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ORIGINAL RESEARCH

Third-order derivative-free methods in Banach spaces for nonlinear ill-posed equations

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Abstract
We develop three third order derivative-free iterative methods to solve the nonlinear ill-posed oprerator equation $F(x) = f$ approximately. The methods involve two steps and are free of derivatives. Convergence analysis shows that these methods converge cubically. The adaptive scheme introduced in Pereverzyev and Schock (SIAM J Numer Anal 43(5):2060–2076, 2005) has been employed to choose regularization parameter. These methods are applied to the inverse gravimetry problem to validate our developed results.

Keywords Iterative methods · Cubic convegence · Nonlinear ill-posed equation · Derivative free method · Lavrentiev regularization method · Adaptive method

Mathematics Subject Classification 41H25 · 65F22 · 65J15 · 65J22 · 47A52

1 Introduction

Let $\mathbb{B}$ be a Banach space and $\mathbb{B}^*$ be its dual space. Let $F : D(F) \subseteq \mathbb{B} \to \mathbb{B}$ be an m-accretive, single valued and Fréchet differentiable nonlinear mapping. In this paper, we develop three third order derivative free methods to solve the nonlinear ill-posed operator equation

$$F(x) = f. \tag{1.1}$$

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Througt out this paper, let $F'(x)$ denotes the Fréchet derivative of F at x, $\| \, . \, \|$ denotes the norm on $\mathbb{B}$ and $\mathbb{B}^*$ and we write $\langle x, j \rangle$ instead of $j(x)$ for $j \in \mathbb{B}^*$ and $x \in \mathbb{B}$.

Definition 1.1 [7,10,13] The operator F is said to be m-accretive in $\mathbb{B}$, if

1. for the dual mapping J on $\mathbb{B}$, $\langle F(x) - F(y), J(x-y) \rangle \geq 0$.
2. $R(F + \lambda I) = \mathbb{B}$, $\lambda \geq 0$ where I and $R(F)$ denote the identity mapping on $\mathbb{B}$ and the range of F, respectively.

In otherwords,

$$F(x) + \alpha(x - x_0) = f^\delta, \quad \| f^\delta - f \| \leq \delta \longrightarrow 0 \tag{1.2}$$

has a solution x_α^δ, which is unique, for $\alpha \geq 0$ and $f^\delta \in \mathbb{B}$ if F is m-accretive [10,13,17].

Throughout this paper, Let x_0 be the initial approximation and $\hat{x}$ be the exact solution of (1.1).

Example 1.2 An example of (1.1) is the parameter identification problem [3]. Consider the boundary value problem

$$\begin{aligned} \Delta x + \xi(x) = q \quad &\in \quad \Omega \\ x = 0 \quad &\in \quad \partial\Omega \end{aligned} \tag{1.3}$$

where $\xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz continuously differentiable monotonically increasing function and Ω is a smooth domain in $\mathbb{R}^3$. Finding the source term q in (1.3) from measurement of x in Ω is nonlinear ill-posed equation. The corresponding forward operator is $F : H^2(\Omega) \longrightarrow H^2(\Omega)$ defined by

$$F(q) = x.$$

Then F is monotone [2].

In the earlier studies such as [1,4,10,13,19], the convergence rate is of optimal order for $\|x_\alpha^\delta - \hat{x}\|$ using the Hölder type assumption

$$x_0 - \hat{x} = F'(\hat{x})v. \tag{1.4}$$

George et al. [2] considered source condition

$$x_0 - \hat{x} = F'(x_0)^\nu v \quad 0 < \nu \leq 1 \tag{1.5}$$

in Banach Space and obtained the optimal error estimate for $\|x_\alpha^\delta - \hat{x}\|$. Observe that (1.4) is depends on $\hat{x}$ which is unknown but (1.5) is depends on the known initial guess x_0.

The remaining part of the paper is ordered in the following way. Preliminaries in Sect. 2. Convergence analysis of the method is considered in Sect. 3. A priori choice, adaptive choice and its implementation are considered in Sect. 4. Lastly, in Sect. 5 the derived techniques are applied to inverse gravimetry problem to validate the results developed.

2 Preliminaries

Let x_α satisfies:

$$F(x) + \alpha(x - x_0) = f. \tag{2.1}$$

and x_α^δ be the solution of (1.2). Then (see [2]),

$$\|x_\alpha^\delta - x_\alpha\| \leq \frac{\delta}{\alpha} \tag{2.2}$$

and

$$\|x_\alpha - \hat{x}\| \leq \|\hat{x} - x_0\|. \tag{2.3}$$

Furthermore [4], if $F : \mathbb{B} \to \mathbb{B}$ is an accretive and Fréchet differentiable on $\mathbb{B}$, then $F'(x) + \alpha I$ is invertible,

$$\|(F'(x) + \alpha I)^{-1}\| \leq \frac{1}{\alpha} \tag{2.4}$$

and

$$\|(F'(x) + \alpha I)^{-1} F'(x)\| \leq 2 \tag{2.5}$$

for real number $\alpha > 0$ and $x \in \mathbb{B}$. From (2.4) we have,

$$\|\alpha(F'(x) + \alpha I)^{-1}\| \leq 1.$$

It is known that (cf [2]), if $F : \mathbb{B} \to \mathbb{B}$ is a monotone and Fréchet differentiable operator, then for $x \in \mathbb{B}$ and $0 < \nu \leq 1$,

$$\|\alpha(F'(x) + \alpha I)^{-1} F'(x)^\nu\| \leq C_0 \alpha^\nu \tag{2.6}$$

for some constant $C_0 > 0$.

The following Assumption is used extensively in our analysis.

Assumption 2.1 (see [1,15,16]) There exists a constant $k_0 \geq 0$ such that for every $x \in B(x_0, r)$ and $v \in \mathbb{B}$ there exists an element $\Phi(x, x_0, v) \in \mathbb{B}$ such that $[F'(x) - F'(x_0)]v = F'(x_0)\Phi(x, x_0, v)$, $\|\Phi(x, x_0, v)\| \leq k_0 \|v\| \|x - x_0\|$.

Let Assumption 2.1 and (1.5) hold. If $3k_0 r < 1$, then

$$\|x_\alpha - \hat{x}\| \leq \frac{C_0}{1 - 3k_0 r} \alpha^\nu \tag{2.7}$$

where v is as in (1.5) [2].

3 Convergence analysis

Let $\mathbb{B}$ be a real Banach algebra and $F : \mathbb{B} \longrightarrow \mathbb{B}$ be a differentiable operator in the sense of Fréchet with bounded $F'(x)$ and $F''(x), \forall x \in D(F)$.

i.e.,

$$\|F'(x)\| \leq \beta_1, \qquad \forall x \in D(F)$$

and

$$\|F''(x)\| \leq \beta_2, \qquad \forall x \in D(F)$$

where β_1 and β_2 are constants.

Iterative methods are used to approximately solve (1.2). George et al. [2], considered an iterative method which is derivative and inverse free and converges qudratically to the solution x_α^δ of (1.2). We study similar methods where Fréchet derivative of the operator F is not involved and the methods converges cubically to the solution x_α^δ of (1.2).

Remark 3.1 Let (x_n) be a sequence in $\mathbb{B}$ which converges to x^*. Then (x_n) is said to have order of convergence $p > 1$, if there are reals b, r such that

$$\|x_n - x^*\| \leq be^{-rp^n}$$

for all $n \in N$ and $b, r > 0$. If the sequence (x_n) is such that $\|x_n - x^*\| \leq bq^n$, $0 < q < 1$, then (x_n) is said to be converges linearly. One can refer to [5], for more discussion of the convergence rate.

For $\alpha > 0$ let

$$R_\alpha(x) := F(x) + \alpha(x - x_0) - f^\delta \tag{3.1}$$

and let

$$R'_\alpha(.)h := F'(.)h + \alpha h. \tag{3.2}$$

Motivated by the third order Modified Derivative-free family of the Chebyshev Method (MDCM) considered in [6], we propose the third order method

$$y_{n,\alpha}^\delta = x_{n,\alpha}^\delta - \frac{R_\alpha(x_{n,\alpha}^\delta)}{H_i(x_{n,\alpha}^\delta)}, \tag{3.3}$$

$$x_{n+1,\alpha}^\delta = y_{n,\alpha}^\delta - \frac{R_\alpha(y_{n,\alpha}^\delta)}{H_i(x_{n,\alpha}^\delta)}, \quad i = 1, 2, 3 \tag{3.4}$$

where $x_{0,\alpha}^{\delta} := x_0$ is the initial guess, for approximately solving (1.2). Here

$$H_1(x_{n,\alpha}^{\delta}) := \frac{R_\alpha(x_{n,\alpha}^{\delta}) - R_\alpha(x_{n,\alpha}^{\delta} - R_\alpha(x_{n,\alpha}^{\delta}))}{R_\alpha(x_{n,\alpha}^{\delta})} \tag{3.5}$$

$$H_2(x_{n,\alpha}^{\delta}) := \frac{R_\alpha(x_{n,\alpha}^{\delta} + R_\alpha(x_{n,\alpha}^{\delta})) - R_\alpha(x_{n,\alpha}^{\delta})}{R_\alpha(x_{n,\alpha}^{\delta})} \tag{3.6}$$

and

$$H_3(x_{n,\alpha}^{\delta}) := \frac{R_\alpha(x_{n,\alpha}^{\delta} + R_\alpha(x_{n,\alpha}^{\delta})) - R_\alpha(x_{n,\alpha}^{\delta} - R_\alpha(x_{n,\alpha}^{\delta}))}{2R_\alpha(x_{n,\alpha}^{\delta})}. \tag{3.7}$$

As in earlier papers [9–14] etc., we choose the parameter $\alpha = \alpha_i$ from finite set

$$D_N = \{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\},$$

using the adaptive scheme considered in [15].

Remark 3.2 (a) Out of the three methods above, first one requires evaluation of the following three functions: $F(x_{n,\alpha}^{\delta}),\ F(x_{n,\alpha}^{\delta} - R(x_{n,\alpha}^{\delta})),\ F(y_{n,\alpha}^{\delta})$, the second one requires evaluation of the functions: $F(x_{n,\alpha}^{\delta}),\ F(x_{n,\alpha}^{\delta} + R(x_{n,\alpha}^{\delta})),\ F(y_{n,\alpha}^{\delta})$ and the third one requires the evaluation of $F(x_{n,\alpha}^{\delta}),\ F(x_{n,\alpha}^{\delta} - R(x_{n,\alpha}^{\delta})),\ F(x_{n,\alpha}^{\delta} + R(x_{n,\alpha}^{\delta}))$ and $F(y_{n,\alpha}^{\delta})$.

(b) The method (3.4) with $H_3(x_{n,\alpha}^{\delta})$ is considered in this paper. In the following sections all results are valid for the method (3.4) with $H_1(x_{n,\alpha}^{\delta})$ and $H_2(x_{n,\alpha}^{\delta})$ as defined in (3.5) and (3.6), respectively.

Let

$$C_\beta := \min\left\{\frac{\|F(x_0) - f^\delta\|}{(2 + \beta_1/\alpha_0)(\beta_1 + \alpha_N)}, 2\right\},$$

$$\delta < \frac{C_\beta}{2}\alpha_0$$

and

$$\|\hat{x} - x_0\| \le \rho \ \text{ with } \rho < \min\left\{\frac{1}{3k_0}, \frac{1}{2}\left(\frac{C_\beta}{2} - \frac{\delta}{\alpha_0}\right)\right\}. \tag{3.8}$$

For convenience we use the notation

$$e_n^x = x_{n,\alpha}^{\delta} - x_\alpha^{\delta} \ \text{ for } \ n = 0, 1, 2, \ldots, \tag{3.9}$$

and

$$e_n^y = y_{n,\alpha}^\delta - x_\alpha^\delta \text{ for } n = 0, 1, 2, \dots. \tag{3.10}$$

Here x_α^δ is the solution of $R_\alpha(x) = 0$.

Using the above notation, we state our main result below.

Theorem 3.3 *Let $F'(x)$ and $F''(x)$ exists for all $x \in D(F)$. Then the iterative method defined in (3.4) have cubic converges to x_α^δ. Also,*

$$\|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| = O(e^{-\gamma 3^{n+1}})$$

where $\gamma = -ln(\|e_0^x\|)$.

We prove the following lemmas and a proposition to prove the above theorem.

Lemma 3.4 *Let e_n^x be as in (3.9). Then*

$$\|e_0^x\| \leq 2\rho + \frac{\delta}{\alpha_0}.$$

Proof Note that, $e_0^x = x_0 - x_\alpha^\delta$ and since x_α^δ is the solution of (1.2),

$$F(x_\alpha^\delta) - F(\hat{x}) + \alpha(x_\alpha^\delta - \hat{x}) = f^\delta - f + \alpha(x_0 - \hat{x}).$$

Since F is monotone, by taking innerproduct with $x_\alpha^\delta - \hat{x}$ we have,

$$\|x_\alpha^\delta - \hat{x}\| \leq \frac{\delta}{\alpha} + \|x_0 - \hat{x}\|. \tag{3.11}$$

Also, by triangle inequality

$$\|x_\alpha^\delta - x_0\| \leq \|x_\alpha^\delta - \hat{x}\| + \|\hat{x} - x_0\|. \tag{3.12}$$

The result now follows from (3.8), (3.11) and (3.12). □

Define the operators $M(x)$, $M_1(x)$ and $M_2(x)$ as follows.

$$M(x) = \int_0^1 R_\alpha''(x_\alpha^\delta + t(x - x_\alpha^\delta))(1-t)dt, \quad x \in D(F), \tag{3.13}$$

$$M_1(x) = \int_0^1 R_\alpha''(x_\alpha^\delta + t(x + R_\alpha(x) - x_\alpha^\delta))(1-t)dt, \quad x \in D(F) \tag{3.14}$$

and

$$M_2(x) = \int_0^1 R_\alpha''(x_\alpha^\delta + t(x - R_\alpha(x) - x_\alpha^\delta))(1-t)dt, \quad x \in D(F). \tag{3.15}$$

Let

$$\Gamma_1 := \frac{[M_1(x_{n,\alpha}^\delta) - M_2(x_{n,\alpha}^\delta)][(e_n^x)^2 + (R_\alpha(x_{n,\alpha}^\delta))^2]}{2R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}, \tag{3.16}$$

and

$$\Gamma_2 := \frac{[M_1(x_{n,\alpha}^\delta) + M_2(x_{n,\alpha}^\delta)]e_n^x R_\alpha(x_{n,\alpha}^\delta)}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}. \tag{3.17}$$

Lemma 3.5 *Let R'_α be as in (3.2), Γ_1 and Γ_2 be as above. Then*

$$R_\alpha(x_{n,\alpha}^\delta + R_\alpha(x_{n,\alpha}^\delta)) - R_\alpha(x_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)) = 2R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)[1 + \Gamma_1 + \Gamma_2].$$

Proof By Taylor expansion around the solution x_α^δ of $R_\alpha(x) = 0,$ we have

$$R_\alpha(x_{n,\alpha}^\delta) = R'_\alpha(x_\alpha^\delta)(x_{n,\alpha}^\delta - x_\alpha^\delta) + M(x_{n,\alpha}^\delta)(x_{n,\alpha}^\delta - x_\alpha^\delta)^2. \tag{3.18}$$

The Taylor expansion of $R_\alpha(x_{n,\alpha}^\delta + R_\alpha(x_\alpha^\delta))$ and $R_\alpha(x_{n,\alpha}^\delta - R_\alpha(x_\alpha^\delta))$ around the solution x_α^δ of $R_\alpha(x) = 0$ we get

$$\begin{aligned} R_\alpha(x_{n,\alpha}^\delta + R_\alpha(x_{n,\alpha}^\delta)) &= R'_\alpha(x_\alpha^\delta)(x_{n,\alpha}^\delta - x_\alpha^\delta + R_\alpha(x_{n,\alpha}^\delta)) \\ &\quad + M_1(x_{n,\alpha}^\delta)(x_{n,\alpha}^\delta - x_\alpha^\delta + R_\alpha(x_{n,\alpha}^\delta))^2 \\ &= R'_\alpha(x_\alpha^\delta)[(x_{n,\alpha}^\delta - x_\alpha^\delta) + R_\alpha(x_{n,\alpha}^\delta)] + M_1(x_{n,\alpha}^\delta)[(x_{n,\alpha}^\delta - x_\alpha^\delta)^2 \\ &\quad + (R_\alpha(x_{n,\alpha}^\delta))^2 + 2(x_{n,\alpha}^\delta - x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)] \\ &= R'_\alpha(x_\alpha^\delta)[e_n^x + R_\alpha(x_{n,\alpha}^\delta)] + M_1(x_{n,\alpha}^\delta)[(e_n^x)^2 \\ &\quad + (R_\alpha(x_{n,\alpha}^\delta))^2 + 2e_n^x R_\alpha(x_{n,\alpha}^\delta)] \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} R_\alpha(x_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)) &= R'_\alpha(x_\alpha^\delta)(x_{n,\alpha}^\delta - x_\alpha^\delta - R_\alpha(x_{n,\alpha}^\delta)) \\ &\quad + M_2(x_{n,\alpha}^\delta)(x_{n,\alpha}^\delta - x_\alpha^\delta - R_\alpha(x_{n,\alpha}^\delta))^2 \\ &= R'_\alpha(x_\alpha^\delta)\left[(x_{n,\alpha}^\delta - x_\alpha^\delta) - R_\alpha(x_{n,\alpha}^\delta)\right] + M_2(x_{n,\alpha}^\delta)[(x_{n,\alpha}^\delta - x_\alpha^\delta)^2 \\ &\quad + (R_\alpha(x_{n,\alpha}^\delta))^2 - 2(x_{n,\alpha}^\delta - x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)] \\ &= R'_\alpha(x_\alpha^\delta)[e_n^x - R_\alpha(x_{n,\alpha}^\delta)] + M_2(x_{n,\alpha}^\delta)[(e_n^x)^2 \\ &\quad + (R_\alpha(x_{n,\alpha}^\delta))^2 - 2e_n^x R_\alpha(x_{n,\alpha}^\delta)]. \end{aligned} \tag{3.20}$$

From (3.19) and (3.20) we have

$$\begin{aligned} &R_\alpha(x_{n,\alpha}^\delta + R_\alpha(x_{n,\alpha}^\delta)) - R_\alpha(x_{n,\alpha}^\delta - R_\alpha(x_{n,\alpha}^\delta)) \\ &\quad = 2R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta) \end{aligned}$$

$$
\begin{aligned}
&+[M_1(x_{n,\alpha}^\delta) - M_2(x_{n,\alpha}^\delta)]((e_n^x)^2 + (R_\alpha(x_{n,\alpha}^\delta))^2)\\
&+2[M_1(x_{n,\alpha}^\delta) + M_2(x_{n,\alpha}^\delta)]e_n^x R_\alpha(x_{n,\alpha}^\delta)\\
&= 2R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)[1+\Gamma_1+\Gamma_2]. \qquad (3.21)
\end{aligned}
$$

□

Lemma 3.6 *Let R_α, R'_α, Γ_1 and Γ_2 be as in (3.1), (3.2), (3.16) and (3.17) respectively. Then*

(i)

$$\|R_\alpha(x_{n,\alpha}^\delta)\| \leq (\beta_1+\alpha)\|e_n^x\| + \frac{\beta_2+\alpha}{2}\|e_n^x\|^2$$

(ii)

$$\|(R_\alpha(x_{n,\alpha}^\delta))^2(\Gamma_1+\Gamma_2)\| = O(\|e_n^x\|^3).$$

Proof Observe that,

$$\|R'_\alpha(x_\alpha^\delta)\| \leq \beta_1+\alpha \ \text{ and } \ \|M(x)\| \leq \frac{\beta_2+\alpha}{2}. \qquad (3.22)$$

Now (i) follows from (3.18) and (3.22). To prove (ii),

$$
\begin{aligned}
\|R_\alpha(x_{n,\alpha}^\delta)\| &= \|R'_\alpha(x_\alpha^\delta)^{-1}R'_\alpha(x_\alpha^\delta)(R_\alpha(x_{n,\alpha}^\delta))\|\\
&\leq \frac{1}{\alpha}\|R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)\| \qquad (3.23)
\end{aligned}
$$

and also

$$
\begin{aligned}
\|(R_\alpha(x_{n,\alpha}^\delta))^2(\Gamma_1+\Gamma_2)\| &\leq \|\frac{1}{\alpha}R_\alpha(x_{n,\alpha}^\delta)\left\{2[M_1(x_{n,\alpha}^\delta) - M_2(x_{n,\alpha}^\delta)][(e_n^x)^2\right.\\
&\quad + (R_\alpha(x_{n,\alpha}^\delta))^2]\\
&\quad \left.+[M_1(x_{n,\alpha}^\delta) + M_2(x_{n,\alpha}^\delta)]e_n^x R_\alpha(x_{n,\alpha}^\delta)\right\}\|\\
&= O(\|e_n^x\|^3).
\end{aligned}
$$

From (i), (3.22) and the inequality $\|M_i(x)\| \leq \frac{\beta_2+\alpha}{2}$ for $i = 1, 2.$ we get the result (ii). □

Lemma 3.7 *Let R_α and R'_α be as in (3.1) and (3.2) respectively. If $\|x_{n,\alpha}^\delta - x_0\| < \frac{\|F(x_0)-f^\delta\|}{\beta_1+\alpha}$ for each $n = 1, 2, \ldots$. Then*

$$\frac{1}{\|R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)\|} \leq \frac{1}{\alpha(\|F(x_0)-f^\delta\| - (\beta_1+\alpha)\|x_{n,\alpha}^\delta - x_0\|)} \ \text{ for } \ n = 1, 2, \ldots.$$

Proof Note that

$$\begin{aligned} R_\alpha(x_{n,\alpha}^\delta) &= F(x_{n,\alpha}^\delta) - f^\delta + \alpha(x_{n,\alpha}^\delta - x_0) \\ &= F(x_0) - f^\delta + F(x_{n,\alpha}^\delta) - F(x_0) + \alpha(x_{n,\alpha}^\delta - x_0) \\ &= F(x_0) - f^\delta + \left[\int_0^1 F'(x_0 + t(x_{n,\alpha}^\delta - x_0)dt + \alpha I\right](x_{n,\alpha}^\delta - x_0). \end{aligned}$$

So,

$$\begin{aligned} \|R_\alpha(x_{n,\alpha}^\delta)\| &\geq \|F(x_0) - f^\delta\| - \|\left[\int_0^1 F'(x_0 + t(x_{n,\alpha}^\delta - x_0)dt + \alpha I\right](x_{n,\alpha}^\delta - x_0)\| \\ &\geq \|F(x_0) - f^\delta\| - (\beta_1 + \alpha)\|x_{n,\alpha}^\delta - x_0\| \end{aligned} \tag{3.24}$$

for $n = 1, 2, \ldots$. From (3.23) and (3.24) the result follows. □

Proposition 3.8 *Let R_α be as in (3.1) and x_α^δ be the solution of $R_\alpha(x) = 0$. Further the assumptions in Lemmas 3.4–3.7 hold, $F'(x)$ and $F''(x)$ exists $\forall x \in D(F)$. Then*

$$\|y_{n,\alpha}^\delta - x_\alpha^\delta\| \leq \frac{2(\beta_1 + \alpha)^2}{\alpha(\|F(x_0) - f^\delta\| - (\beta_1 + \alpha)\|x_{n,\alpha}^\delta - x_0\|)}\|e_n^x\|^2 + O(\|e_n^x\|^3).$$

Proof Let $\Theta = \Gamma_1 + \Gamma_2$. Then by (3.3), (3.21) and (3.23), we have

$$\begin{aligned} e_n^y &= e_n^x - \frac{(R_\alpha(x_{n,\alpha}^\delta))^2}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)(1+\Theta)} \\ &= e_n^x - \frac{(R_\alpha(x_{n,\alpha}^\delta))^2}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}[I - \Theta + \Theta^2 - \cdots] \\ &= e_n^x - \frac{(R_\alpha(x_{n,\alpha}^\delta))^2}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}(I - \Theta) \\ &\quad - \frac{(R_\alpha(x_{n,\alpha}^\delta))^2}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)} \times \text{higher order terms in } \Theta \\ &= \frac{1}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}\Big[R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)e_n^x - R_\alpha(x_{n,\alpha}^\delta)^2(I - \Theta) \\ &\quad - (R_\alpha(x_{n,\alpha}^\delta))^2 \times \text{higher order terms in } \Theta\Big]. \end{aligned} \tag{3.25}$$

The result now follows from (3.22), Lemmas 3.6 and 3.7. □

Proof of Theorem 3.3. Let $\Theta = \Gamma_1 + \Gamma_2$. Then by (3.4), (3.21) and (3.23), we have

$$e_{n+1}^x = e_n^y - \frac{R_\alpha(x_{n,\alpha}^\delta)R_\alpha(y_{n,\alpha}^\delta)}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)(1+\Theta)}$$

$$
\begin{aligned}
&= e_n^y - \frac{(R_\alpha(x_{n,\alpha}^\delta)R_\alpha(y_{n,\alpha}^\delta)}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}[I - \Theta + \Theta^2 - \cdots] \\
&= e_n^y - \frac{(R_\alpha(x_{n,\alpha}^\delta)R_\alpha(y_{n,\alpha}^\delta)}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}(I - \Theta) \\
&\quad - \frac{(R_\alpha(x_{n,\alpha}^\delta)R_\alpha(y_{n,\alpha}^\delta)}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)} \times \text{higher order terms in} \quad \Theta \\
&= \frac{1}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}[R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)e_n^y - R_\alpha(x_{n,\alpha}^\delta)R_\alpha(y_{n,\alpha}^\delta)(I - \Theta) \\
&\quad - (R_\alpha(x_{n,\alpha}^\delta)R_\alpha(y_{n,\alpha}^\delta) \times \text{higher order terms in} \quad \Theta].
\end{aligned} \tag{3.26}
$$

Hence, we have

$$
\begin{aligned}
\|e_{n+1}^x\| \le \| &\frac{1}{R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)}\|[\|R'_\alpha(x_\alpha^\delta)R_\alpha(x_{n,\alpha}^\delta)\|\|e_n^y\| \\
&+ \|R_\alpha(x_{n,\alpha}^\delta)R_\alpha(y_{n,\alpha}^\delta)\| + \|R_\alpha(x_{n,\alpha}^\delta)R_\alpha(y_{n,\alpha}^\delta)\|[\|\Theta\| \\
&+ \text{higher order terms in} \quad \|\Theta\|]].
\end{aligned}
$$

If $\|x_{n,\alpha}^\delta - x_0\| < \frac{\|F(x_0)-f^\delta\|}{\beta_1+\alpha}$, then analogous to the proof of Proposition 3.8 (using Lemmas 3.5–3.7) one can prove that

$$
\|e_{n+1}^x\| \le \frac{2(\beta_1+\alpha)^2}{\alpha(\|F(x_0) - f^\delta\| - (\beta_1+\alpha)\|x_{n,\alpha}^\delta - x_0\|)}\|e_n^x\|\|e_n^y\| + O(\|e_n^x\|^2\|e_n^y\|).
$$

Therefore by Proposition 3.8,

$$
\|e_{n+1}^x\| \le \frac{4(\beta_1+\alpha)^4}{\alpha^2(\|F(x_0) - f^\delta\| - (\beta_1+\alpha)\|x_{n,\alpha}^\delta - x_0\|)^2}\|e_n^x\|^3 + O(\|e_n^x\|^4). \tag{3.27}
$$

Now it remains to show that $\|x_{n,\alpha}^\delta - x_0\| < \frac{\|F(x_0)-f^\delta\|}{\beta_1+\alpha}$. This can be shown as follows; since $\frac{2(\beta_1/\alpha+1)(\beta_1+\alpha)}{\|F(x_0)-f^\delta\|}\|e_0^x\| < 1$ (by 3.8), we have by (3.27)

$$
\begin{aligned}
\|x_{1,\alpha}^\delta - x_0\| &\le \|x_{1,\alpha}^\delta - x_\alpha^\delta\| + \|x_\alpha^\delta - x_0\| \\
&\le \left(\frac{2(\beta_1/\alpha+1)(\beta_1+\alpha)}{(\|F(x_0) - f^\delta\|}\right)^2 \|e_0^x\|^3 + O(\|e_0^x\|^4) + \|x_\alpha^\delta - x_0\| \\
&\le 2\|e_0^x\| < C_\beta < \frac{\|F(x_0) - f^\delta\|}{\beta_1+\alpha}
\end{aligned}
$$

(by avoiding higher order terms in $\|e_0^x\|$). By (3.27) and (3.8) we get,

$$
\|x_{2,\alpha}^\delta - x_0\| \le \|x_{2,\alpha}^\delta - x_\alpha^\delta\| + \|x_\alpha^\delta - x_0\|
$$

$$
\begin{aligned}
&\leq \left(\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(x_0)-f^\delta\| - (\beta_1+\alpha)\|x_{1,\alpha}^\delta - x_0\|)} \right)^2 \|e_1^x\|^3 \\
&\quad + O(\|e_1^x\|^4) + \|x_\alpha^\delta - x_0\| \\
&\leq \left(\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(x_0)-f^\delta\| - (\beta_1+\alpha)2\|e_0^x\|)} \right)^8 \|e_0^x\|^9 \\
&\quad + O(\|e_0^x\|^{10}) + \|x_\alpha^\delta - x_0\| \\
&\leq 2\|x_\alpha^\delta - x_0\| = 2\|e_0^x\| < C_\beta < \frac{\|F(x_0)-f^\delta\|}{\beta_1+\alpha}.
\end{aligned}
$$

(by ignoring higher order terms in $\|e_0^x\|$) which shows $\|x_{n,\alpha}^\delta - x_0\| < \frac{\|F(x_0)-f^\delta\|}{\beta_1+\alpha}$ for $n = 2$. By simply repacing $x_{2,\alpha}^\delta$ by $x_{k+1,\alpha}^\delta$ in the above estimates we arrive at $\|x_{k+1,\alpha}^\delta - x_0\| < \frac{\|F(x_0)-f^\delta\|}{\beta_1+\alpha}$. Thus by induction $\|x_{n,\alpha}^\delta - x_0\| < \frac{\|F(x_0)-f^\delta\|}{\beta_1+\alpha}$ for $n > 0$. From the preceding relation it follows that

$$
\begin{aligned}
\|x_{n+1,\alpha}^\delta - x_\alpha^\delta\| &\leq \left(\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(x_0)-f^\delta\| - (\beta_1+\alpha)2\|e_0^x\|)} \right)^2 \|e_n^x\|^3 \\
&\quad + O(\|e_n^x\|^4) \\
&= O(e^{-\gamma 3^{n+1}})
\end{aligned}
\tag{3.28}
$$

where $\gamma := -\ln(\|e_0^x\|)$. Hence the proof. □

Remark 3.9 Note that, by (3.28) we have

$$
\begin{aligned}
\|e_{n+1}^x\| &\leq \left(\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(x_0)-f^\delta\| - (\beta_1+\alpha)2\|e_0^x\|)} \right)^2 \|e_n^x\|^3 \\
&\quad + O(\|e_n^x\|^4).
\end{aligned}
\tag{3.29}
$$

and by the repeated application of (3.29) we have the following estimate

$$
\begin{aligned}
\|e_{n+1}^x\| &\leq \left(\left(\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(x_0)-f^\delta\| - (\beta_1+\alpha)2\|e_0^x\|)} \right)^2 \right)^{\frac{1}{2}(3^{n+1}-1)} \|e_0^x\|^{3^{n+1}} \\
&\quad + O(\|e_0^x\|^{3^{n+1}+4}).
\end{aligned}
\tag{3.30}
$$

Since $\|e_0^x\| < 1$, ignoring the terms of order $\|e_0^x\|^{3^{n+1}+4}$ and taking

$$
\|e_{n+1}^x\| \leq C_{\alpha_N} e^{-\gamma 3^{n+1}}
\tag{3.31}
$$

where $C_{\alpha_N} := \left((\frac{2(\beta_1+\alpha)^2}{\alpha(\|F(x_0)-f^\delta\| - (\beta_1+\alpha)2\|e_0^x\|)})^2 \right)^{\frac{1}{2}(3^{n+1}-1)}$, $\gamma = -\ln(\|e_0^x\|)$. Note that $C_{\alpha_N} e^{-\gamma 3^{n+1}} = [C_{\alpha_N} e^{-2\gamma 3^n}] e^{-\gamma 3^n}$, and for large values of n, $C_{\alpha_N} e^{-2\gamma 3^n} \leq C_1$ for any constant $C_1 > 0$. Therefore, for large values of n, from (3.31), (2.2) and by (2.7) we have

$$\|x^{\delta}_{n+1,\alpha} - \hat{x}\| \leq C_1 e^{-\gamma 3^n} + \frac{\delta}{\alpha} + \frac{C_0}{1-3k_0 r}\alpha^{\nu}.$$

Let

$$n_\delta := \min\{n : e^{-\gamma 3^n} \leq \frac{\delta}{\alpha} \ \& \ C_{\alpha_N} e^{-\gamma 3^n} \leq C_1\} \tag{3.32}$$

for constant C_1. From above Remark, we have the following Theorem.

Theorem 3.10 *Let $x^{\delta}_{n_\delta+1,\alpha}$ be as in (3.4) and n_δ be as in (3.32). Using the assumptions of Theorem 3.3 we have the following relation:*

$$\|x^{\delta}_{n_\delta+1,\alpha} - \hat{x}\| \leq \tilde{C}\left(\alpha^{\nu} + \frac{\delta}{\alpha}\right) \tag{3.33}$$

where $\tilde{C} = \max\{C+1, \frac{C_0}{1-3k_0 r}\}$.

4 A priori choice of the parameter

If $\alpha_\delta := \alpha(\delta)$ satisfies $\alpha_\delta^{1+\nu} = \delta$ then the error $\alpha^{\nu} + \frac{\delta}{\alpha}$ in (3.33) is of optimal order. That is $\alpha^{\delta} = \delta^{\frac{1}{1+\nu}}$. Then the following Theorem follows by (3.33).

Theorem 4.1 *For $\delta > 0$, let $\alpha := \alpha_\delta = \delta^{\frac{1}{1+\nu}}$. Let n_δ be defined as in (3.32). Then using the assumptions of Theorem 3.10,*

$$\|x^{\delta}_{n_\delta,\alpha} - \hat{x}\| = O(\delta^{\frac{\nu}{1+\nu}}).$$

4.1 Adaptive scheme and its implementation

The adaptive scheme of Pereverzyev and Schock [15] involve the following steps.

Table 1 Relative error and residual for the method (3.4) with H_3

δ	α_k	m_1	m_2	Δ_1	Δ_2
0.01	0.0104	23	24	0.0864	0.0126
		22	23	0.0711	0.0114
		21	22	0.0542	0.0098
0.05	0.0520	23	24	0.0864	0.0096
		22	23	0.0711	0.0114
		21	22	0.0541	0.0096

For simplicity, take $x_i^\delta := x_{n_i,\alpha_i}^\delta$. Choose $i \in \{0, 1, 2, \ldots, N\}$ and $\alpha_i = \mu^i \alpha_0$ where $\mu > 1$ and $\alpha_0 = \delta$.

Let

$$l := \max\left\{i : \alpha_i^\nu \leq \frac{\delta}{\alpha_i}\right\} < N \quad \text{and} \tag{4.1}$$

$$k := \max\left\{i : \|x_i^\delta - x_j^\delta\| \leq 4\tilde{C}\frac{\delta}{\alpha_j}\right\}, \; j = 0, 1, 2, \ldots, i \tag{4.2}$$

where $\tilde{C}$ is as in Theorem 3.10.

Theorem 4.2 *Assume that there exists* $i \in \{0, 1, \ldots, N\}$ *such that* $\alpha_i^\nu \leq \frac{\delta}{\alpha_i}$ *and let* l *and* k *be as in* (4.1) *and* (4.2)*, respectively. If assumptions of Theorem* 3.10 *and* 4.1 *are fulfilled, Then* $l \leq k$; *and*

$$\|\hat{x} - x_k^\delta\| \leq 6\bar{C}\mu\delta^{\frac{\nu}{\nu+1}}.$$

Proof The proof of the Theorem is similar to the proof of Theorem 4.4 in [8]. □

4.2 Implementation of adaptive choice rule

The choice of the regularization parameter stated in Theorem 4.2 has the following steps:

- Choose $\alpha_0 > 0$ such that $\delta_0 < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \ldots, N$.

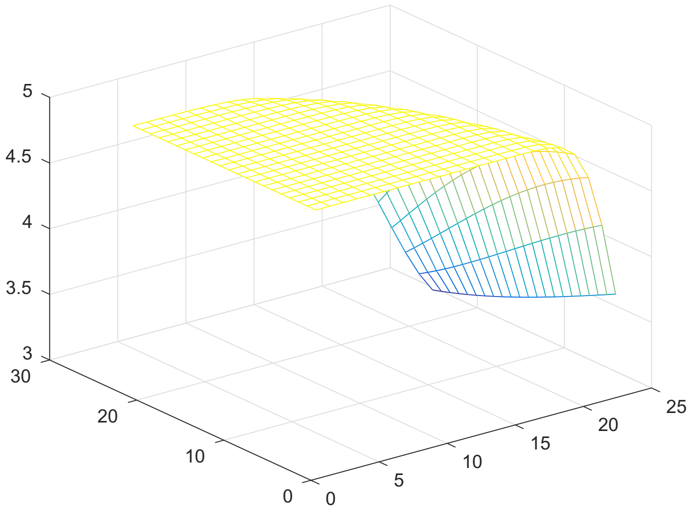

Fig. 1 Exact solution for $m_1 = 22, m_2 = 23$

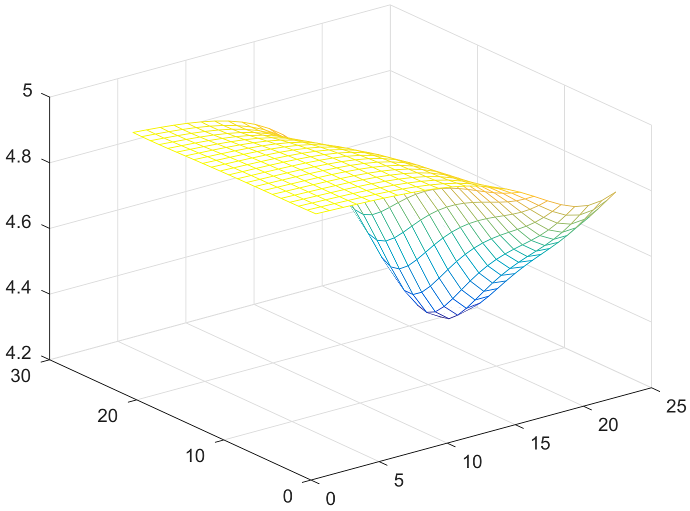

Fig. 2 Approximate solution for the method (3.4) with H_1, for $m_1 = 22$, $m_2 = 23$ and $\delta = 0.01$

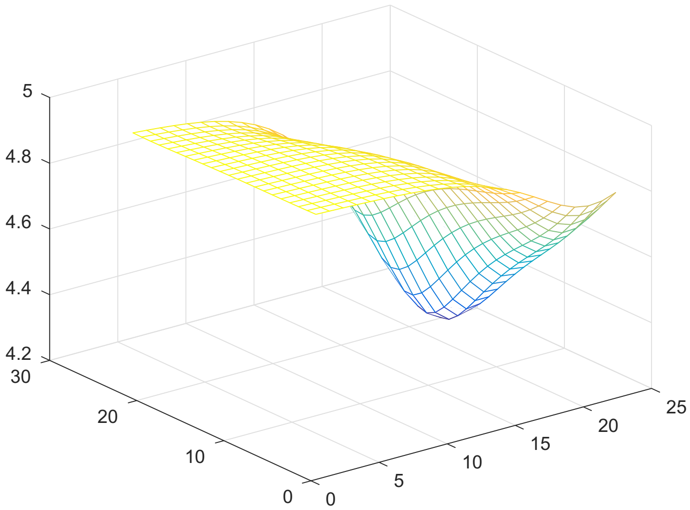

Fig. 3 Approximate solution for the method (3.4) with H_2, for $m_1 = 22$, $m_2 = 23$ and $\delta = 0.01$

Algorithm

1. Set $i = 0$.
2. Select $n_i := \min\left\{n : e^{-\gamma 3^n} \leq \frac{\delta}{\alpha_i}, \& \, C_{\alpha_N} e^{-\gamma 3^n} \leq C_1\right\}$.
3. Solve $x_i := x^{\delta}_{n_i,\alpha_i}$ by using the iteration (3.4).
4. If $\|x_i - x_j\| > 4\tilde{C}\frac{\delta}{\alpha_j}$, $j < i$, then take $k = i - 1$ and return x_k.
5. Else set $i = i + 1$ and go to 2.

5 Numerical example

For the implementation of the method we consider the following integral equation

$$F(t) = -\int\int_D \frac{1}{[(u-u')^2 + (v-v')^2 + t^2(u',v')]^{1/2}} du'dv' = f(u,v). \quad (5.1)$$

Here $f(u,v) = \Delta g(u,v) + F(H)$ and $F : H'(\Omega) \subseteq L^2(\Omega) \to L^2(\Omega), \Omega = [0,m] \times [0,m]$. These problem arise from Inverse gravimetry problem (see [18] and References in it):

$F'(t_0)$ denotes the derivative of F at $t_0(u,v)$ and is given by

$$F'(t_0)h = \int\int_D \frac{t_0(u',v')\,h(u',v')}{[(u-u')^2 + (v-v')^2 + (t_0(u',v'))^2]^{3/2}} du'dv'. \quad (5.2)$$

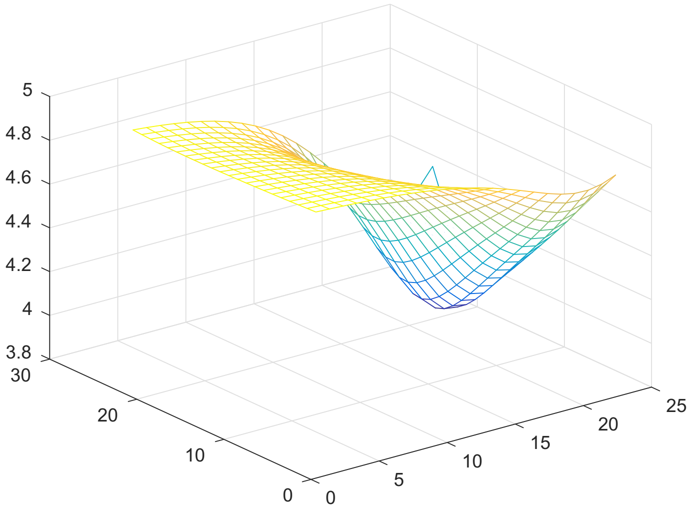

Fig. 4 Approximate solution for the method (3.4) with H_3, for $m_1 = 22, m_2 = 23$ and $\delta = 0.01$

Table 2 Mesh size and corresponding errors for the method (3.4) with $H_1 and H_2$

δ	α_k	m_1	m_2	Δ_1	Δ_2
0.01	0.0104	23	24	0.0728	0.0082
		22	23	0.0650	0.0070
		21	22	0.0581	0.0059
0.05	0.0520	23	24	0.0724	0.0081
		22	23	0.0648	0.0069
		21	22	0.0577	0.0059

Applying 2-dimensional analogy of rectangles's formula with uniform grid for every variable to the integral equation (5.1), we obtain the following system of non-linear equation:

$$\sum_{i=1}^{m_2}\sum_{j=1}^{m_1}\frac{1}{[(u_k-u'_j)^2+(v_l-v'_i)^2+t^2(u'_j,v'_i)]^{1/2}}\Delta u\Delta v=f(u_k,v_l)$$

$(k=1,2,...,m_1, l=1,2,...,m_2)$ for the unknown vector $\{t_{j,i}=t(u_j,u_i),\ j=1,2,...,m_1, i=1,2,...,m_2\}$. In vector-matrix form this system amounts to:

$$F_n(t_n)=f_n, \tag{5.3}$$

where t_n, f_n are vectors of dimension $n=m_1 m_2$. Since F_n is positive definite matrix we apply our method to F_n.

We take exact solution as

$$\begin{aligned}\hat{t}(u,v)=5-2\exp-[(u/10-3.5)^2(v/10-2.5)^2]\\ -3\exp-[(u/10-5.5)^2(v/10-4.5)^2],\end{aligned}$$

where $\hat{t}(u,v)$ is given on the domain $D=\{0\le u\le m_1,\ 0\le v\le m_2\}$. Let $\Delta u=\Delta v=1,\ n=m_1m_2,\ \Delta\sigma=0.25,\ t_0=H=5$.

The results of numerical experiments for various values of m_1 and m_2 are given in Table 1. Here x^{δ}_{n,α_k} is the numerical solution obtained by method (3.4); the relative error of solution and residual are

$$\Delta_1=\frac{\|\hat{x}-x^{\delta}_{n,\alpha_k}\|}{\|x^{\delta}_{n,\alpha_k}\|},\qquad \Delta_2=\frac{\|F_n(x^{\delta}_{n,\alpha_k})-f_n\|}{\|f_n\|}.$$

In Fig. 1 we display the exact solution for $m_1=22, m_2=23$ and in Figs. 2, 3 and 4 we display the approximate solution for the method (3.4) with H_1, H_2 and H_3 respectively; for $m_1=22, m_2=23$ and $\delta=0.01$.

Remark 5.1 During the computation we observe that the method (3.4) with H_1, H_2 gives the same numerical results as given in the Table 2.

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