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Pole Assignment for Multi-input Multi-output Systems Using Output Feedback*

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Key Words—Pole placement; output feedback; constant gain controller; multi-input multi-output system; shifting of unassigned poles.

Abstract—In this paper the problem of pole assignment using constant gain output feedback is studied for MIMO system with system order $n > m + l - 1$, where m and l are the number of inputs and outputs, respectively. A new procedure is presented to design a constant gain output feedback matrix which assigns $(m + l - 2)$ poles exactly to the desired locations and shifts all the unassigned poles to suitable locations using root locus techniques.

1. *Introduction*

IT IS well known (Munro and Novin Hirbod, 1979) that for linear multivariable systems with m-inputs and l-outputs, a maximum of $(m + l - 1)$ poles can be arbitrarily assigned by constant gain output feedback. When $(m + l - 1)$ is less than the order of the system n, the remaining $(n - m - l + 1)$ unassigned poles may move to undesirable locations. In order to ensure satisfactory performance of the closed-loop system, the unassigned poles should be shifted to the stable region.

Recently Paplinski and Gibbard (1985) and Rajagopalan and Appukuttan (1988) have proposed methods for finding the most suitable locations of unassigned poles for single-input multi-output (SIMO) systems. Subsequently Chen *et al.* (1988) reported a method of finding the most suitable locations of the unassigned poles for MIMO systems. In this method, as one of the vectors in the dyadic feedback matrix is prespecified, the approach is essentially similar to the one reported for SIMO system by Rajagopalan and Appukuttan (1988).

In this paper, a new procedure for pole assignment using constant gain output feedback applicable to MIMO systems is presented. Here, $(m + l - 2)$ poles are arbitrarily assigned and the remaining design freedom is utilized to shift the unassigned poles to the best possible locations using the root locus method. Also, the procedure to obtain the polynomial with the set of unassigned poles in one degree of freedom is different from the procedure reported in other papers.

2. *Design procedure*

Consider a controllable and observable system described by

$$\dot{x} = Ax + Bu; \quad y = Cx \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector; $u \in \mathbb{R}^m$ is the input and $y \in \mathbb{R}^l$ is the output. Without loss of generality it is assumed that $l \geq m$. The objective is to design a feedback control law $u = -Ky$ so that $(m + l - 2)$ poles are assigned arbitrarily and the remaining unassigned poles are shifted to suitable locations.

The procedure consists of finding the feedback control law $u = -(K_1 + K_2)y$ in two stages. In the first stage, the feedback matrix K_1 is determined to assign $(m - 1)$ poles, namely, $\beta_1, \ldots, \beta_{m-1}$ using unity rank design (Seraji, 1979) or full rank design (Munro and Hirbod, 1979). Let the closed-loop system matrix at the end of stage one be

$$A_1 = A - BK_1C, \quad K_1 \in \mathbb{R}^{m \times l}. \qquad (2)$$

In the second stage, the feedback matrix K_2 is written as

$$K_2 = k_2 f_2^{\mathrm{T}}, \quad K_2 \in \mathbb{R}^{m \times l}, \quad k_2 \in \mathbb{R}^m, \quad f_2 \in \mathbb{R}^l. \qquad (3)$$

The closed-loop characteristic polynomial of the system with K_2 is given by

$$H_2(s) = \det(sI - A_1 + Bk_2 f_2^{\mathrm{T}} C) = H_1(s) + f_2^{\mathrm{T}} W_1(s) k_2 \qquad (4)$$

where $H_1(s) = \det(sI - A_1)$ and $W_1(s) = C \operatorname{adj}(sI - A_1)B$. The vector k_2 is determined to preserve the $(m - 1)$ poles assigned in the first stage (Munro and Hirbod, 1979). Then, the closed-loop characteristic polynomial (4) can be written as

$$H_2(s) = [f_2^{\mathrm{T}} \quad 1] N_p s_{n+1}^* \qquad (5)$$

where $N_p = \begin{bmatrix} 0 & N_{1k} \\ 1 & h_1^{\mathrm{T}} \end{bmatrix}$ is the $(l+1) \times (n+1)$ coefficient matrix, $N_{1k} \in \mathbb{R}^{l \times n}$ is the coefficient matrix of $W_1(s)k_2$, $H_1(s) = s^n + h_1^{\mathrm{T}} s_n^*$ and $s_n^* = [s^{n-1}, \ldots, s \quad 1]^{\mathrm{T}}$.

The vector f_2^{T} is computed to assign $(l - 1)$ additional poles and to shift the unassigned poles to the stable locations.

Let $n_1^{\mathrm{T}}, n_2^{\mathrm{T}}, \ldots, n_{l+1}^{\mathrm{T}}$ be the $1 \times (n + 1)$ row vectors of N_p. Let β_m be one of the additional poles to be assigned in the second stage. Substituting $s = \beta_m$ in (5) we get

$$[f_2^{\mathrm{T}} \quad 1]\delta = [f_{21}, \ldots, f_{2l} \quad 1]\delta = 0 \qquad (6)$$

where $\delta = [\delta_1, \ldots, \delta_{l+1}]^T$ and δ_is are the constants obtained as the inner product of the row vector n_i^{T} and the vector $(\beta_m)_{n+1}^*$. Using (6), any one parameter of f_2^{T} (say, f_{21}) can be expressed in terms of the other parameters. The first parameter f_{21} can be written as

$$f_{21} = -[f_{22}, \ldots, f_{2l} \quad 1]\alpha \qquad (7)$$

where

$$\alpha = [\alpha_2, \ldots, \alpha_{l+1}]^{\mathrm{T}} \quad \text{and} \quad \alpha_i = \delta_i / \delta_1, \; i = 2, \ldots, l+1.$$

Substituting (7) in (5)

$$H_2(s) = [f_{22}, \ldots, f_{2l} \quad 1] N_{p1} s_{n+1}^* \qquad (8)$$

where the row reduced coefficient matrix

$$N_{p1} = [n_2, \ldots, n_{l+1}]^{\mathrm{T}} - \alpha n_1^{\mathrm{T}}, \quad N_{p1} \in \mathbb{R}^{l \times (n+1)}. \qquad (9)$$

Thus the closed-loop characteristic polynomial $H_2(s)$ has only $(l - 1)$ free design parameters. This procedure is repeated to assign additional $(l - 2)$ poles. At the end of the computation, the closed-loop characteristic polynomial can be expressed as

$$H_2(s) = (s - \beta_1) \cdots (s - \beta_{m+l-2}) H_{2,(m+l-2)}(s) \qquad (10)$$

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where

$$H_{2,(m+l-2)}(s) = [f_{2l} \quad 1]\begin{bmatrix} 0 & t^{\mathrm{T}} \\ 1 & g^{\mathrm{T}} \end{bmatrix} s^*_{n'+1}$$

$n' = n - m - l + 2$ and $t \in \mathbb{R}^{n'}$ and $g \in \mathbb{R}^{n'}$.

The variation of the unassigned poles as a function of f_{2l} can be obtained by finding the roots of the equation $H_{2,(m+l-2)}(s) = 0$. This equation is rearranged to plot the root locus as

$$1 + \frac{f_{2l} t^{\mathrm{T}} s^*_n}{s^n + g^{\mathrm{T}} s^*_n} = 0. \quad (11)$$

From the root locus plot of (11), the value of f_{2l} corresponding to the best possible location of the unassigned poles can be obtained. Once f_{2l} is known, other constants f_{2i} can be computed from the relation similar to (7) obtained in each step of the additional pole placement procedure. The final feedback matrix is $K = K_1 + K_2$.

3. *Numerical example*

Consider the MIMO system described by (1). Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Here, $n = 4$, $m = 2$ and $l = 2$. The design procedure explained in the paper is used to assign two poles at -2 and -3 and shift the other two poles to the best possible locations. In the first stage, the pole, -2, is assigned using $K_1 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$. In the second stage, to preserve the assigned pole at $s = -2$, $k_2 = [1 \quad -4]^{\mathrm{T}}$.

Assigning the additional pole at $s = -3$, (7) and (10) become

$$f_{2l} = -1.9 f_{22} + 0.9$$

and

$$H_2(s) = (s+2)(s+3)[f_{22} \quad 1]\begin{bmatrix} 0 & -0.9 & -1.4 \\ 1 & -0.1 & -0.6 \end{bmatrix} s^*_3.$$

Therefore, the equation for plotting the root locus as function of f_{22} is written as

$$1 + \frac{(-0.9) f_{22}(s + 1.556)}{(s - 0.825)(s + 0.725)} = 0.$$

The root locus plot is shown in Fig. 1. From the plot the best location for both the unassigned poles is -2.96 which occurs for $f_{22} = -6.69$. Therefore the feedback matrix

$$K = \begin{bmatrix} 17.61 & -6.69 \\ 54.4 & 26.76 \end{bmatrix}$$

gives the closed-loop poles -2, -3, -2.96 and -2.96.

On the other hand, if the method given by Chen *et al.* (1988) is applied to this example, only one pole can be assigned exactly. After assigning the pole at $s = -2$, from the root locus plot it can be noted that the remaining three unassigned poles cannot be shifted using their method to the left of the line $\sigma = -0.5$ in the s-plane. The corresponding pole locations may be unacceptable in most cases. Therefore for this design example the present method gives much better results than the method given by Chen *et al.* (1988).

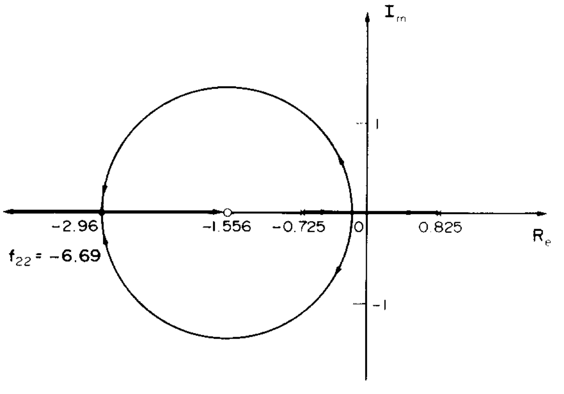

FIG. 1. Root locus plot.

4. *Conclusion*

A new procedure for pole placement using constant gain output feedback applicable to MIMO systems has been presented. Using this procedure, $(m + l - 2)$ poles are assigned exactly and a reduced-order polynomial corresponding to the unassigned poles as a function of the free parameter of the feedback gain is obtained. From the root locus plot the free parameter is selected corresponding to the best possible location of the unassigned poles and then the feedback gain matrix is computed. The method is simple and can be easily programmed on a digital computer. Also, for a general MIMO system, as the number of unassigned poles in the present method is always less than that in the method proposed by Chen *et al.* (1988) the present method may always result in better locations for the unassigned poles.

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