = -[F'(x_0)^*(F(x_k) - y^\delta) + \alpha(x_k - x_0)]$ and $\alpha_k = \beta$ for each $k = 1, 2, \ldots$. Since $\alpha_k = \beta$ one need not have to compute α_k in each step as in the earlier studies such as [13,14,16]. In other words, the computational work is reduced considerably in the proposed method (1.3).

Note that method (1.3) coincide with the method considered in [17] when $\beta = 1$, but our convergence analysis is different from that of [17] and is based on the property of the norm of a self adjoint operator in the Hilbert space (see Section 2). Moreover the condition on the radius of the convergence ball in [17] is too restrictive than the condition in this study. The numerical experiments (see comparison Table 1) also show that the method considered in this paper provides better error estimate than that of the method considered in [17]. We also consider the finite dimensional realization of the method (1.3) in Section 3. The error analysis and the algorithm for implementing the method (1.3) are given in Section 4. Finally the numerical results are given in Section 5.

2. Convergence analysis of method (1.3)

Denote by $B_r(x)$, $\overline{B}_r(x)$ the open and closed ball in X, respectively, with center $x \in X$ and of radius $r > 0$. Throughout this paper we assume that the operator F satisfies the following assumptions.

Assumption 2.1.

(a) There exists a constant $k_0 > 0$ such that for every $x \in D(F)$ and $v \in X$, there exists an element $\Phi(x, x_0, v) \in X$ satisfying

$$[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v), \quad \|\Phi(x, x_0, v)\| \le k_0\|v\|\|x - x_0\|.$$

(b)

$$\|F'(x_0)\| \le M.$$

Notice that in the literature the stronger than (a) condition (a)′

$$[F'(x) - F'(z)]v = F'(z)\xi(x, z, v), \quad \|\xi(x, z, v)\| \le K\|v\|\|x - z\|$$

is used for some $\xi(x, z, v) \in X$. However,

$$k_0 \le K$$

holds in general and $\frac{K}{k_0}$ can be arbitrarily large [18]. It is also worth noticing that $(a)'$ implies (a) but not necessarily vice versa and element ξ is less accurate and more difficult to find than Φ (see the numerical example in [17]).

Assumption 2.2. There exists a continuous, strictly monotonically increasing function $\varphi : (0, a] \to (0, \infty)$ with $a \ge \|F'(x_0)\|^2$ satisfying;

- $$\lim_{\lambda \to 0} \varphi(\lambda) = 0$$
- $$\sup_{\lambda \ge 0} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \le \varphi(\alpha), \qquad \forall \alpha \in (0, a].$$

- there exists $v \in X$ with $\|v\| \le 1$ such that

$$x_0 - \hat{x} = \varphi(F'(x_0)^* F'(x_0))v.$$

It is known that for $\alpha > 0$,

$$F'(x_0)^*(F(x) - y^\delta) + \alpha(x - x_0) = 0 \tag{2.1}$$

has a unique solution x_α^δ in $B_r(x_0)$ provided $0 < r < \frac{1}{k_0}$ [17, Theorem 2.] (see also [19, Section 3]). Also it is known (cf. [17, Theorem 4]) that if Assumptions 2.1 and 2.2 are satisfied, then

$$\|x_\alpha^\delta - \hat{x}\| \le \frac{1}{1 - k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right). \tag{2.2}$$

Let $\delta_0 > 0,\ a_0 > 0$ be some constants with $\delta_0^2 < a_0$ and $\|x_0 - \hat{x}\| \le r$. Let $\delta \in (0, \delta_0]$ and $\alpha \in [\delta^2, a_0]$. Further, let $\beta,\ q_{\alpha,\beta}$ be parameters such that

$$\beta \le \frac{1}{M^2 + a_0} \tag{2.3}$$

and

$$q_{\alpha,\beta} = 1 - \alpha\beta + \frac{3\beta M^2 k_0}{2} r. \tag{2.4}$$

Remark 2.3.

1. Suppose $0 < r < \frac{2\alpha}{3M^2 k_0}$. Then, we have

$$\begin{aligned} q_{\alpha,\beta} &= 1 - \alpha\beta + \frac{3\beta M^2 k_0}{2} r \\ &< 1 - \alpha\beta + \frac{3\beta M^2 k_0}{2}\frac{2\alpha}{3M^2 k_0} = 1, \end{aligned}$$

i.e., $q_{\alpha,\beta} < 1$ for all $0 < r < \frac{2\alpha}{3M^2 k_0}$.

2. Notice that, if $\alpha \longrightarrow 0$, then $r \longrightarrow 0$ and in this case $x_0 = \hat{x}$. Further, in practice, we choose α from a set $\{0 < \alpha_0 < \alpha_1, \dots, \alpha_N < 1\}$ (see Section 4.2) and hence $r > 0$.

Hereafter, we assume that $0 < r < \min\{\frac{1}{2k_0}, \frac{2\alpha}{3M^2 k_0}\}$.

Theorem 2.4. *Let x_n be as in (1.3) and let $0 < r < \min\{\frac{1}{2k_0}, \frac{2\alpha}{3M^2 k_0}\}$. Then for each $\delta \in (0, \delta_0]$, $\alpha \in [\delta^2, a_0]$, the sequence $\{x_n\}$ is in $B_{2r}(x_0)$ and converges to x_α^δ as $n \to \infty$. Further,*

$$\|x_{n+1} - x_\alpha^\delta\| \le q_{\alpha,\beta}^{n+1} \|x_0 - x_\alpha^\delta\|, \tag{2.5}$$

where $q_{\alpha,\beta}$ is as in (2.4).

Proof. Clearly, $x_0 \in \overline{B_{2r}(x_0)}$. Let $A_n := \int_0^1 F'(x_\alpha^\delta + t(x_n - x_\alpha^\delta))dt$. Since $x_\alpha^\delta \in B_r(x_0)$, A_0 is well defined. Assume that for some $n > 0$, $x_n \in B_{2r}(x_0)$ and A_n is well defined. Then, since x_α^δ satisfies Eq. (2.1), we have

$$\begin{aligned} x_{n+1} - x_\alpha^\delta &= x_n - x_\alpha^\delta - \beta[F'(x_0)^*(F(x_n) - F(x_\alpha^\delta)) + \alpha(x_n - x_\alpha^\delta)] \\ &= x_n - x_\alpha^\delta - \beta[F'(x_0)^* A_n + \alpha I](x_n - x_\alpha^\delta) \\ &= x_n - x_\alpha^\delta - \beta[F'(x_0)^*(A_n - F'(x_0))](x_n - x_\alpha^\delta) \\ &\quad - \beta[F'(x_0)^* F'(x_0) + \alpha I](x_n - x_\alpha^\delta) \\ &= [I - \beta(F'(x_0)^* F'(x_0) + \alpha I)](x_n - x_\alpha^\delta) \\ &\quad - \beta[F'(x_0)^*(A_n - F'(x_0))](x_n - x_\alpha^\delta). \end{aligned} \tag{2.6}$$

Using Assumptions 2.1, we have

$$\begin{aligned} x_{n+1} - x_\alpha^\delta &= [I - \beta(F'(x_0)^* F'(x_0) + \alpha I)](x_n - x_\alpha^\delta) \\ &\quad - \beta F'(x_0)^* F'(x_0) \int_0^1 \Phi(x_\alpha^\delta + t(x_n - x_\alpha^\delta), x_0, x_n - x_\alpha^\delta)dt. \end{aligned}$$

Now since $I - \beta(F'(x_0)^*F'(x_0) + \alpha I)$ is a positive self-adjoint operator,

$$\begin{aligned}
&\|I - \beta(F'(x_0)^*F'(x_0) + \alpha I)\| \\
&\quad = \sup_{\|x\|=1} |\langle (I - \beta(F'(x_0)^*F'(x_0) + \alpha I))x, x\rangle| \\
&\quad = |\sup_{\|x\|=1} (1 - \beta\alpha)\langle x, x\rangle - \beta\langle F'(x_0)^*F'(x_0)x, x\rangle| \\
&\quad \le 1 - \alpha\beta.
\end{aligned} \tag{2.7}$$

The last step follows from relation

$$\beta|\langle F'(x_0)^*F'(x_0)x, x\rangle| \le \beta\|F'(x_0)\|^2 \le \beta M^2 \le \frac{1}{M^2+\alpha}M^2 = 1 - \frac{\alpha}{M^2+\alpha} \le 1 - \beta\alpha.$$

Hence, by Assumption 2.1, we have

$$\begin{aligned}
\|x_{n+1} - x_\alpha^\delta\| &\le (1 - \alpha\beta)\|x_n - x_\alpha^\delta\| \\
&\quad + \beta M^2 k_0 \int_0^1 ((1-t)\|x_\alpha^\delta - x_0\| + t\|x_n - x_0\|)dt\|x_n - x_\alpha^\delta\| \\
&\le \left(1 - \alpha\beta + \beta\frac{3k_0M^2r}{2}\right)\|x_n - x_\alpha^\delta\| \\
&\le q_{\alpha,\beta}\|x_n - x_\alpha^\delta\|.
\end{aligned} \tag{2.8}$$

Since $q_{\alpha,\beta} < 1$ (see Remark 2.3), we have

$$\|x_{n+1} - x_\alpha^\delta\| < \|x_0 - x_\alpha^\delta\| \le r$$

and

$$\|x_{n+1} - x_0\| \le \|x_{n+1} - x_\alpha^\delta\| + \|x_0 - x_\alpha^\delta\| \le 2r$$

i.e., $x_{n+1} \in B_{2r}(x_0)$. Also, for $0 \le t \le 1$,

$$\|x_\alpha^\delta + t(x_{n+1} - x_\alpha^\delta) - x_0\| = \|(1-t)(x_\alpha^\delta - x_0) + t(x_{n+1} - x_\alpha^\delta)\| < 2r.$$

Hence, $x_\alpha^\delta + t(x_{n+1} - x_\alpha^\delta) \in B_{2r}(x_0)$ and A_{n+1} is well defined. Thus, by induction x_n is well defined and remains in $B_{2r}(x_0)$ for each $n = 0, 1, 2, \ldots$. By letting $n \to \infty$ in (1.3), we obtain the convergence of x_n to x_α^δ. The estimate (2.5) now follows from (2.8). □

Remark 2.5.

1. If Assumption 2.1 is fulfilled only for all $x \in B_r(x_0) \cap Q \neq \emptyset$, where Q is an convex closed a priori set, for which $\hat{x} \in Q$, then we can modify method (1.3) by the following way

$$x_{n+1,\alpha}^\delta = P_Q(T(x_{n,\alpha}^\delta))$$

   to obtain the same estimate in the following Theorem 2.4; here P_Q is the metric projection onto the set Q and T is the step operator in (1.3).
2. Instead of Assumption 2.1, if we use the following Lipschitz condition:

$$\|F'(x_1) - F'(x_2)\| \le L_0\|x_1 - x_2\| \tag{2.9}$$

   then from (2.6) and (2.7), one can prove that (2.5) holds with $\bar{q}_{\alpha,\beta} := 1 - \alpha\beta + \beta M L_0 r$ instead of $q_{\alpha,\beta}$, provided $0 < r < \frac{\alpha}{ML_0}$.
3. Also by using (2.9) instead of Assumption 2.1, one can prove (2.1) has a unique solution if $0 < r < \frac{\sqrt{\alpha}}{L_0}$ and

$$\|x_\alpha^\delta - \hat{x}\| \le \frac{1}{1 - \frac{L_0 r}{\sqrt{\alpha}}}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right).$$

3. Finite dimensional realization of method (1.3)

For implementing method (1.3) one needs numerical calculations in finite dimensional spaces. One of the approaches in this regard is through discretization (see [1, page 63]). Here the regularization is achieved by a finite dimensional approximation alone. Regularization of ill-posed problems by projection methods can be found in the literature, for e.g in [20–23]. This section is concerned with the finite dimensional realization of the method (1.3). Precisely, our aim in this section is to obtain an approximation for x_α^δ, in the finite dimensional space $R(P_h)$ of X. Here $\{P_h\}_{h>0}$ is a family of orthogonal projections of X onto $R(P_h)$, the range of P_h. For the results that follow, we impose the following conditions. Let

$$\epsilon_h := \|F'(x_0)(I - P_h)\|$$

and

$$b_h := \|(I - P_h)\hat{x}\|.$$

We assume that $\lim_{h\longrightarrow 0}\epsilon_h = 0$ and $\lim_{h\to 0} b_h = 0$. The above assumption is satisfied if $P_h \to I$ point-wise and if $F'(\cdot)$ is compact operator. Further, we assume that there exist $\varepsilon_0 > 0$, $b_0 > 0$ and $\delta_0 > 0$ such that $\epsilon_h < \epsilon_0$ and $b_h < b_0$.

We have taken the discretized version of (1.3) as

$$x_{k+1,\alpha}^{h,\delta} = x_{k,\alpha}^{h,\delta} - \beta P_h[F'(x_0)^*(F(x_{k,\alpha}^{h,\delta}) - y^\delta) + \alpha(x_{k,\alpha}^{h,\delta} - x_0^{h,\delta})] \tag{3.1}$$

where $x_0^{h,\delta} =: P_h x_0$. Let

$$(\delta_0 + \varepsilon_0)^2 < \bar{a}_0.$$

Next we prove that, for $\alpha > 0$

$$P_h F'(x_0)^*(FP_h(x) - y^\delta) + \alpha P_h(x - x_0) = 0 \tag{3.2}$$

has a unique solution $x_\alpha^{h,\delta}$ in $B_r(x_0) \cap R(P_h)$.

Theorem 3.1. *Let $\hat{x}$ be a solution of (1.1), Assumption 2.1 satisfied and let $F : D(F) \subseteq X \to Y$ be Fréchet differentiable in a ball $B_r(x_0) \cap R(P_h) \subseteq D(F)$ with $0 < r < \frac{1}{2k_0}$. Then (3.2) possesses a unique solution $x_\alpha^{h,\delta}$ in $B_r(x_0) \cap R(P_h)$.*

Proof. For $x \in B_r(x_0) \cap R(P_h)$, let

$$M_h = \int_0^1 F'(\hat{x} + t(x - \hat{x}))dt.$$

If $P_h F'(x_0)^* M_h P_h + \alpha I$ is invertible, then

$$\begin{aligned}(P_h F'(x_0)^* M_h P_h + \alpha I)(x - P_h\hat{x}) = \alpha P_h(x_0 - \hat{x}) + P_h F'(x_0)^*(y^\delta - y)\\ + P_h F'(x_0)^* M_h (I - P_h)\hat{x}\end{aligned} \tag{3.3}$$

has a unique solution $x_\alpha^{h,\delta} \in R(P_h)$. Observe that

$$F(P_h x) - y^\delta = F(P_h x) - F(\hat{x}) + y - y^\delta = M_h(P_h x - \hat{x}) + y - y^\delta$$

and hence

$$\begin{aligned}&P_h F'(x_0)^*(FP_h(x) - y^\delta) + \alpha P_h(x - x_0)\\ &= P_h F'(x_0)^*(M_h(P_h x - \hat{x}) + y - y^\delta) + \alpha P_h(x - x_0)\\ &= (P_h F'(x_0)^* M_h P_h + \alpha I)P_h(x - \hat{x}) - \alpha P_h(x_0 - \hat{x})\\ &\quad - P_h F'(x_0)^* M_h(I - P_h)\hat{x} - P_h F'(x_0)^*(y^\delta - y).\end{aligned}$$

Therefore by (3.3) $P_h F'(x_0)^*(FP_h(x) - y^\delta) + \alpha P_h(x - x_0) = 0$ has a unique solution $x_\alpha^{h,\delta}$. Clearly, $x_\alpha^{h,\delta} \in B_r(x_0) \cap R(P_h)$. So, it remains to show that $P_h F'(x_0)^* M_h P_h + \alpha I$ is invertible for $x \in B_r(x_0) \cap R(P_h)$. Note that by Assumption 2.1, we have

$$\begin{aligned}&\|(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)^{-1} P_h F'(x_0)^*(M_h - F'(x_0))P_h\|\\ &= \sup_{\|v\|\le 1} \|(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)^{-1} P_h F'(x_0)^*(M_h - F'(x_0))P_h v\|\\ &\le \sup_{\|v\|\le 1} \left\|(P_h F'(x_0)^* F'(x_0) P_h + \alpha P_h)^{-1} P_h F'(x_0)^* \int_0^1 (F'(\hat{x} + t(x - \hat{x})) - F'(x_0))dt P_h v\right\|\\ &\le \sup_{\|v\|\le 1} \left\|(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)^{-1} P_h F'(x_0)^* F'(x_0) \int_0^1 \Phi(\hat{x} + t(x - \hat{x}), x_0, P_h v)dt\right\|\\ &\le \sup_{\|v\|\le 1} \left\|(P_h F'(x_0)^* F'(x_0) P_h + \alpha I)^{-1} P_h F'(x_0)^* F'(x_0)[P_h + I - P_h] \int_0^1 \Phi(\hat{x} + t(x - \hat{x}), x_0, P_h v)dt\right\|\\ &\le \left[k_0 + k_0 \frac{\varepsilon_h}{\sqrt{\alpha}}\right] \int_0^1 \|\hat{x} + t(x - \hat{x}) - x_0\|dt\end{aligned}$$

$$\leq \left[k_0 + k_0 \frac{\varepsilon_h}{\sqrt{\alpha}}\right] \int_0^1 [(1-t)\|\hat{x} - x_0\| + t\|x - x_0\|]dt$$
$$\leq k_0 \left(1 + \frac{\varepsilon_h}{\sqrt{\alpha}}\right) \frac{r+r}{2} < 2k_0 r < 1.$$

Therefore, $I + (P_hF'(x_0)^*F'(x_0)P_h + \alpha I)^{-1}P_hF'(x_0)^*(M_h - F'(x_0))P_h$ is invertible. Now from the relation

$$\begin{aligned} &P_hF'(x_0)^*M_hP_h + \alpha I \\ &\quad = (P_hF'(x_0)^*F'(x_0)P_h + \alpha I) \\ &\qquad [I + (P_hF'(x_0)^*F'(x_0)P_h + \alpha I)^{-1}P_hF'(x_0)^*(M_h - F'(x_0))P_h] \end{aligned}$$

it follows that $P_hF'(x_0)^*M_hP_h + \alpha I$ is invertible. □

Theorem 3.2. *Let $x_{n,\alpha}^{h,\delta}$ be as in (3.1) and let $0 < r < \min\{\frac{2\alpha}{3M^2k_0}, \frac{1}{2k_0}\}$. Then for each $\delta \in (0, \delta_0]$, $\alpha \in ((\delta + \varepsilon_h)^2, \bar{a_0}]$, $\varepsilon_h \leq \varepsilon_0$ the sequence $\{x_{n,\alpha}^{h,\delta}\}$ is in $B_{2r}(x_0) \cap R(P_h)$ and converges to $x_\alpha^{h,\delta}$ as $n \to \infty$. Further,*

$$\|x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \leq q_{\alpha,\beta}^{n+1}\|P_hx_0 - x_\alpha^{h,\delta}\|, \tag{3.4}$$

where $q_{\alpha,\beta}$ is as in (2.4).

Proof. Since $x_\alpha^{h,\delta}$ satisfies Eq. (3.2), we have

$$\begin{aligned} x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta} &= x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta} - \beta[P_hF'(x_0)^*(F(x_{n,\alpha}^{h,\delta}) - F(x_\alpha^{h,\delta})) + \alpha P_h(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta})] \\ &= x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta} - \beta[P_hF'(x_0)^*A_n^h + \alpha P_h](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\ &= x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta} - \beta[P_hF'(x_0)^*(A_n^h - F'(x_0))](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\ &\quad - \beta[P_hF'(x_0)^*F'(x_0)P_h + \alpha P_h](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\ &= [I - \beta(P_hF'(x_0)^*F'(x_0)P_h + \alpha I)](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\ &\quad - \beta[P_hF'(x_0)^*(A_n^h - F'(x_0))](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}), \end{aligned}$$

where $A_n^h =: \int_0^1 F'(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}))dt$. Using Assumption 2.1 we have

$$\begin{aligned} x_{n+1,\alpha}^{h,\delta} - x_\alpha^{h,\delta} &= [I - \beta(P_hF'(x_0)^*F'(x_0)P_h + \alpha I)](x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}) \\ &\quad - \beta[P_hF'(x_0)^*F'(x_0)] \int_0^1 \Phi(x_\alpha^{h,\delta} + t(x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}), x_0, (x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}))dt. \end{aligned}$$

Now since $I - \beta(P_hF'(x_0)^*F'(x_0)P_h + \alpha I)$ is a positive self-adjoint operator, as in (2.7)

$$\|I - \beta(P_hF'(x_0)^*F'(x_0)P_h + \alpha I)\| \leq 1 - \beta\alpha.$$

Hence,

$$\begin{aligned} \|x_{n+1}^{h,\delta} - x_\alpha^{h,\delta}\| &\leq (1 - \alpha\beta)\|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \\ &\quad + \beta M^2k_0 \int_0^1 \left((1-t)\|x_\alpha^{h,\delta} - x_0\| + t\|x_{n,\alpha}^{h,\delta} - x_0\|dt\right) \|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\| \\ &\leq \left(1 - \alpha\beta + \beta\frac{3k_0M^2r}{2}\right) \|x_{n,\alpha}^{h} - x_\alpha^{h,\delta}\| \\ &\leq \left(1 - \alpha\beta + \beta\frac{3k_0M^2r}{2}\right) \|x_{n,\alpha}^{h,\delta} - x_\alpha^{h,\delta}\|. \end{aligned}$$

The rest of the proof is analogous to the proof of Theorem 2.4. □

Remark 3.3. Instead of Assumption 2.1, if we use (2.9) then Theorem 3.1 holds with $0 < r < \frac{\sqrt{\alpha}}{L_0}$ and Theorem 3.2 holds with $\bar{q}_{\alpha,\beta} := 1 - \alpha\beta + \beta ML_0r$ instead of $q_{\alpha,\beta}$, provided $0 < r < \frac{\alpha}{ML_0}$.

4. Error bounds under source conditions

Note that by (2.1), we have

$$P_hF'(x_0)^*(F(x_\alpha^\delta) - y^\delta) + \alpha P_h(x_\alpha^\delta - x_0) = 0. \tag{4.1}$$

So, by (3.2) and (4.1), we obtain

$$P_hF'(x_0)^*(F(x_\alpha^{h,\delta})-F(x_\alpha^\delta))+\alpha P_h(x_\alpha^{h,\delta}-x_\alpha^\delta)=0.$$

That is,

$$\begin{aligned}(P_hF'(x_0)^*F'(x_0)P_h+\alpha I)(x_\alpha^{h,\delta}-P_hx_\alpha^\delta) &= P_hF'(x_0)^*(F'(x_0)-T)(x_\alpha^{h,\delta}-x_\alpha^\delta)\\ &\quad +P_hF'(x_0)^*F'(x_0)(I-P_h)x_\alpha^\delta,\end{aligned}$$

where $T=\int_0^1 F'(x_\alpha^\delta+t(x_\alpha^{h,\delta}-x_\alpha^\delta))dt$. So,

$$\begin{aligned}&\|x_\alpha^{h,\delta}-P_hx_\alpha^\delta\|\\ &= \big\|(P_hF'(x_0)^*F'(x_0)P_h+\alpha P_h)^{-1}[P_hF'(x_0)^*(F'(x_0)-T)(x_\alpha^{h,\delta}-x_\alpha^\delta)\\ &\quad +P_hF'(x_0)^*F'(x_0)(I-P_h)x_\alpha^\delta]\big\|\\ &\le \left\|(P_hF'(x_0)^*F'(x_0)P_h+\alpha P_h)^{-1}\left[P_hF'(x_0)^*\right.\right.\\ &\quad \left.\left.\times\int_0^1[F'(x_\alpha^\delta+t(x_\alpha^{h,\delta}-x_\alpha^\delta))-F'(x_0)]dt(x_\alpha^{h,\delta}-x_\alpha^\delta)\right]\right\|+\frac{\varepsilon_h}{\sqrt{\alpha}}\|x_\alpha^\delta\|\\ &\le \left\|(P_hF'(x_0)^*F'(x_0)P_h+\alpha P_h)^{-1}\left[P_hF'(x_0)^*F'(x_0)[P_h+I-P_h]\right.\right.\\ &\quad \left.\left.\times\int_0^1\varphi(x_\alpha^\delta+t(x_\alpha^{h,\delta}-x_\alpha^\delta)),x_0,x_\alpha^{h,\delta}-x_\alpha^\delta\right)\right\|dt+\frac{\varepsilon_h}{\sqrt{\alpha}}\|x_\alpha^\delta\|\\ &\le k_0\left(1+\frac{\varepsilon_h}{\sqrt{\alpha}}\right)\int_0^1[(1-t)\|x_\alpha^\delta-x_0\|+t\|x_\alpha^{h,\delta}-x_0\|]dt\|x_\alpha^{h,\delta}-x_\alpha^\delta\|\\ &\quad +\frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_\alpha^\delta-x_0\|+\|x_0\|)\\ &\le 2k_0r\|x_\alpha^{h,\delta}-x_\alpha^\delta\|+\frac{\varepsilon_h}{\sqrt{\alpha}}(r+\|x_0\|)\\ &\le 2k_0r[\|x_\alpha^{h,\delta}-P_hx_\alpha^\delta\|+\|(I-P_h)x_\alpha^\delta\|]+\frac{\varepsilon_h}{\sqrt{\alpha}}(r+\|x_0\|),\end{aligned}$$

hence

$$\|x_\alpha^{h,\delta}-P_hx_\alpha^\delta\|\le\frac{1}{1-2k_0r}\left[2k_0r\|(I-P_h)x_\alpha^\delta\|+\frac{\varepsilon_h}{\sqrt{\alpha}}(r+\|x_0\|)\right]. \tag{4.2}$$

Further, we observe that

$$\|P_hx_0-x_\alpha^{h,\delta}\|\le\|P_h(x_0-x_\alpha^{h,\delta})\|\le r. \tag{4.3}$$

Combining the estimates in (2.2), (4.2), (4.3) and Theorem 3.2 we obtain the following:

Theorem 4.1. *Let the assumptions in Theorem 3.2 hold and let $x_{n,\alpha}^{h,\delta}$ be as in (3.2). Then*

$$\begin{aligned}\|x_{n,\alpha}^{h,\delta}-\hat{x}\| &\le q_{\alpha,\beta}^n r+\frac{1}{1-2k_0r}\left[b_h+\frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right]\\ &\quad +\frac{1}{1-2k_0r}2\left(\frac{\delta}{\sqrt{\alpha}}+\varphi(\alpha)\right).\end{aligned} \tag{4.4}$$

Further if $n_\delta:=\min\{n: q_{\alpha,\beta}^n<\frac{\delta+\varepsilon_h}{\sqrt{\alpha}}\}$ *and* $b_h\le\frac{\delta+\varepsilon_h}{\sqrt{\alpha}}$ *then*

$$\|x_{n_\delta,\alpha}^h-\hat{x}\|\le C\left(\frac{\delta+\varepsilon_h}{\sqrt{\alpha}}+\varphi(\alpha)\right) \tag{4.5}$$

where $C:=r+\frac{1}{1-2k_0r}[1+\max\{r+\|x_0\|,2\}]$.

Proof. By triangle inequality, we have $\|x_{n,\alpha}^{h,\delta}-\hat{x}\|\le\|x_{n,\alpha}^{h,\delta}-x_\alpha^{h,\delta}\|+\|x_\alpha^{h,\delta}-x_\alpha^\delta\|+\|x_\alpha^\delta-\hat{x}\|$. Therefore, from (4.2), (4.3), Theorem 3.2 and (2.2), we obtain that

$$\begin{aligned}\|x_{n,\alpha}^{h,\delta}-\hat{x}\| &\le q_{\alpha,\beta}^n r+\|x_\alpha^{h,\delta}-P_hx_\alpha^\delta\|+\|(I-P_h)x_\alpha^\delta\|+\|x_\alpha^\delta-\hat{x}\|\\ &\le q_{\alpha,\beta}^n r+\frac{1}{1-2k_0r}\left[2k_0r\|(I-P_h)x_\alpha^\delta\|+\frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right]\end{aligned}$$

$$
\begin{aligned}
&+ \|(I-P_h)x_\alpha^\delta\| + \frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right)\\
\le\ & q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[\|(I-P_h)x_\alpha^\delta\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right]\\
&+ \frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right)\\
\le\ & q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[\|(I-P_h)(x_\alpha^\delta - \hat{x} + \hat{x})\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right]\\
&+ \frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right)\\
\le\ & q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[\|x_\alpha^\delta - \hat{x}\| + \|(I-P_h)\hat{x}\| + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right]\\
&+ \frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right)\\
\le\ & q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[b_h + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right]\\
&+ \left(1+\frac{1}{1-2k_0 r}\right)\frac{1}{1-k_0 r}\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right)\\
\le\ & q_{\alpha,\beta}^n r + \frac{1}{1-2k_0 r}\left[b_h + \frac{\varepsilon_h}{\sqrt{\alpha}}(\|x_0\|+r)\right]\\
&+ \frac{1}{1-2k_0 r}2\left(\frac{\delta}{\sqrt{\alpha}} + \varphi(\alpha)\right).
\end{aligned}
$$

This proves (4.4), and (4.5) follows from (4.4). □

4.1. *A priori choice of the parameter*

Note that the estimate $\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} + \varphi(\alpha)$ in Theorem 4.1 is of optimal order for the choice $\alpha := \alpha_{\delta,h}$ which satisfies $\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} = \varphi(\alpha)$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \le a$. Then $\alpha_{\delta,h} = \varphi^{-1}[\psi^{-1}(\delta+\varepsilon_h)]$ satisfies $\frac{\delta+\varepsilon_h}{\sqrt{\alpha}} = \varphi(\alpha)$.

In view of the above observation, Theorem 4.1 leads to the following:

Theorem 4.2. *Let $\psi(\lambda) = \lambda\sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \le a$ and assumptions in Theorem 4.1 hold. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}[\psi^{-1}(\delta+\varepsilon_h)]$ and let n_δ be as in Theorem 4.1. Then*

$$\|x_{n_\delta,\alpha_\delta}^{h,\delta} - \hat{x}\| = O(\psi^{-1}(\delta+\varepsilon_h)).$$

4.2. *Balancing principle*

Note that the best function φ measuring the rate of convergence in Theorem 4.1 is usually unknown. Therefore, in practical applications different parameters $\alpha = \alpha_i$ are often selected from some finite set

$$D := \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N < 1\},$$

and corresponding elements $x_{n,\alpha_i}^{h,\delta}$, $i = 1, 2, \ldots, N$ are studied on line. Let

$$n_i := \min\left\{n : q_{\alpha,\beta}^n \le \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}\right\}$$

and let $x_{\alpha_i}^h := x_{n_i,\alpha_i}^{h,\delta}$. Then from Theorem 4.1, we have

$$\|x_{\alpha_i}^h - \hat{x}\| \le C\left(\frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}} + \varphi(\alpha_i)\right), i = 1, 2, \ldots, N.$$

We choose the regularization parameter α from the set D_N defined by

$$D_N := \{\alpha_i = \mu^{2i}\alpha_0 < 1, i = 1, 2, \ldots, N\},$$

where $\alpha_0 = (\delta + \varepsilon_h)^2$ (see [24,25]) and $\mu > 1$. Using the ideas in [24], we consider all possible functions φ satisfying Assumption 2.1 and $\varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}$. Any of such functions is called admissible for $\hat{x}$ and it can be used as a measure for the convergence of $x^h_{\alpha_i} \to \hat{x}$ (see [26]).

The main result of this section is the following theorem, proof of which is analogous to the proof of Theorem 4.4 in [10].

Theorem 4.3. *Assume that there exists $i \in \{0, 1, \ldots, N\}$ such that $\varphi(\delta) \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}$. Let assumptions of Theorem 4.1 be satisfied and let*

$$l := \max\left\{i : \varphi(\alpha_i) \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}\right\} < N,$$

$$k = \max\left\{i : \forall j = 1, 2, \ldots, i;\ \|x^h_{\alpha_i} - x^h_{\alpha_j}\| \leq 4C\frac{\delta+\varepsilon_h}{\sqrt{\alpha_j}}\right\},$$

where C is as in Theorem 4.1. Then $l \leq k$ and

$$\|x^h_{\alpha_k} - \hat{x}\| \leq 6C\mu\psi^{-1}(\delta+\varepsilon_h).$$

Remark 4.4. The balancing algorithm associated with the choice of the parameter specified in Theorem 4.1 involves the following steps:

- Choose $\alpha_0 = (\delta+\varepsilon_h)^2$ and $\mu > 1$.
- Choose $\alpha_i := \mu^{2i}\alpha_0$, $i = 0, 1, 2, \ldots, N$.

1. Set $i = 0$.
2. Choose $n_i := min\left\{n : q^n_{\alpha_i,\beta} \leq \frac{\delta+\varepsilon_h}{\sqrt{\alpha_i}}\right\}$.
3. Solve $x_i := x^{h,\delta}_{n_i,\alpha_i}$ by using the iteration (3.1).
4. If $\|x_i - x_j\| > 4C\frac{1}{\mu^j}$, $j < i$, then take $k = i - 1$ and return x_k.
5. Else, set $i = i + 1$ and go to 2.

5. Numerical examples

Let $V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots$ be a sequence of finite-dimensional subspaces of X with $\overline{\bigcup_{m\in\mathbb{N}} V_m} = X$ and P_h, $(h = \frac{1}{m})$ is the orthogonal projector of X onto $R(P_h) := V_m \subset D(F)$. Precisely, we choose the orthonormal system of box function $\Phi_i(t,\tau) = \Psi_k(t)\Psi_l(\tau)$, $i = (k-1)m + l$, $k = 1, 2, 3, \ldots, m_1$, $l = 1, 2, 3, \ldots, m_1$, $i = 1, 2, \ldots, m (= m_1^2)$. where $\Psi_k(t)$, $\Psi_l(\tau)$ are L_2-orthonormalized characteristic functions of the intervals $[k-1, k]$, $[l-1, l]$ [26], respectively, as a basis of V_m in $\Omega = [0, m_1] \times [0, m_1]$.

We consider the following integral equation (inverse gravimetry problem (see [27] and references in it)) for the implementation of the method (3.1).

Let $F : H^1(\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$ defined by

$$F(u) \equiv -\iint_\Omega \frac{1}{[(x-x')^2 + (y-y')^2 + u^2(x',y')]^{1/2}}dx'dy' = f(x,y), \tag{5.1}$$

where $\Omega = [0, m_1] \times [0, m_1]$. The Fréchet derivative of the operator F at the point $u_0(x,y)$ is expressed by the formula

$$F'(u_0)h = \iint_\Omega \frac{u_0(x',y')\,h(x',y')}{[(x-x')^2 + (y-y')^2 + (u_0(x',y'))^2]^{3/2}}dx'dy'. \tag{5.2}$$

Applying to the integral equations (5.1) the two-dimensional analogy of rectangle's formula with uniform grid for every variable, we obtain the following system of nonlinear equations:

$$\sum_{i=1}^{m_1}\sum_{j=1}^{m_1} \frac{1}{[(x_k - x'_j)^2 + (y_l - y'_i)^2 + u^2(x'_j, y'_i)]^{1/2}}\Delta x\Delta y = f(x_k, y_l);$$

$(k = 1, 2, \ldots, m_1, l = 1, 2, \ldots, m_1)$. The discrete variant of the derivative $F'(u_0)$ has the form

$$\{F^0_n h_n\}_{k,l} = \sum_{i=1}^{m_1}\sum_{j=1}^{m_1} \frac{\Delta x\Delta y\, u_0(x'_j, y'_i)h(x'_j, y'_i)}{[(x_k - x'_j)^2 + (y_l - y'_i)^2 + u_0^2(x'_j, y'_i)]^{3/2}}, \tag{5.3}$$

where $u_0(x,y) \equiv H$ is constant.

Table 1
Comparison table for relative and residual error for method (1.3) and method in [17].

δ	Relative error and residual error for the method (1.3)			Relative error and residual error for the method in [17]		
	α_k	Δ_1	Δ_2	α_k	Δ_1	Δ_2
0.01	8.3521e−5	1.8576e−4	3.8057e−5	5.0625e−4	8.1214e−4	7.9872e−4
0.001	8.3521e−7	1.8577e−4	3.8123e−5	5.0625e−6	8.1214e−4	7.9871e−4
0.0002	7.3324e−8	1.8578e−4	3.8129e−5	2.0250e−7	8.1214e−4	7.9871e−4
0.0001	8.3521e−9	1.8579e−4	3.8130e−5	5.0625e−8	8.1214e−4	7.9871e−4

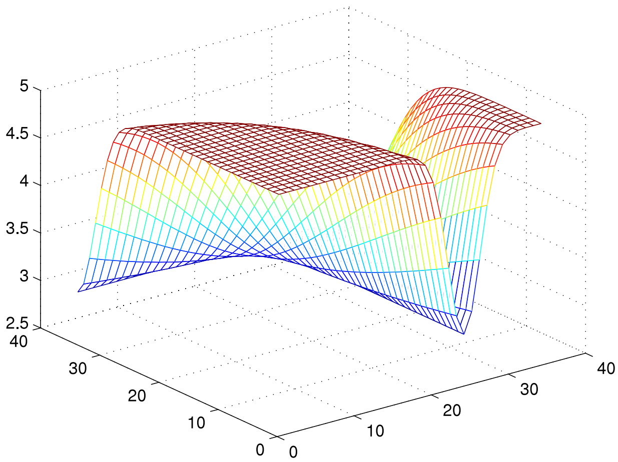
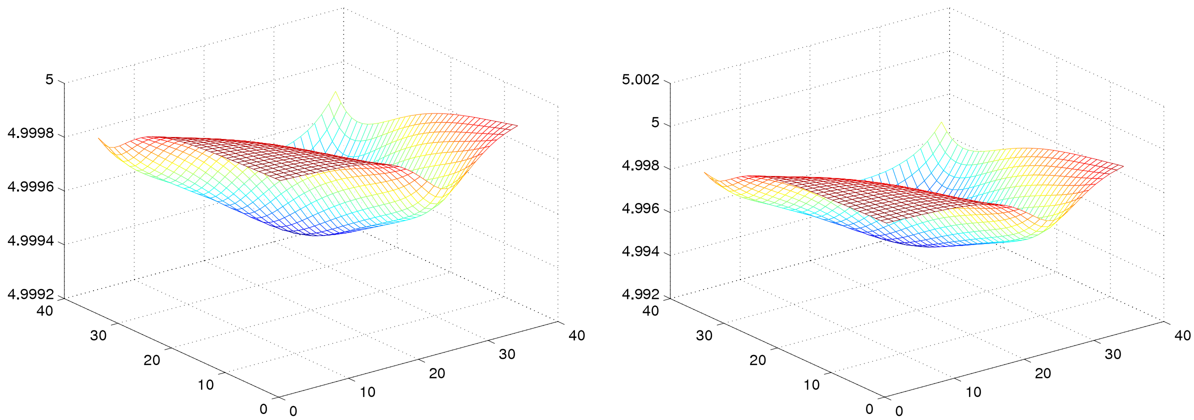


Fig. 1. Exact solution.



(a) Approximate solution for $\delta = 0.01$.

(b) Approximate solution for $\delta = 0.0001$.

Fig. 2. Approximate solution.

We take the exact solution as

$$\hat{u}(x, y) = 5 - 2\exp^{(-[(x/10-3.5)^2(y/10-2.5)^2])} - 3\exp^{(-[(x/10-5.5)^2(y/10-4.5)^2])},$$

and $f^\delta = F(\hat{u}) + \delta$. Let $\Delta x = \Delta y = 1,\ \ m_1 = 35,\ \ H \equiv 5$.

Note that on the set

$$Q = \{1.0 \leq u(x, y) \leq 10.0\}$$

$\|F'(u) - F'(u_0)\| \leq L_0\|u - u_0\|$ (see [27,28]).

The results of numerical experiments are presented in Table 1. Here $\tilde{u}_n$ is the numerical solution obtained by our method; the relative error of solution and residual are

$$\Delta_1 = \frac{\|\hat{u} - \tilde{u}_n\|}{\|\hat{u}_n\|}, \qquad \Delta_2 = \frac{\|F_n(\tilde{u}_n) - f_n\|}{\|f_n\|}$$

respectively, for a noisy right-hand side.

Comparison Table 1 shows that the relative and residual error for method (1.3) is smaller than that of the method in [17] for a given data error.

Fig. 1, gives the exact solution, Fig. 2(a) gives the approximate solution for $\delta = 0.01$ and Fig. 2(b) gives the approximate solution for $\delta = 0.0001$.

6. Conclusion

In this study, we considered a discretized frozen steepest descent method for numerical approximation of a Tikhonov type regularizer. The balancing principle considered by Pereverzev and Schock in [24] was used for choosing the regularization parameter. We provide a numerical example to verify the theoretical results.

References

[1] H.W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, in: Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht, 2000.
[2] F. Hettlich, W. Rundell, A second degree method for nonlinear inverse problems, SIAM J. Numer. Anal. 37 (2000) 587–620.
[3] V. Isakov, Inverse Problems for Partial Differential Equations, second ed. in: Applied Mathematical Sciences, vol. 127, Springer, New York, 2006.
[4] M. Hanke, A. Neubauer, O. Scherzer, A convergence analysis of the Landweber iteration for nonlinear ill-posed problems, Numer. Math. 72 (1995) 21–37.
[5] R. Ramlau, A modified Landweber-method for inverse problems, Numer. Funct. Anal. Optim. 20 (1999) 79–98.
[6] M. Hanke, A regularizing Levenberg–Marquardt scheme, with application to inverse ground-water filtration problems, Inverse Problems 13 (1997) 79–95.
[7] A.W. Bakushinskii, The problem of the convergence of the iteratively regularized Gauss–Newton method, Comput. Math. Phys. 32 (1992) 1353–1359.
[8] B. Blaschke, A. Neubauer, O. Scherzer, On convergence rates for the iteratively regularized Gauss–Newton method, IMA J. Numer. Anal. 17 (1997) 421–436.
[9] M. Hanke, Regularizing properties of a truncated Newton-cg algorithm for nonlinear ill-posed problems, Numer. Funct. Anal. Optim. 18 (1997) 971–993.
[10] S. George, On convergence of regularized modified Newton's method for nonlinear ill-posed problems, J. Inverse Ill-Posed Probl. 18 (2010) 133–146.
[11] B. Kaltenbacher, Some Newton-type methods for the regularization of nonlinear ill-posed problems, Inverse Problems 13 (1997) 729–753.
[12] R. Ramlau, TIGRA — an iterative algorithm for regularizing nonlinear ill-posed problems, Inverse Problems 19 (2003) 433–465.
[13] O. Scherzer, A convergence analysis of a method of steepest descent and two-step algorithm for nonlinear ill-posed problems, Numer. Funct. Anal. Optim. 17 (1996) 197–214.
[14] A. Neubauer, O. Scherzer, A convergence rate result for a steepest descent method and a minimal error method for the solution of nonlinear ill-posed problems, J. Math. Anal. Appl. 14 (1995) 369–377.
[15] S.F. Gilyazov, Iterative solution methods for inconsistent linear equations with nonself-adjoint operators, Moscow Univ. Comput. Math. Cybernet. 1 (1997) 8–13.
[16] V. Vasin, Irregular nonlinear operator equations: tikhonov's regularization and iterative approximation, J. Inverse Ill-Posed Probl. 21 (2013) 109–123.
[17] I.K. Argyros, S. George, P. Jidesh, Inverse free iterative methods for nonlinear ill-posed operator equations, Int. J. Math. Math. Sci. , 2014, Article ID 754154, 8 pages, http://dx.doi.org/10.1155/2014/754154.
[18] I.K. Argyros, Convergence and Applications of Newton-Type Iterations, Springer, New York, 2008.
[19] Q. Jin, On a class of frozen regularized Gauss–Newton methods for nonlinear inverse problems, Math. Comp. 79 (2010) 2191–2211.
[20] C.W. Groetsch, A. Neubauer, Convergence of a general projection method for an operator equation of the first kind, Houston J. Math. 14 (1988) 201–208.
[21] B. Kaltenbacher, A. Neubauer, O. Scherzer, Iterative Regularization Methods for Nonlinear Ill-Posed Problems, De Gruyter, Berlin, 2008.
[22] A. Krisch, An Introduction To the Mathematical Theory of Inverse Problems, Springer, New York, 1996.
[23] S.V. Perverzev, S. Probdorf, On the characterization of self-regularization properties of a fully discrete projection method for symms integral equation, J. Integral Equations Appl. 12 (2000) 113–130.
[24] S. Pereverzev, E. Schock, On the adaptive selection of the parameter in regularization of ill-posed problems, SIAM J. Numer. Anal. 43 (2005) 2060–2076.
[25] E.V. Semenova, Lavrentiev regularization and balancing principle for solving ill-posed problems with monotone operators, Comput. Methods Appl. Math. 4 (2010) 444–454.
[26] S. Lu, S.V. Pereverzev, Sparsity reconstruction by the standard Tikhonov method, RICAM-Report No. 2008-17, 2008.
[27] V.V. Vasin, I.I. Prutkin, L.Yu. Timerkhanova, Retrieval of a three-dimensional relief of geological boundary from gravity data, Izv. Phys. Solid Earth 32 (1996) 58–62.
[28] V.V. Vasin, Modified processes of Newton type generating Fejer approximations of regularized solutions of nonlinear equations, in: Proceedings in Mathematics and Mechanics, URORAN, Tom 18, 3, 2012, (in Russian).