Strong (weak) edge-edge domination number of a graph

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Strong (Weak) Edge-Edge Domination Number of a Graph

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Abstract

For any edge $\boldsymbol{x = uv}$ of an isolate free graph $\boldsymbol{G(V,E)}, \langle \boldsymbol{N[x]} \rangle$ is the subgraph induced by the vertices adjacent to u and v in G. We say that an edge x, *e*-dominates an edge y if $\boldsymbol{y} \in \langle \boldsymbol{N[x]} \rangle$. A set $\boldsymbol{L} \subseteq \boldsymbol{E}$ is an Edge-Edge Dominating Set (EED-set) if every edge in $\boldsymbol{E - L}$ is e-dominated by an edge in *L*. The *edge-edge domination number* $\boldsymbol{\gamma_{ee}(G)}$ is the cardinality of a minimum EED-set. We find the relation ship between the new parameter and some known graph parameters.

Keywords: Edge-Edge Dominating sets (EED sets), Strong Edge-Edge Dominating sets (SEED sets)

1 Introduction

For any undefined terminologies refer [3]. Degree of an edge $\boldsymbol{x = uv}$, deg (x) is the number of edges adjacent to the edge x. Equivalently deg(x) = deg (u) + deg (v) - 2. Unless specified otherwise by a graph we mean a simple undirected isolate free and isolate edge free graph. For any edge $\boldsymbol{x = uv, N(x) = \{y \in E | y \text{ } is \text{ } adjacent \text{ } to \text{ } x\}}$ and $\boldsymbol{N[x] = N(x) \cup \{x\}}$. And $\boldsymbol{\langle N[x] \rangle}$ is the sub graph induced by $\boldsymbol{N(u) \cup N(v)}$.

The strong weak domination was first introduced by Sampathkumar and Pushpalatha [12]. For any two adjacent vertices u and v in a graph $G(V, E)$, u strongly (weakly) dominates v if deg (u) ≥ deg(v) (deg (u) ≤ deg (v)). A set $D \subseteq V$ is a *dominating set* (*strong dominating set* [SD-set], *weak dominating set* [WD-set] respectively) of G if every $v \in V - D$ is dominated (strongly dominated, weakly dominated respectively) by some $u \in D$. The *domination number* $\gamma(G)$ (*strong domination number* $\gamma_s(G)$, *weak domination number* $\gamma_w(G)$ respectively) is the minimum cardinality of a dominating set (SD- set, WD-set respectively) of *G*. Similarly, this concept is extended to coverings , independent sets and matchings by S.S.Kamath and R.S.Bhat [2, 4 and 5]. Sampathkumar and P.S.Neeralagi [10, 11] defined the neighbourhood sets and line neighbourhood sets as follows. A set $S \subseteq V$ is a *neighbourhood set* (*n- set*) if $G = \bigcup_{v \in S} \langle N[v] \rangle$. A set $L \subseteq E$ is a *Line neighbourhood set* (*ln- set*) if $G = \bigcup_{x \in L} \langle N[x] \rangle$. The *neighbourhood number* $n_0 = n_0(G)$ [*line neighbourhood number* $n'_0 = n'_0(G)$] is the cardinality of a minimum n-set [ln-set] of *G*. Mixed domination was introduced in 1985 by R.Laskar and Ken Peters [8] and then in 1992 by, Sampathkumar and S.S.Kamath [9]. An edge *x*, *m- dominates* a vertex v if $v \in N[x]$. A set $L \subseteq E$ is an *Edge Vertex Dominating set* (EVD-set) if every vertex in G is *m-dominated* by an edge in *L*. The *edge vertex domination number* $\gamma_{ev}(G)$ is the minimum cardinality of an EVD-set. Strong (weak) Edge vertex domination studied by R.S.Bhat et.al [1] and Vertex Edge domination is studied by S.S. Kamath and R.S.Bhat [6].

2. Strong /Weak Edge Edge Dominating sets

Let $x, y \in E$, of an isolate free graph $G(V, E)$ then the edge *x*, *e*-dominates an edge y if $y \in \langle N[x] \rangle$. An edge x strongly (weakly) *e*-dominates an edge y if $y \in \langle N[x] \rangle$ and deg (x) ≥ deg (y) (deg (x) ≤ deg(y)).

A set $L \subseteq E$ is an *Edge-Edge Dominating set* (EED-set) if every edge in *E* - *L* is *e*-dominated by an edge in *L*. The *edge-edge domination number* $\gamma_{ee}(G)$ is the minimum cardinality of an EED-set. A set $L \subseteq E$ is a *Strong Edge-Edge Dominating set* (SEED-set) [*Weak Edge-Edge Dominating set* (WEED-set)] if every edge in $E - L$ is strongly (weakly) *e*-dominated by an edge in *L*. The *strong* (*weak*) *edge-edge domination number* $\gamma_{see}(G)$ $(\gamma_{wee}(G))$ is the minimum cardinality of a SEED-set (WEED-set).

We observe that the definition of EED set is a restatement of the definition of line neighbourhood set and hence we have $\gamma_{ee} = n'_0$. In [9] it is proved that $\gamma_{ev} \le \gamma_{ee}$. Therefore we have $\gamma_{ev} \le \gamma_{ee} = n'_0$. Since every SEED set and WEED set is an EED set we have $\gamma_{ee} \le \gamma_{see}$ and $\gamma_{ee} \le \gamma_{wee}$.

A set $L \subset E$ is said to be *Full Edge-Edge dominating* set (FEED set) if every edge in L is e-dominated by an edge in $E - L$. A set $L \subset E$ is said to be a *Full Strong Edge-Edge Dominating set* (FSEED-set) [*Full Weak Edge-Edge Dominating set* (FWEED-set)] if every edge in L, is weakly (strongly) e-dominated by an edge in $E - L$. The *FEED number* $f_{ee}(G)$ (*FSEED number* $f_{see}(G)$, *FWEED number* $f_{wee}(G)$ respectively) is the maximum cardinality of a FEED set (FSEED set, FWEED set respectively).

Example 1. Here, $\gamma_{ee}(G_1) = \gamma_{see}(G_1) = \gamma_{wee}(G_1) = 3$. The dotted edges in Fig.1a, Fig.1b represent the both γ_{ee}- set as well as γ_{see} -set and the dotted edges in Fig.1c is a γ_{wee} -set. More over $f_{ee}(G_1) = f_{wee}(G_1) = f_{see}(G_1) = 6$. The dark edges in each figure form f_{ee} -set, f_{wee} -set and f_{see} -set respectively. We also observe that $\gamma_{ev}(G_1) = 2 < 3 = \gamma_{ee}(G_1)$.

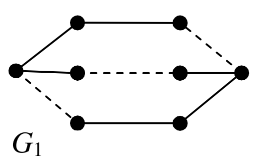

Fig 1.a

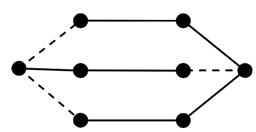

Fig 1.b

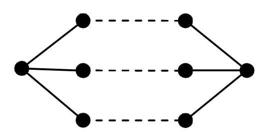

Fig 1.c

For any u, v in G the distance between u and $v, d(u, v)$ is the length of a shortest path between u and v. Let $, y \in E, a \in V$, and $x = uv$ then the distance between the vertex a and edge x, is defined as $d(x, a) = \min\{d(u, a), d(v, a)\}$ and the distance between the two edges $d(x, y) = \min\{d(y, u), d(y, v)\}$. This concept of distance between a vertex and an edge plays an important role in EED sets.

Remark 1. If *L* is a minimal EED set of an isolate edge free graph *G,* then *E* - *L* is also an EED set of *G.*

3. Main Results

Our first result gives a necessary and sufficient condition for a set $L \subseteq E$ to be an EED set of *G* in terms of distance between a vertex and an edge.

Proposition 1. *Let $G(V, E)$ be any graph without isolated edges. A set $L \subseteq E$ is an*

EED set of G if, and only if, for every edge $y = uv$ in $E - L$, there exists an edge $x \in L$ such that $d(x,u) \leq 1$ and $d(x,v) \leq 1$.

Proof. (⇒) Let *L* be an EED set of *G*. Then since *G* is a graph without isolated edges, every edge $y = uv$ in $E - L$ is *e*-dominated by some edge $x \in L$. Hence both $u, v \in N[x]$. This implies that both $d(x,u) \leq 1$ and $d(x,v) \leq 1$ hold as desired.

(⇐) Let $L \subseteq E$ and for every edge $y = uv$ in $E - L$, there exists an edge $x \in L$ such that $d(x,u) \leq 1$ and $d(x,v) \leq 1$. Suppose *L* is not an EED set of *G*, then there exists at least one edge $z = ab \in E - L$ such that z is not *e*-dominated by any edge in *L*. Then at least one of the conditions $d(x,a) \geq 2$ or $d(x,b) \geq 2$ holds, a contradiction to our assumption.

We shall next give a necessary and sufficient condition for minimality of an EED set.

Theorem 2.
(a) *A set L is a minimal EED set of G if, and only if, for any $x \in L$ one of the following two conditions holds.*
(i) *No edge in L e-dominates the edge x.* (ii)*There exists a $y \in E - L$ such that y is uniquely e -dominated by the edge x.*

(b) *A set L is a minimal SEED set* (*WEED set*) *of G if, and only if, for any $x \in L$ one of the following two conditions holds.*
(i) *No edge in L strongly* (*weakly*) *e-dominates the edge x.* (ii) *There exists an edge $y \in E - L$ which is uniquely strongly* (*weakly*) *e-dominated by the edge x.*

Proof. Assume *L* is a minimal EED set. Then for every $x \in L, L - \{x\}$ is not an EED set. This means that there exists a $y \in E - L$ such that *y* is not *e*-dominated by any edge in $L - \{x\}$. Then either $y = x$ *or* $y \in E - L$.

Case 1. If $y = x$: Then *x* is not *e*-dominated by any edge in *L*. Hence condition (i)holds. **Case 2.** If $y \in E - L$: Then *y* is not *e*-dominated by any edge in $L - \{x\}$ and *y* is *e*-dominated by *L*, together imply that *y* is uniquely *e*-dominated by the edge *x*. Hence condition (ii) holds. Conversely, suppose *L* is an EED set and for any $x \in L$ one of the following two conditions stated in the Proposition holds. We show that *L* is a minimal EED set. Then there exists a $x \in L$, such that $L - \{x\}$ is an EED set. This implies that *x* is *e*-dominated by an edge in *L*. That is *x* does not satisfy condition (i). Also if $L - \{x\}$ is an EED set, then every edge in $E - L$ is *e*-dominated by some edge in $L - \{x\}$. This implies that *x* does not uniquely *e*-dominate any edge in $E - L$. That is *x* does not satisfy condition (ii)-a contradiction to our assumption. Part (b) can be proved with the similar argument, hence we omit the proof.

Proposition 3. *Let $G(V,E)$ be any graph. For any set $L \subset E$,*
(i) *L is an EED set if, and only if, $E - L$ is a FEED set.*
(ii) *L is an SEED (WEED) set if, and only if, $E - L$ is a FWEED (FSEED) set.*

Proof. We prove (i) only. The proof of (ii) is similar. If L is an EED set then $E - L$ is a FEED set follows from Remark 2. Conversely if L is a FEED set then every edge in L is *e*-dominated by some edge in $E - L$. Clearly the edges in $E - L$ are *e*-dominated by them selves. Hence $E - L$ is an EED set.

Proposition 4. *Let $G\ (p,q)$ be any graph. Then*

$$\gamma_{ee} + f_{ee} = q \quad (1)$$

$$\gamma_{see} + f_{wee} = q \quad (2)$$

$$\gamma_{wee} + f_{see} = q \quad (3)$$

Proof. Let L be a minimum EED set of *G*. Then from Proposition 4, we have $E - L$ is a FEED set. Therefore $f_{ee} \geq |E - L| = q - \gamma_{ee}$ ... (i). On the other hand if L is a maximum FEED set, again from Proposition 4, $E - L$ is an EED set of *G*. Hence $\gamma_{ee} \leq |E - L| = q - f_{ee}$ ... (ii). Now (1) follows from (i) and (ii). Similarly the results (2) and (3) follow.

4. EE-degree, SEE-degree and WEE-degree

Several types of new degree are defined in [7]. The *Edge-Edge degree* (EE-degree) of an edge $x \in E$, $d_{ee}(x)$ is the number of edges *e* dominated by *x*. Equivalently $d_{ee}(x)$ is the number of edges in $N\langle[x]\rangle$. *Strong Edge-Edge degree* (SEE-degree) of an edge $x \in E$, $d_{see}(x)$ is the number of edges strongly *e*-dominated by *x*. Similarly WEE-deg (*x*) is $d_{wee}(x)$ defined. With respect to these degrees we get the following new graph parameters. Maximum EE-degree $\Delta_{ee}(x)$, minimum EE-degree $\delta_{ee}(x)$, Maximum SEE-degree $\Delta_{see}(x)$, minimum SEE-degree $\delta_{see}(x)$, Maximum WEE-degree $\Delta_{wee}(x)$, minimum WEE-degree $\delta_{wee}(x)$. An Edge *x* is called *SEE-Silent* (*WEE-Silent*), if $d_{see}(x) = 0$ ($d_{wee}(x) = 0$). A set $L \subset E$ is said to be *SEE-Silent set* (*WEE-Silent set*) if for every edge $x \in L$, $d_{see}(x) = 0$ ($d_{wee}(x) = 0$). The *SEE-Silent* (*WEE-Silent*) *number* $\eta_{see} = \eta_{see}(G)$ ($\eta_{wee} = \eta_{wee}(G)$) is the maximum cardinality of a SEE-silent set of G.

5. Bounds on γ_{ee}, γ_{see} and γ_{wee}

We now get some bounds in terms of $\Delta_{ee}$and Δ_{wee}.

Proposition 5. *For any (p,q) graph G,*

$$\left\lceil \frac{q}{\Delta_{ee}} \right\rceil \leq \gamma_{ee} \leq \gamma_{see} \leq q - \eta_{see} \quad (4)$$

$$\left\lceil \frac{q}{\Delta_{wee}} \right\rceil \leq \gamma_{wee} \leq q - \eta_{wee} \quad (5)$$

Further the above bounds are sharp.

Proof. The lower bound in (4) is proved in [7]. Let $L \subseteq E$ be a η_{see} -set of G. Since every edge in L is a SEE-Silent, no edge in L strongly e-dominate any edge in G. Therefore $E - L$ is a SEED set of G. Hence $\gamma_{see} \leq |E_L| = q - \eta_{see}$. With similar argument we can prove the upper bound in (5). Since an edge in G can weakly e-dominate at most Δ_{wee} edges and it self, we need at least $\frac{q}{\Delta_{wee}}$ edges to weakly e-dominate all the edges. This implies the lower bound in (5). The above bounds are sharp as the upper bound in (4) is attained for P_4 and P_5 and the upper bound in (5) is attained for P_4.

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