A STUDY ON ITERATIVE ROOT PROBLEM

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the research thesis entitled **"A STUDY ON ITERATIVE ROOT PROBLEM"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

Place : NITK, Surathkal
Date : 25-10-2017

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CERTIFICATE

This is to **certify** that the research thesis entitled **"A STUDY ON ITERATIVE ROOT PROBLEM"** submitted by **M. Suresh Kumar**, (Register Number MA12P02) as the record of the research work carried out by him, is *accepted as the research thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. V. Murugan
Research Guide

Chairman - DRPC

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ABSTRACT

The iterative root problem is one of the classical problem in the theory of iterative functional equations and is described as follows: Given a non-empty X, a self map F on X and a fixed positive integer n, to find another self map f on X such that $f^n = F$. If such a function f exists, then it is called an n^{th} *iterative root* of F. Existence of iterative roots for strictly monotone continuous functions are well-studied. Among the piecewise monotone continuous (PM) functions, the existence of iterative roots of functions with height less than two is also well-studied.

In this thesis, we develop the method of characteristic interval to any continuous functions and discuss the properties of non-isolated forts of any continuous functions on a compact interval. This helps us to derive the conditions on the existence of iterative roots for a class of PM functions with non-monotonicity height greater than one and a class of continuous functions with infinitely many forts. As an application we obtain a new class of functions which is dense in the space of all continuous functions from a compact interval into itself.

We also provide sufficient conditions on the existence of solutions of series-like iterative functional equation for a class of PM functions. We conclude the thesis with results on the uniqueness of iterative roots of order preserving homeomorphisms by using the set of points of coincidence.

Mathematics Subject Classification (AMS-2010): *39B12, 39B22.*

Keywords: *Iterative Roots, Fractional iterates, Forts, Isolated forts, Non-isolated Forts, Functional equations, PM Functions, Height, Characteristic Interval, Homeomorphisms, Commuting functions, Subcommuting functions, Comparable functions.*

Table of Contents

Chapter 1

Introduction

The term functional equation, in a simple manner, can be defined as follows: *Functional equation is an equation involving independent functions whose unknowns are functions.* The theory of functional equations is a classical tool in mathematics to solve many mathematical models which arises in applied mathematics and engineering. The algebraic, analytical and topological structures of functional equations not only helps us to study the mathematical models, but also provides wide scope in pure mathematics. Functional equations finds applications widely in the study of mechanics, dynamical systems, economics, game theory, geometry, neural networks, artificial intelligence, probability and statistics (cf. (Aczél, 1966; Castillo et al., 2005; Iannella and Kindermann, 2005; Kindermann, 1998)).

One of the oldest example of a functional equation is

$$f(x+y) - f(x-y) = g(x)h(y). \tag{1.0.1}$$

J. D' Alembert reduced a problem of vibrating string to the functional equation (1.0.1) and it is the first ever functional equation in the modern theory of functional equations (D'Alembert, 1747).

Some well-known examples of functional equations:

1. $f(x+y) + f(x-y) = 2f(x)f(y)$ (D' Alembert)

2. $f(x+y) = f(x) + f(y)$ (Cauchy)

3. $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ (Jensen)

The existence of solutions and other analytical properties of above functional equations can be found in (Kuczma et al., 1990; Aczél, 1966; Castillo et al., 2005;

Kannappan, 2009).

1.1 Iterative Functional Equations

Functional equations which involve iterates or compositions of unknown functions are called *iterative functional equations.* The study of iterative functional equation is rooted in the classical works of Abel (Abel, 1826), Babbage (Babbage, 1815), Schröder (Schröder, 1870) and many other well-known mathematicians.

The first ever study of iterative functional equation was due to Charles Babbage (Babbage, 1815). He discussed the functional equation of the form

$$f^2(x) = id(x), \tag{1.1.2}$$

where id denotes the identity function. The equation (1.1.2) named after him as *Babbage Functional Equation.* Some other classical examples of iterative functional equations are

1. $f(h(x)) = h(x+1)$ (Abel),

2. $h(f(x)) = g(h(x))$ (Schröder).

There are many other examples of iterative functional equations, however, we mostly concentrate on the generalization of Babbage's functional equation.

1.2 Iterative Root Problem

Given a non-empty set X and a function $f : X \to X$, define $f^0(x) = x$ for all $x \in X$ and for $n \in \mathbb{N}$, $f^n(x) = f(f^{n-1}(x))$ for all $x \in X$. The function f^n is called the n^{th} *iterate* of f. *Iterative root problem* is one of the classical problem in iterative functional equations and is described as follows: Let $F : X \to X$ be any function and $n \in \mathbb{N}$ be fixed. The iterative root problem is to find function $f : X \to X$ such that

$$f^n(x) = F(x) \text{ for all } x \in X. \tag{1.2.3}$$

The function f, if it exists, is called an *iterative root of order* n or a *fractional iterate of order* n of the given function F.

1.2.1 Examples

Example 1.2.1. *The function* $f(x) = x^2$ *is a square iterative root of* $F(x) = x^4$ *on* $\mathbb{R}$.

Example 1.2.2. *For each fixed positive real number* α*, we define the function* $F : \mathbb{R} \to \mathbb{R}$ *by* $F(x) = \alpha x$ *for all* $x \in \mathbb{R}$*. Then for each positive integer* n*, the function* $f : \mathbb{R} \to \mathbb{R}$ *defined by* $f(x) = \beta x$ *for all* $x \in \mathbb{R}$ *satisfies the functional equation*

$$f^n(x) = F(x) \text{ for all } x \in \mathbb{R},$$

where β *is the positive real number such that* $\beta^n = \alpha$*. Hence* f *is an iterative root order* n *of* F*.*

Example 1.2.3. *The function* $F : [0,1] \to [0,1]$ *defined by* $F(x) = 1 - x$ *for all* $x \in \mathbb{R}$ *does not have any continuous iterative root of order* $2n$ *for all* $n \in \mathbb{N}$*.*

For $n \in \mathbb{N}$*, let* $f : [0,1] \to [0,1]$ *be any continuous function such that* $f^{2n}(x) = F(x)$ *for all* $x \in [0,1]$*. Since the function* F *is bijective, we see that* f *is also bijective (see Łojasiewicz (1951)). Therefore,* f *is either monotonically increasing or decreasing, in either case we see that* f^{2n} *is always increasing on* $[0,1]$*, however* F *is always decreasing on* $[0,1]$*. Thus* F *has no continuous iterative root of order* $2n$ *for all* $n \in \mathbb{N}$*.*

1.2.2 Applications

Iterative functional equations find applications in embedding flow problem (Fort, 1955), invariant curves (Kuczma et al., 1990), neural networks (Kindermann, 1998; Iannella and Kindermann, 2005) and several engineering applications require solutions of the iterative root problem (see the monographs (Aczél, 1966) and (Castillo et al., 2005)). We give a brief introduction about how the iterative root problem can be applied in finding invariant curves and to the embedding flow problem.

Invariant Curves Problem: Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be any map. A subset M of $\mathbb{R}^n$ is said to be invariant under F, if $F(M) \subseteq M$.

Given a map $F : \mathbb{R}^n \to \mathbb{R}^n$ and a curve M in $\mathbb{R}^n$, the invariant curve problem is to determine the condition that the curve is invariant under F. For the simplicity, we discuss this problem in $\mathbb{R}^2$.

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be any map and let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be its corresponding coordinate functions. Let $\varphi : [0,1] \to \mathbb{R}$ be any curve and $\Omega = \{(x, \varphi(x)) \in \mathbb{R}^2 \mid t \in [0,1]\}$ be the graph of φ. Now the condition that the curve φ is invariant under F (i.e., $F(\Omega) \subseteq \Omega$) reduces to the following iterative functional equation:

$$\varphi(f(x, \varphi(x))) = g(x, \varphi(x)) \text{ for all } x \in [0,1]. \tag{1.2.4}$$

If $f(x,y) = x + y,\ g(x,y) = \alpha y,$ for all $(x,y) \in \mathbb{R}^2$ and α is a fixed real number, then the above functional equation (1.2.4) reduces to the functional equation

$$\varphi(x + \varphi(x)) = \alpha\varphi(x). \tag{1.2.5}$$

If $\alpha = 1$ then the equation (1.2.5) is known as *Euler's functional equation*. Further, if $f(x,y) = y,\ g(x,y) = x$, then the functional equation (1.2.4) reduces to the Babbage functional equation

$$\varphi^2(x) = x.$$

A detailed discussion on invariant curve problems can be found in the books by Nitecki (Nitecki, 1971) and Kuczma (Kuczma et al., 1990).

Let X be a topological space. A *(topological) flow* on X is a continuous function $F : X \times \mathbb{R} \to X$ such that

(a) For each $t \in \mathbb{R}$, the function $F_t(x) = F(x,t)$ is a homeomorphism from X onto X, and

(b) $F(x, t+s) = F(F(x,s), t)$ for all $x \in X$ and $t, s \in \mathbb{R}$.

Embedding Flow Problem: *For a given topological space X and a given homeomorphism f from X onto itself, does there exist a flow on X for which $F_1 = f$?*

If such a flow F exists, then f is said to be embedded in F. Suppose, for a given homeomorphism $f : X \to X$ there exists a flow F on X. Then, for each positive integer n, property (b) reduces into the following iterative functional equation

$$f^n(x) = F_n(x) \text{ for all } x \in X,$$

where f^n denote the n^{th} iterate of f. Therefore, f is the n^{th} iterative root of F_n becomes necessary condition for solving embedding flow problem.

In fact, the existence of solutions of embedding flow problem on an interval was proved by Fort.

Theorem 1.2.4. *(Fort, 1955) Any order preserving homeomorphism of an interval onto itself can be embedded in a flow.*

A detailed results on embedding flow problem can be found in (Fort, 1955; Zdun, 2014)

1.3 Basic Results on Iterative Root Problem

Mathematicians like Bödewadt (Bödewadt, 1944), Łojasiewicz (Łojasiewicz, 1951), Haidukov (Haidukov, 1958), Kuczma (Kuczma, 1961; Kuczma et al., 1990) and Zhang (Zhang, 1997) have made contribution to the significant growth of the study on the iterative root problem. Recent works in this field are due to Zhang (Liu and Zhang, 2011; Liu et al., 2012), Jarczyk (Baron and Jarczyk, 2001), Lin (Lin, 2014; Lin et al., 2017), Liu (Liu and Gong, 2017) and many others.

We would like to emphasize few existence and nonexistence of solutions of iterative root problems in our context.

1.3.1 Iterative Roots of Monotone Functions

Iterative roots of strictly monotone functions are well-studied and some of the basic results are the following:

Theorem 1.3.1. *(Babbage, 1815) Let f be a particular solution of the functional equation*

$$f^n(x) = x \text{ for all } x \in \mathbb{R}. \tag{1.3.6}$$

Then for any invertible function h on $\mathbb{R}$ the function $h^{-1} \circ f \circ h$ is also a solution of the functional equation (1.3.6).

Theorem 1.3.2. *(Isaacs, 1950) Let $F : X \to X$ be a function such that $F(a) = b$ and $F(b) = a$ for some $a, b \in X$ with $a \neq b$. If, for any $x \in X$, the equality $F^2(x) = x$ implies that $x \in \{a, b, F(x)\}$, then the equation*

$$f^2(x) = F(x) \text{ for all } x \in X$$

has no solutions.

Proposition 1.3.3. *(Łojasiewicz, 1951) Let $f : X \to X$ be any function satisfies the equation $f^n = F$ on X for some $n \geq 1$. Then*

(i) f is one-one if and only if F is one-one.

(ii) f is onto if and only if F is onto.

(iii) f is bijective if and only if F is bijective.

The functional equation (1.2.3) need not possess solution when the function F is not continuous.

Example 1.3.4. *(Dikof and Graw, 1980) Let $F : \mathbb{R} \to \mathbb{R}$ be the strictly increasing function $F(x) = 2(x + [x]) + \frac{5}{2}$, $x \in \mathbb{R}$, where $[x]$ is the integer part of x. Then the function F has no iterative root of any order.*

Throughout the thesis, we fix $I = [a, b]$ and $C(I)$ be the set of all continuous self mappings from I into I, unless otherwise stated. One of the fundamental result in the theory of existence of iterative roots was proved by Bödewadt.

Theorem 1.3.5. *(Bödewadt, 1944) Let $F : I \to I$ be any strictly increasing homeomorphism. Then for each $n \in \mathbb{N}$, there is a strictly increasing homeomorphism $f : I \to I$ such that $f^n = F$ on I.*

An another classical result on the existence of solutions of the functional equation (1.2.3) is due to Kuczma. The following theorem is often used in many of our results, so we sketch the proof of this theorem.

Theorem 1.3.6. *(Kuczma et al., 1990) If $F : I \to I$ is a continuous strictly increasing function, then the equation (1.2.3) has continuous strictly increasing solution for all $n \in \mathbb{N}$.*

Proof. Put $\mathbb{F} = \{x \in [a, b] \mid F(x) = x\}$. Then $I = \mathbb{F} \cup (\bigcup_{\alpha,\beta\in\mathbb{F}} I_{\alpha\beta})$, where $I_{\alpha\beta}$ is a pairwise disjoint interval of the form $[\alpha, \beta]$ with $\alpha, \beta \in \mathbb{F}$ or $\alpha = a$ or $\beta = b$. Clearly $F|_{I_{\alpha\beta}} : I_{\alpha\beta} \longrightarrow I_{\alpha\beta}$ is a strictly increasing continuous function and either

$$\alpha < F(x) < x < \beta \text{ for all } x \in (\alpha, \beta)$$

or

$$\alpha < x < F(x) < \beta \text{ for all } x \in (\alpha, \beta).$$

Suppose on each $I_{\alpha\beta}$ there is a strictly increasing continuous function $f_{\alpha\beta}$ such that

$$f^n_{\alpha\beta}(x) = F(x) \text{ for all } x \in I_{\alpha\beta},$$

then the function $f : I \to I$ defined by

$$f(x) := \begin{cases} f_{\alpha\beta}(x), & \text{if } x \in I_{\alpha\beta} \\ x, & \text{if } x \in \mathbb{F}, \end{cases}$$

is a continuous strictly increasing function and satisfies the functional equation

$$f^n(x) = F(x) \text{ for all } x \in I.$$

Therefore to prove the result, it is enough if we prove F has an iterative root of order n on each $I_{\alpha\beta} = [\alpha, \beta]$. Without loss of generality we may assume that $\alpha < F(x) < x < \beta$ for all $x \in (\alpha, \beta)$. Fix arbitrarily a point $x_0 \in (\alpha, \beta)$ and $n \in \mathbb{N}$, choose any points $x_1 > x_2 > \cdots > x_{n-1}$ from the interval $(F(x_0), x_0)$. Put

$$x_n = F(x_0)\ ,\ x_{n+1} = F(x_1)\ ,\ x_{n+2} = F(x_2), \ldots,$$

and

$$x_{-1} = F^{-1}(x_{n-1})\ ,\ x_{-2} = F^{-1}(x_{n-2})\ ,\ x_{-3} = F^{-1}(x_{n-3}), \ldots\ .$$

Then it is easy to observe that

$$\cdots < x_n < \cdots < x_2 < x_1 < x_0 < x_{-1} < x_{-2} < \cdots < x_{-n} < \cdots\ .$$

Put $I_k = [x_{k+1}, x_k]$ for all $k \in \mathbb{Z}$. Now for each $k \in \{0, 1, \ldots n-2\}$, let f_k be the arbitrary but fixed strictly increasing homeomorphism from I_k onto I_{k+1}. For $k \geq n-1$, put

$$f_k(x) := F \circ f_{k-n+1}^{-1} \circ \cdots \circ f_{k-1}^{-1}(x) \text{ for all } x \in I_k$$

and for $k \leq -1$, put

$$f_k(x) := f_{k+1}^{-1} \circ \cdots \circ f_{k+n-1}^{-1}(x) \circ F(x) \text{ for all } x \in I_k.$$

Thus for each $k \in \mathbb{Z}$, the function f_k is a strictly increasing homeomorphism from I_k onto I_{k+1}. Therefore the function $f : [\alpha, \beta] \longrightarrow [\alpha, \beta]$ defined by

$$f(x) := \begin{cases} \alpha, & \text{if } \ x = \alpha \\ f_k(x), & \text{if } \ x \in I_k \\ \beta, & \text{if } \ x = \beta \end{cases}$$

is also a strictly increasing homeomorphism on $[\alpha, \beta]$. Also for each $x \in I_k$ and $k \geq 0$,

$$\begin{aligned} f^n(x) &= f_{k+n-1} \circ f_{k+n-2} \circ \cdots \circ f_{k+1} \circ f_k(x) \\ &= F \circ f_k^{-1} \circ f_{k+1}^{-1} \circ \cdots \circ f_{k+n-2}^{-1} \circ f_{k+n-2} \circ \cdots \circ f_{k+1} \circ f_k(x) \\ &= F(x). \end{aligned}$$

On the other hand, if $k \leq -1$ we have,

$$\begin{aligned} f^n(x) &= f_{k+n-1} \circ f_{k+n-2} \circ \cdots \circ f_{k+1} \circ f_k(x) \\ &= f_{k+n-1} \circ f_{k+n-2} \circ \cdots \circ f_{k+1} \circ f_{k+1}^{-1} \circ \cdots \circ f_{k+n-1}^{-1}(x) \circ F(x) \\ &= F(x). \end{aligned}$$

Thus the function f satisfies the functional equation (1.2.3) on $[\alpha, \beta]$. $\square$

We remark here that the solution constructed above depends on strictly increasing homeomorphisms from an interval into itself and there are infinitely many such homeomorphisms. Therefore, iterative roots of strictly increasing continuous functions are not necessarily unique.

Theorem 1.3.7. *(Kuczma et al., 1990) Let $F : I \to I$ is a continuous strictly decreasing onto function. Then, for each odd $n \in \mathbb{N}$ there exists a strictly increasing and continuous function $f : I \to I$ such that $f^n = F$ on I.*

The following theorem gives the continuous strictly monotone solutions of the Babbage functional equation which is discussed in (Vincze, 1959; McShane, 1961).

Theorem 1.3.8. *(McShane, 1961) If a self mapping f of a real interval I is a continuous solution of equation $f^n(x) = x$ for all $x \in I$, then either f itself is the identity mapping or n has to be even and f is a strictly decreasing involution.*

Let $W(n) = \{f^n \mid f \in C(I)\}$ and $W = \cup_{n=2}^{\infty} W(n)$.

Theorem 1.3.9. *(Simon, 1989) The set W is of first category and $\overline{W} \neq C(I)$.*

Theorem 1.3.10. *(Blokh, 1992) The set W is nowhere dense in $C(I)$.*

Even though the set of all continuous functions possessing continuous iterative roots on a compact interval are topologically small, in the sense that they does not contain any open ball in $C(I)$, developing a theory of the existence of iterative root for the class of non-monotone functions is challenging and interesting.

1.3.2 Iterative Roots of PM Functions

As in (Zhang, 1997), we present some notations and basic results for the study of PM functions.

Definition 1.3.11. *(Zhang, 1997) Let $F : I \to I$ be a continuous function. A point $\alpha \in int\ I$ is called a fort of F, if F is not strictly monotone in any neighborhood of α.*

Note that, the point $\alpha \in int\ I$ is a fort of F if and only if for each $\epsilon >$ there exist two distinct points $x_1, x_2 \in N_\epsilon(\alpha) = \{x \in I \mid |x - \alpha| < \epsilon\}$ such that $F(x_1) = F(x_2)$.

We call a continuous function $F : I \to I$ is a *piecewise monotone (PM) function*, if F has only finitely many forts. The collection of all *PM functions* from I into I is denoted by $PM(I)$.

If $S(F)$ and $N(F)$ denotes the set of all forts and the number of forts of F respectively, then it is easy to observe that every fort of F is a fort of F^n for all $n \in \mathbb{N}$ and hence $\{N(F^n)\}$ is a non-decreasing sequence of non-negative integers.

Proposition 1.3.12. *(Zhang, 1997) Let $F \in PM(I)$. If $N(F^m) = N(F^{m+1})$, then $N(F^m) = N(F^{m+i})$ for all $i \in \mathbb{N}$ and F is strictly monotone on the range of F^m.*

Definition 1.3.13. *(Zhang, 1997) The height of a PM function $F \in PM(I)$, denoted by $H(F)$, is defined to be the least non-negative integer m such that $N(F^m) = N(F^{m+1})$, if it exists. Otherwise, $H(F) = \infty$.*

Example 1.3.14. *Any strictly increasing continuous function from an interval onto itself is of height zero.*

Example 1.3.15. *Consider $f : [0, 1] \to [0, 1]$ defined by*

$$f(x) := \begin{cases} x, & \text{if } x \in [0, \frac{1}{4}) \\ \frac{3}{8} - \frac{x}{2}, & \text{if } x \in [\frac{1}{4}, \frac{3}{4}) \\ 2x - \frac{3}{2}, & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

It is easy to observe that

$$f^2(x) = f^3(x) = \begin{cases} x, & \text{if } \ x \in [0, \frac{1}{4}) \\ \frac{3}{8} - \frac{x}{2}, & \text{if } \ x \in [\frac{1}{4}, \frac{3}{4}) \\ 2x - \frac{3}{2}, & \text{if } \ x \in [\frac{3}{4}, \frac{7}{8}) \\ \frac{9}{8} - x, & \text{if } \ x \in [\frac{7}{8}, 1]. \end{cases}$$

Here $S(f) = \{\frac{1}{4}, \frac{3}{4}\}$ *and* $S(f^2) = \{\frac{1}{4}, \frac{3}{4}, \frac{7}{8}\}$*. Moreover* $N(f) < N(f^2) = N(f^3)$*. This shows that* $H(f) = 2$*.*

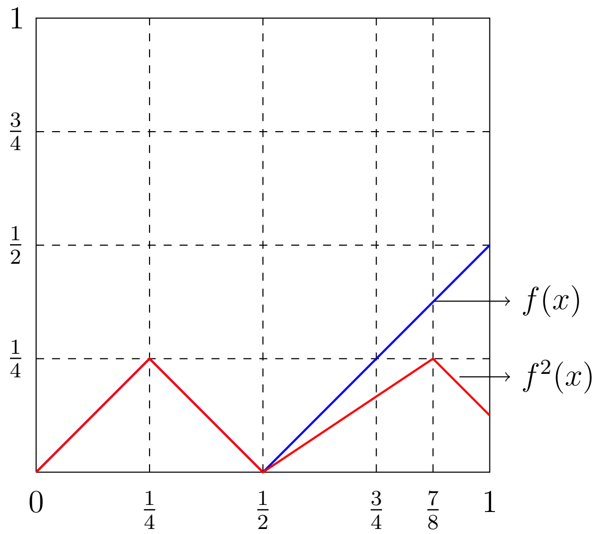

Figure. 1.3.1

Example 1.3.16. *Let* $T : [0, 1] \to [0, 1]$ *be the tent map defined by*

$$T(x) = \begin{cases} 2x, & \text{if } \ x \in [0, \frac{1}{2}), \\ 2 - 2x, & \text{if } \ x \in [\frac{1}{2}, 1]. \end{cases}$$

Then $S(T^m) = \{\frac{1}{2^m}, \frac{2}{2^m}, \ldots, \frac{2^m-1}{2^m}\}$ *for all* $m \in \mathbb{N}$*, whence* $H(T) = \infty$*.*

Zhang and Yang (Zhang and Yang, 1983) defined the characteristic interval of PM functions of height less than or equal to one written in Chinese, however Zhang (Zhang, 1997) published it in English as follows:

Let $F : I \to I$ be a PM function of height less than or equal to one. Then, by Proposition 1.3.12, F is strictly monotone on the range of F. Now, by extending the interval on which F is monotone, there exist two points a', b' such that

(i) $[a', b'] \supseteq R(F)$,

(ii) a' and b' are either forts or endpoints,

(iii) there is no fort inside (a', b').

Definition 1.3.17. *(Zhang, 1997) The unique interval $[a', b']$ defined above is the characteristic interval of F, denoted by Ch_F.*

Characteristic interval plays an important role on the existence of iterative roots of a PM function of height less than or equal to one. The monotonicity of F on the characteristic interval gives the existence of solution of the iterative functional equation (1.2.3) on the characteristic interval based on Theorem 1.3.6 and Theorem 1.3.7. Therefore, to study the existence of iterative root on the whole interval I, it is enough to study the possible extension of the iterative root of the function from the characteristic interval to the whole interval. The following theorem gives one such extension:

Theorem 1.3.18. *(Zhang, 1997) Let $F \in PM(I)$ with $H(F) \leq 1$ and $F_0 = F|_{Ch_F}$. Suppose*

(i) there exists a continuous function f_0 such that $f_0^n = F_0$ on Ch_F and

(ii) $F(I) \subseteq F(Ch_F)$.

Then there exists a continuous function f from I into I such that $f(x) = f_0(x)$ for all $x \in Ch_F$ and $f^n = F$ on I.

Theorem 1.3.19. *(Liu and Zhang, 2011) Every continuous iterative root of a PM function F with $H(F) \leq 1$ is an extension of an iterative root of F of the same order from the characteristic interval of F.*

We now discuss the iterative roots of a PM function of height greater than one. Li and Chen generalized Theorem 1.3.19 for any PM functions with finite height.

Theorem 1.3.20. *(Li and Chen, 2014) Let $F \in PM(I)$ and $H(F) = k$. Then every continuous iterative root of F is an extension of an iterative root of F of the same order from the characteristic interval of F^k.*

The following theorem gives the nonexistence of iterative roots of PM functions of height greater than one.

Theorem 1.3.21. *(Zhang, 1997) Let $F \in PM(I)$ and $H(F) > 1$. Then F has no continuous iterative roots of order n, for $n > N(F)$.*

Having proved that a PM function F with $H(F) \geq 2$ has no continuous iterative root of order n for $n > N(F)$, Zhang raised the following problem (Zhang, 1997):

Problem 1.3.22. *Does there exist iterative roots of order n of a PM function $F \in PM(I)$ with $H(F) \geq 2$ for $n \leq N(F)$?*

The article by Liu et al. (Liu et al., 2012) offers a necessary and sufficient condition for the existence of iterative root of F for the case $n = N(F)$, which we will discuss in Chapter 2. The results on iterative roots of non-PM functions, i.e., functions having infinitely many forts, can be found in (Lin, 2014; Lin et al., 2017). In both papers the authors discussed the existence and nonexistence of iterative roots of non-PM functions which are constant on some subinterval and strictly monotone elsewhere. One can refer (Baron and Jarczyk, 2001; Zdun and Solarz, 2014) for a detailed survey of recent results on iterative roots.

1.3.3 Some Generalizations

There are many iterative functional equations which is a generalization of iterative root problem. Zhao (Zhao, 1983) discussed the existence and uniqueness of solutions the following functional equation of the form

$$\lambda_1 f(x) + \lambda_2 f^2(x) = F(x). \tag{1.3.7}$$

Mukherjea and Ratti (Mukherjea and Ratti, 1983) studied the functional equation of the form

$$\sum_{i=0}^{n} c_i f^i(x) = 0, \tag{1.3.8}$$

where c_i's are positive real numbers.

W. Zhang (Zhang, 1988) further generalized the iterative functional equation (1.2.3) into the following functional equation

$$\sum_{i=1}^{n} \lambda_i f^i(x) = F(x), \tag{1.3.9}$$

where $F : I \to I$ is a given map, $f : I \to I$ is an unknown map, and all λ_i $(i = 1, \ldots, n)$ are real constants. The functional equation (1.3.9) is known as *polynomial-like iterative functional equation*. It is easy to observe that the problem of finding solution of the functional equation (1.3.9) reduces to the iterative root problem when $\lambda_n = 1$ and $\lambda_i = 0$ for $1 \leq i \leq n-1$.

Zhang (Zhang, 1988) proved the existence and uniqueness of continuous solutions of the polynomial-like iterative functional equation for the class of continuous strictly increasing functions using fixed point theory.

By generalizing the polynomial-like iterative functional equation (1.3.9), Jarczyk (Jarczyk, 1997) considered the following functional equation of the form

$$\sum_{i=1}^{\infty} a_i f^i(x) = x, \tag{1.3.10}$$

however a more generalized functional equation of the polynomial-like iterative functional equation (1.3.9) was considered by Murugan and Subrahmanyam (Murugan and Subrahmanyam, 2005). The authors considered the functional equation of the form

$$\sum_{i=1}^{\infty} \lambda_i f^i(x) = F(x) \tag{1.3.11}$$

and discussed the existence and uniqueness of continuous solutions of the above functional equation for the class of continuous strictly increasing functions using fixed point theory. The equation (1.3.11) is known as *series-like iterative functional equation.*

For a more detailed study on the existence of continuous, differentiable solutions of the functional equation (1.3.11) and its generalization, one can refer (Jarczyk, 1987; Murugan and Subrahmanyam, 2005, 2009). It is notable that the existence of the solutions of the polynomial-like and series-like functional equations has been studied only for the class of strictly monotone functions. Recently Liu et al., (Liu and Gong, 2017), proved the existence of solutions of polynomial-like iterative functional equations for PM functions of height less than or equal to one, however, the problem on the existence of solutions of series-like iterative functional equations for PM functions is unsolved.

1.4 Outline of the Remaining Chapters

Chapter 2 deals with the iterative root problem for the class of PM functions with height greater than one. Our main contributions are the following:

- We defined characteristic interval for any PM functions by generalizing the definition of characteristic interval.
- We proved results on existence of iterative roots of order less than the number of forts of PM function of height greater than one on its characteristic interval.
- We also obtained theorem on extension of iterative roots from the characteristic interval to the whole interval.

Thus Chapter 2 provides an affirmative answer to the Problem 1.3.22.

In Chapter 3, we considered the iterative root problem for the class of Non-PM functions. We discussed the iterative roots for the class of Non-PM functions which are non-constant in any interval, but having infinitely many forts. The main results are the following:

- Extension of iterative roots of continuous functions having infinitely many forts from the characteristic interval to the whole interval, if it exists on the characteristic interval.
- By generalizing the method of characteristic interval, we proved results on nonexistence of iterative roots of continuous functions having infinitely many forts for a special class of functions.
- We also proved that the set of continuous functions from I into itself which do not possess iterative roots are dense in $C(I)$.

In Chapter 4, our main result describes the existence of solutions of series-like iterative functional equation of the form

$$\sum_{i=1}^{\infty} \lambda_i f^i(x) = F(x),$$

for the class of PM functions of height less than or equal to one using the method of characteristic interval.

Chapter 5 concludes our thesis with the discussion of uniqueness of iterative roots of order preserving homeomorphism. Our main results provide some sufficient conditions on uniqueness of iterative roots of order preserving homeomorphism. Indeed, we proved that an order preserving homeomorphism from an interval onto itself does not possess different iterative roots which are subcommuting or comparable using the points of coincidence of functions.

Chapter 2

Iterative Roots of PM Functions

In this chapter, we investigate iterative root problem for the class of PM functions of height greater than one. For any $F \in PM(I)$ with $H(F) \geq 2$ does not possess iterative roots of order n, $n > N(F)$ (cf. Theorem 1.3.21). Liu et al. (Liu et al., 2012) gave a necessary and sufficient condition for the existence of iterative root of F for the case $n = N(F)$.

2.1 Introduction

We begin our discussion with few results from (Liu et al., 2012).

Definition 2.1.1. *(Liu et al., 2012) A strictly increasing function ϕ on I into itself is said to be a reversing correspondence, if there exists a fixed point ξ of ϕ and a strictly decreasing function ψ maps the fixed points of ϕ onto itself which fixes ξ such that for every consecutive fixed points ξ_1 and ξ_2 of ϕ the expression $\phi(x) - x$ has opposite signs in the intervals (ξ_1, ξ_2) and $(\psi(\xi_2), \psi(\xi_1))$.*

An n^{th} iterative root f of a PM function F is called *type τ_1 (type τ_2)*, if f is strictly increasing (decreasing) on the smallest closed interval containing all the forts of F.

Theorem 2.1.2. *(Liu et al., 2012) Let $F \in PM(I)$ with $H(F) \geq 2$ and assume that $N(F) \geq 2$. Suppose that $c_1, c_2, \ldots, c_n$ are forts of F with $c_1 < c_2 < \cdots < c_n$. Then F has a continuous iterative root of order n, $n = N(F)$, of type τ_1 if and only if one of the following conditions is fulfilled:*

(i) n is even, $F|_{[a,c_1]}$ is a reversing correspondence,

$$\begin{aligned} F(a) &\geq F(c_2) \geq \cdots \geq F(c_{n-2}) \geq F(c_n) \geq a, \\ F(c_1) &\leq F(c_3) \leq \cdots \leq F(c_{n-3}) \leq F(c_{n-1}) \leq c_1 \end{aligned} \tag{2.1.1}$$

and either all inequalities of (2.1.1) are equalities, or at most one of them, namely $F(a) \geq F(c_2)$ or $F(c_n) \geq a$, is an equality ;

(ii) n is odd, $F|_{[a,c_1]}$ is decreasing,

$$\begin{aligned} F(a) &\leq F(c_2) \leq \cdots \leq F(c_{n-1}) \leq c_1, \\ F(c_1) &\geq F(c_3) \geq \cdots \geq F(c_n) \geq a \end{aligned} \tag{2.1.2}$$

and either all inequalities of (2.1.2) are equalities, or at the most one of them, namely $F(a) \leq F(c_2)$ or $F(c_n) \geq a$, is an equality;

(iii) n is even, $F|_{[c_n,b]}$ is a reversing correspondence,

$$\begin{aligned} F(b) &\leq F(c_{n-1}) \leq \cdots \leq F(c_1) \leq b, \\ F(c_n) &\geq F(c_{n-2}) \geq \cdots \geq F(c_2) \geq c_n \end{aligned} \tag{2.1.3}$$

and either all inequalities of (2.1.3) are equalities, or at the most one of them, namely $F(b) \leq F(c_{n-1})$ or $F(c_1) \leq b$, is an equality;

(iv) n is odd, $F|_{[c_n,b]}$ is decreasing,

$$\begin{aligned} F(b) &\geq F(c_{n-1}) \geq \cdots \geq F(c_2) \geq c_n, \\ F(c_n) &\leq F(c_{n-2}) \geq \cdots \geq F(c_1) \leq b \end{aligned} \tag{2.1.4}$$

and either all inequalities of (2.1.4) are equalities, or at the most one of them, namely $F(b) \geq F(c_{n-1})$ or $F(c_1) \leq b$, is an equality.

Now we remark here that, the existence of n^{th} iterative roots of PM functions F of height greater than two has been solved only for the case $n = N(F)$. So we observe the the following problems.

Problem 2.1.3. *(Liu et al., 2012) Does any $F \in PM(I)$ with $H(F) \geq 2$ have a type τ_2 iterative root of order n, for $n = N(F)$?*

Problem 2.1.4. *(Liu et al., 2012) Does any $F \in PM(I)$ with $H(F) \geq 2$ have an iterative root of order n, for $n < N(F)$?*

This chapter focuses Problem 2.1.4. We provide necessary conditions on the existence of iterative roots of order $n < N(F)$ for PM functions F with $H(F) \geq 2$. At first we generalize the definition of characteristic interval which defined for the class of PM functions of height less than two to the class of all PM functions. Then, by producing iterative roots of F in its characteristic interval, we extend that iterative root of F to the whole interval.

2.2 Properties of Forts

The problem of finding n^{th} iterative root of PM function F depends on the forts, it is worth studying the properties of forts under iteration. It is known that, for any $f \in PM(I)$, $S(f^{n-1}) \subseteq S(f^n)$ for all $n \in \mathbb{N}$. It is possible that $S(f^{n-1})$ can be a proper subset of $S(f^n)$, the following proposition describes how the new forts have been generated under the iteration of f.
For $x \in S(f)$, $n \in \mathbb{N}$, we define

$$S_x^n(f) := (\cup_{m=0}^{n-1} f^{-m}(x)) \cap I^0, \tag{2.2.5}$$

where $f^{-m}(x) = \{y \in I \mid f^m(y) = x\}$ and I^0 is the interior of I. It is easy to observe that $S_x^1(f) = \{x\}$ and $S(f) = \cup_{x \in S(f)} S_x^1$. The following proposition gives a more general result.

Proposition 2.2.1. *$S(f^n) = \cup_{x \in S(f)} S_x^n(f)$ for $f \in PM(I)$ and $n \in \mathbb{N}$.*

Proof. Let $t \in S_x^n(f)$ and $f^m(t) = x$, for some m with $1 \leq m \leq n-1$. If $t \in \cup_{i=1}^{n-1} S(f^i)$, then obviously $t \in S(f^n)$. Assume $t \notin S(f^i)$ for any $i = 1, 2, \ldots, n-1$

and let $\epsilon > 0$ be given. Note that f^m is strictly monotone at t. Now, by the continuity of f^m at t, choose $\delta_\epsilon > 0$ such that

$$f^m(t-\epsilon, t+\epsilon) \subseteq (f^m(t) - \delta_\epsilon, f^m(t) + \delta_\epsilon).$$

Since $x = f^m(t)$ is a fort for f, there exist

$$y_1 \in (f^m(t) - \delta_\epsilon, f^m(t)) \cap f^m(t-\epsilon, t+\epsilon),$$

and

$$y_2 \in (f^m(t), f^m(t) + \delta_\epsilon) \cap f^m(t-\epsilon, t+\epsilon)$$

such that $f(y_1) = f(y_2)$. Therefore, by intermediate value theorem, there exist $x_1, x_2 \in (t-\epsilon, t+\epsilon)$ such that $x_1 \neq x_2$ and $f^m(x_1) = y_1$ and $f^m(x_2) = y_2$ so that $f^{m+1}(x_1) = f^{m+1}(x_2)$. Hence $t \in S(f^{m+1}) \subseteq S(f^n)$.

On the other hand, suppose $t \in S(f^n)$. Let $m(\leq n)$ be the least positive integer such that $t \in S(f^m)$. We prove $t = f^{-(m-1)}(x)$ for some $x \in S(f)$. If $f^{m-1}(t) \neq x$ for any $x \in S(f)$, then f is monotone at $f^{m-1}(t)$, which in turn implies that f^m is monotone at t, a contradiction. $\square$

2.3 Generalization of Characteristic Interval

J. Zhang and L. Yang defined characteristic interval for PM functions of height less than or equal to one (Zhang and Yang, 1983) (cf. Definition (1.3.17)). We observe that every PM function (not necessarily be of with height ≤ 1) possesses a similar interval. Motivated by J. Zhang and L. Yang (Zhang and Yang, 1983), we define the characteristic interval for any PM function as follows:

Let $F \in PM(I)$ be any function and let $[m, M]$ be its range. Put $a' = a$, if F has no fort on $[a, m]$, otherwise $a' = \sup\{x \in [a, m] \mid x \in S(F)\}$ and put $b' = b$, if F has no forts on $[M, b]$, otherwise $b' = \inf\{x \in [M, b] \mid x \in S(F)\}$. Then the interval $[a', b']$ posses the following properties:

(a) $a', b' \in S(F) \cup \{a, b\}$.

(b) If $[\alpha, \beta]$ any subinterval of I containing $[m, M]$ with $\alpha, \beta \in S(F) \cup \{a, b\}$, then $[a', b'] \subseteq [\alpha, \beta]$.

Suppose there is an interval, say $[t_1, t_2]$, posses the properties (a) and (b). Then $[t_1, t_2] \subseteq [a', b']$ by property (b). Since $t_1 \in S(F) \cup \{a, b\}$ and $t_1 \leq m$, by definition of a', $t_1 \leq a'$. Similarly $b' \leq t_2$. Hence $[a', b'] = [t_1, t_2]$. Therefore the interval $[a', b']$ is unique and has the properties (a) and (b).

Definition 2.3.1. *Let $F : I \to I$ be any PM function. Then the smallest closed interval containing the range of F whose end points are either forts of F or the end points of $[a, b]$, denoted by Ch_F, is called the characteristic interval of F.*

We do call Ch_F as the *characteristic interval* because of the natural generalization of Definition 1.3.17. Indeed, if F is a PM function of height less or equal to one, then Definition 1.3.17 and Definition 2.3.1 are equivalent.

Example 2.3.2. *Let $f : [-\pi, \pi] \to [-\pi, \pi]$ be defined as $f(x) = \sin x$. Then $Ch_F = [\frac{-\pi}{2}, \frac{\pi}{2}]$ (See Figure 2.2.1).*

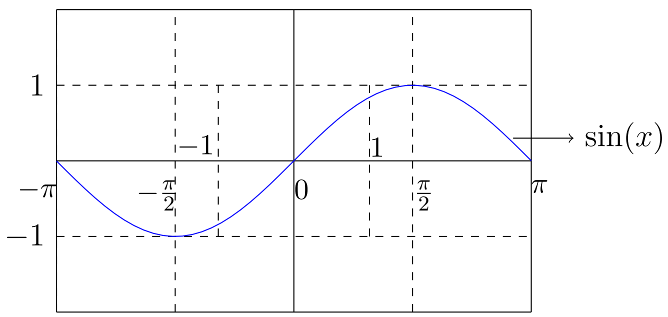

Figure 2.2.1

Example 2.3.3. *The characteristic interval of the tent map $T : [0, 1] \to [0, 1]$ defined by*

$$T(x) = \begin{cases} 2x, & \textit{if} \quad x \in [0, \frac{1}{2}), \\ 2 - 2x, & \textit{if} \quad x \in [\frac{1}{2}, 1], \end{cases}$$

is the interval $[0, 1]$.

2.4 Existence of Iterative Roots

Our aim is to establish the existence of n^{th} iterative root of $F \in PM(I)$ with $H(F) \geq 2$ and $n < N(F)$. The following results describes the behavior of such iterative roots.

Proposition 2.4.1. *Let $F \in PM(I)$ with $H(F) \geq 2$. Suppose there exists a function $f \in PM(I)$ that satisfies the functional equation $f^n = F$ on I. Then $N(f) \leq N(F) - (n-1)$.*

Proof. Since $H(F) \geq 2$, we have

$$N(f) < N(f^2) < \ldots < N(f^{n-1}) < N(f^n) = N(F) < N(F^2). \tag{2.4.6}$$

This implies $N(F) - N(f) \geq n-1$ so that $N(f) \leq N(F) - (n-1)$. □

Lemma 2.4.2. *Let $F \in PM(I)$ with $H(F) \geq 2$. Suppose that a function $f \in PM(I)$ satisfies the functional equation $f^n = F$ on I with $N(f) = N(F) - (n-1)$. Then there exists a fort $c \in S(F)$ such that $S(F) = S(f) \cup S_c^n(f)$. Moreover $|S_c^n(f) \backslash S(f)| = n-1$.*

Proof. Equation (2.4.6) forces that on each iteration of f only one new fort has to be generated, hence $S(f^{i+1}) \backslash S(f^i)$ $(1 \leq i \leq n-1)$ is a singleton set. Let $\{x_i\} = S(f^{i+1}) \backslash S(f^i)$ $(1 \leq i \leq n-1)$. Further, in view of Proposition 2.2.1, we have

$$S(f^l) = S(f) \cup \{x_1, x_2, \ldots, x_{l-1}\} \text{ for } 2 \leq l \leq n,$$

and

$$f^i(x_i) \in S(f) \text{ for } 1 \leq i \leq l-1.$$

To prove this lemma, it is enough to prove $f^l(x_l) = c$, for $2 \leq l \leq n-1$ and for some $c \in S(f)$. We prove this result using induction on l. As $H(f) \geq 2$, for each $x \in S(f)$, $f^{-1}(x) \subseteq S(f)$ is true except at one fort, call it c. Hence

$$S(f^2) = S(f) \cup \{x_1\} \quad \text{where } x_1 \in f^{-1}(c).$$

Suppose $S(f^3) = S(f) \cup \{x_1, x_2\}$ for some $x_2 \in f^{-2}(c')$ where $c' \in S(f)$. Put $f(x_2) = t$. If $c' \neq c$, then $t \in S(f)$. Therefore $x_2 \in f^{-1}(t) \subseteq S(f^2)$, a contradiction. This concludes that $c = c'$ and $S(f^3)$ becomes

$$S(f^3) = S(f) \cup \{x_1, x_2\} \text{ where } f(x_1) = f^2(x_2) = c.$$

Assume,

$$S(f^l) = S(f) \cup \{x_i\}_{i=1}^{l-1} \text{ and } f^i(x_i) = c \text{ for } 1 \leq i < l < n. \tag{2.4.7}$$

Note that $S(f^{l+1}) = S(f) \cup \{x_1, x_2, \ldots, x_{l-1}, x_l\}$, where $f^l(x_l) = c'$ for some $c' \in S(f)$. Put $t = f^{l-1}(x_l)$. If $c \neq c'$, then $t \in S(f)$. Therefore by (2.4.7), $x_l \in f^{-l+1}(t) \subseteq S(f^l)$, a contradiction as $x_l \notin S(f^l)$. Hence $c = c'$ and by induction hypothesis,

$$S(F) = S(f^n) = S(f) \cup \{x_1, x_2, \ldots, x_{n-1}\},$$

where $f^l(x_l) = c,\ c \in S(f)$ for $1 \leq l \leq n-1$. Since $\{x_1, x_2, \ldots, x_{n-1}\} \subseteq S_c^n(f)$ and, by Proposition 2.2.1,

$$S(F) = \cup_{x \in S(f)} S_x^n(f) = S(f) \cup S_c^n(f).$$

Also $S_c^n(f) \backslash S(f) = \{x_1, x_2, \ldots, x_{n-1}\}$ implies $|S_c^n(f) \backslash S(f)| = n-1$. $\square$

Proposition 2.4.1 asserts that any n^{th} iterative root of F, $H(F) \geq 2$ and $n < N(F)$, possesses at least $N(F) - (n-1)$ number of forts. We now concentrate on constructing the n^{th} ($n < N(F)$) iterative root f of a PM function F with $H(F) \geq 2$ such that $N(f) = N(F) - (n-1)$. The following theorem gives a necessary condition for the existence of iterative roots of F.

Theorem 2.4.3. *Let $F \in PM(I)$ with $H(F) \geq 2$. Suppose F has an n^{th} iterative root $f \in PM(I)$ such that $N(f) = N(F) - (n-1)$. Then $N(F|_{Ch_f}) = n$ and $N(f|_{Ch_f}) = 1$.*

Proof. Let $S(f) = \{c_1, c_2, \ldots, c_{N(F)-(n-1)}\}$ with $c_1 < c_2 < \ldots < c_{N(F)-(n-1)}$. By Lemma 2.4.2 we have,

$$S(F) = S(f) \cup \{x_1, x_2, \ldots, x_{n-1}\}, \text{ where } f^j(x_j) = c_i \ (1 \leq j \leq n-1) \tag{2.4.8}$$

for some i. Clearly, f is strictly monotone at all these x_i, in particular at x_1. We prove the result by assuming f is increasing at x_1, the case at which f is strictly decreasing at x_1 can be proved similarly. Note that $c_i \neq x_1$.

Case 1: $c_i < x_1$

In this case we prove $x_j \in (c_i, c_{i+1})$ for $1 \leq j \leq n-1$. If $c_{i+1} < x_1$, then $f[a, x_1] \subseteq [c_{i-1}, c_i]$ and $f[x_1, b] \subseteq [c_i, c_{i+1}]$ so that

$$f^2[a, b] \subseteq [c_{i-1}, c_i].$$

This leads a contradiction to $H(F) \geq 2$ and hence $x_1 < c_{i+1}$. Since $f(x_2) = x_1$, and $f[a, x_1] \subseteq [c_{i-1}, c_i]$, we have $x_1 < x_2$. Suppose that $c_{i+1} < x_2$. Then $f(c_{i+1}) \leq x_1$. If not, there exists an $y \in (x_1, c_{i+1})$ such that $f(y) = x_1$. Therefore, by Proposition 2.2.1, $y \in S(f^3) = S(f) \cup \{x_1, x_2\}$, a contradiction. As $f(c_{i+1}) \leq x_1$, we have $f[a, x_2] \subseteq [c_{i-1}, x_1]$ and $f[x_2, b] \subseteq [x_1, c_{i+1}]$ so that

$$f^3[a, b] \subseteq [c_{i-1}, c_i],$$

again a contradiction to $H(F) \geq 2$. Thus $c_i < x_1 < x_2 < c_{i+1}$ and f is increasing on $[c_i, x_2]$ (see Figure. 2.2.2).

Assume that $c_i < x_1 < x_2 < \ldots < x_m < c_{i+1}$ for $m < n-1$. Note that f is increasing on $[c_i, x_m]$. Since $f(x_{m+1}) = x_m$ and $f[a, x_m] \subseteq [c_{i-1}, x_{m-1}]$ we have $x_m < x_{m+1}$. If $c_{i+1} < x_{m+1}$, then $f(c_{i+1}) \leq x_m$ and $f[a, x_{m+1}] \subseteq [c_{i-1}, x_m]$, $f[x_{m+1}, b] \subseteq [x_m, c_{i+1}]$ which implies that

$$f^{m+2}[a, b] \subseteq [c_{i-1}, c_i],$$

a contradiction to $H(F) \geq 2$. Hence, by induction hypothesis, we have

$$c_i < x_1 < x_2 < \ldots < x_{n-1} < c_{i+1} \tag{2.4.9}$$

and f is increasing on $[c_i, c_{i+1}]$ (see Figure. 2.2.2). In this case, if $f(c_{i+1}) \leq x_{n-1}$ then $f^n[a,b] = F[a,b] \subseteq [c_{i-1}, c_i]$, a contradiction to $H(F) \geq 2$. Therefore $f(c_{i+1}) > x_{n-1}$ and hence the characteristic interval of f is $Ch_f = [c_{i-1}, c_{i+1}]$, $N(F|_{Ch_f}) = n$ and $N(f|_{Ch_f}) = 1$.

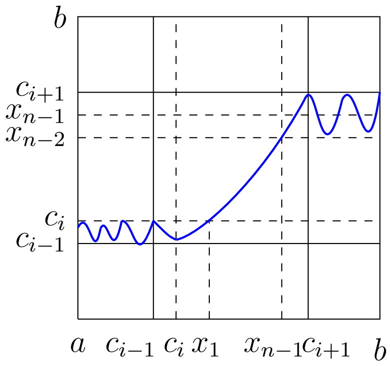

Figure. 2.2.2

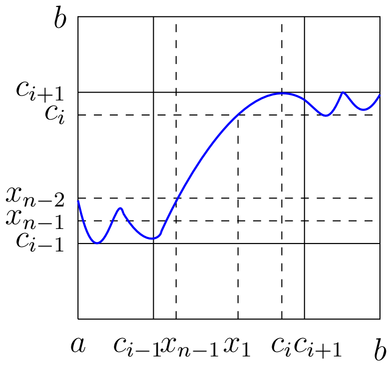

Figure. 2.2.3

Case 2: $c_i > x_1$

In this case we prove $x_j \in (c_{i-1}, c_i)$ for $1 \leq j \leq n-1$. If $x_1 < c_{i-1}$, then $f[a, x_1] \subseteq [c_{i-1}, c_i]$ and $f[x_1, b] \subseteq [c_i, c_{i+1}]$ so that

$$f^2[a,b] \subseteq [c_i, c_{i+1}]$$

which leads a contradiction to the fact that $H(F) \geq 2$. Hence $c_{i-1} < x_1$.

Since $f(x_2) = x_1$, and $f[x_1, b] \subseteq [c_i, c_{i+1}]$, we have $x_2 < x_1$. Suppose that $x_2 < c_{i-1}$. Then $f(c_{i-1}) \geq x_1$. If not, there exists an $y \in (c_{i-1}, x_1)$ such that $f(y) = x_1$. Therefore, by Proposition 2.2.1, $y \in S(f^3) = S(f) \cup \{x_1, x_2\}$, a contradiction. Hence we have $f[a, x_2] \subseteq [c_{i-1}, x_1]$ and $f[x_2, b] \subseteq [x_1, c_{i+1}]$ so that

$$f^3[a,b] \subseteq [c_i, c_{i+1}],$$

again a contradiction to $H(F) \geq 2$. Therefore $c_{i-1} < x_2 < x_1 < c_i$ and f is increasing on $[x_2, c_i]$ (See Figure. 2.2.3).

Assume that $c_{i-1} < x_m < x_{m-1} < \ldots < x_2 < x_1 < c_i$, for $m < n-1$. As above, we have f is increasing on $[x_m, c_i]$. Since $f(x_{m+1}) = x_m$ and $f[x_m, b] \subseteq [x_{m-1}, c_{i+1}]$

we have $x_{m+1} < x_m$. If $x_{m+1} < c_{i-1}$, then $f(c_{i-1}) \geq x_m$ and $f[a, x_{m+1}] \subseteq [c_{i-1}, x_m]$, $f[x_{m+1}, b] \subseteq [x_m, c_{i+1}]$ which implies that

$$f^{m+2}[a, b] \subseteq [c_i, c_{i+1}],$$

a contradiction to $H(F) \geq 2$. Hence by the induction hypothesis we have

$$c_{i-1} < x_{n-1} < \ldots < x_2 < x_1 < c_i \tag{2.4.10}$$

and f is increasing on $[c_{i-1}, c_i]$ (See Figure. 2.2.3). In this case, if $f(c_{i-1}) \geq x_{n-1}$, then $f^n[a, b] = F[a, b] \subseteq [c_i, c_{i+1}]$, a contradiction to $H(F) \geq 2$. Hence $f(c_{i-1}) < x_{n-1}$ and it is clear that the characteristic interval of f is $[c_{i-1}, c_{i+1}]$ and $N(F|_{Ch_f}) = n$ and $N(f|_{Ch_f}) = 1$.

□

We present the following remark as in (Liu et al., 2012).

Remark 2.4.4. *Suppose f be the function described as in case 1 of Theorem 2.4.3. Let $h : I = [a, b] \to I$ be the homeomorphism defined by $h(x) = a + b - x$ for all $x \in I$ and let $g : I \to I$ be the function $g(x) = h^{-1} \circ f \circ h(x)$ for all $x \in I$. If $S(f) = \{c_1, c_2, \ldots, c_{N(F)-(n-1)}\}$, then from equations (2.4.8) and (2.4.9), we have,*

$$\left.\begin{array}{c} f(x_1) = c_i, f(x_2) = x_1, \ldots, f(x_{n-1}) = x_{n-2}, \\ c_i < x_1 < x_2 < \cdots < x_{n-1} < c_{i+1}, \end{array}\right\} \tag{2.4.11}$$

and

$$\left.\begin{array}{rcl} f([a, x_1]) & \subseteq & [c_{i-1}, c_i] \\ f([x_1, c_{i+1}]) & \subseteq & [c_i, f(c_{i+1})] \\ f([c_{i+1}, b]) & \subseteq & [x_{n-2)}, c_{i+1}]. \end{array}\right\} \tag{2.4.12}$$

Moreover, f is strictly increasing on $[c_i, c_{i+1}]$.
Put

$$d_i = a + b - c_i, \textit{ for } 1 \leq i \leq N(F) - (n-1)$$

and

$$y_i = a + b - x_i, \textit{ for } 1 \leq i \leq n-1.$$

Then $S(g) = \{d_i \mid 1 \leq i \leq N(F) - (n-1)\}$ *and*

$$\left.\begin{array}{l} g(y_1) = d_i, g(y_2) = y_1, \ldots, g(y_{n-1}) = y_{n-2}, \\ d_{i-1} < y_{n-1} < y_{n-2} < \cdots < y_2 < y_1 < d_i. \end{array}\right\} \tag{2.4.13}$$

and g *is strictly increasing on* $[d_{i-1}, d_i]$. *Moreover,*

$$\left.\begin{array}{rcl} g([a, y_{n-1}]) & \subseteq & [d_{i-1}, y_{n-2}] \\ g([y_{n-1}, d_i]) & \subseteq & [d_{i-1}, g(d_i)] \\ g([d_i, b]) & \subseteq & [d_i, d_{i+1}]. \end{array}\right\} \tag{2.4.14}$$

Also, it is easy to observe that $g^n = G$, *where,* $G : I \to I$ *the function defined by* $G(x) = h^{-1} \circ F \circ h(x)$ *for all* $x \in I$ *with* $H(G) \geq 2$ *and* $N(g) = N(G) - (n-1)$.

Hence, if f *is the function described in case 1 of Theorem 2.4.3, then the function* g *acts like the function described in case 2 of Theorem 2.4.3. Therefore, in order to study the iterative roots of functions satisfying the hypothesis of Theorem 2.4.3, it is enough to discuss the iterative roots described either in case 1 or case 2.*

Remark 2.4.5. *Suppose, for* $F \in PM(I)$ *with* $H(F) \geq 2$, *there exists* $f \in PM(I)$ *such that* $f^n = F$ *and* $N(f) = N(F) - (n-1)$, $n < N(F)$. *Then by Theorem 2.4.3, there exist* $x_1, x_2, \ldots, x_{n-1} \in S(F)$ *such that* $f(x_1) = c_i$, $f(x_2) = x_1, \ldots,$ $f(x_{n-1}) = x_{n-2}$ *for some* $c_i \in S(F)$. *Using this behavior of the function* f, *we can predict the iteration of the function* f. *In fact, for* $i \in \{1, 2, \ldots, n-1\}$,

$$\left.\begin{array}{rcl} f^i([a, x_i]) & \subseteq & [c_{i-1}, c_i] \\ f^i([x_i, c_{i+1}]) & \subseteq & [c_i, f^i(c_{i+1})] \\ f^i([c_{i+1}, b]) & \subseteq & [x_{n-(i+1)}, f^{i-1}(c_{i+1})] \end{array}\right\} \tag{2.4.15}$$

and

$$\left.\begin{array}{rcl} f^n([a, x_{n-1}]) & \subseteq & [c_{i-1}, c_i] \\ f^n([x_{n-1}, c_{i+1}]) & \subseteq & [c_i, f^n(c_{i+1})] \\ f^n([c_{i+1}, b]) & \subseteq & [x_1, f^{n-1}(c_{i+1})]. \end{array}\right\} \tag{2.4.16}$$

In particular, $F([a, b]) = f^n([a, b]) \subseteq [c_{i-1}, f^{n-1}(c_{i+1})]$. *Therefore, the characteristic interval of* F *is depending on the value of* $f^{n-1}(c_{i+1})$. *Also, it is an* *easy observation that* $Ch_F = Ch_f = [c_{i-1}, c_{i+1}]$ *when* $x_{n-1} < f^{n-1}(c_{i+1})$ *and* $Ch_F = [c_{i-1}, x_k]$ *when* $f^{n-1}(c_{i+1}) \leq x_{n-1}$*, where* k *is the least positive integer such that* $f^{n-1}(c_{i+1}) \leq x_k$*.*

The existence of iterative roots on the characteristic interval becomes necessary for the existence of iterative roots of PM functions, it is necessary to study the existence of iterative roots of PM functions on the characteristic interval.

In the rest of this chapter, we use the following notation: For $F \in PM(I)$ with $H(F) \geq 2$, $n < N(F)$ and $Ch_F = [a', b']$, denote $F_0 = F|_{Ch_F}$. As $H(F) \geq 2$, Ch_F contains at least one fort of F so that $H(F_0) \neq 0$. Also, by Remark 2.4.5, $N(F_0) < n$ when $H(F_0) = 1$ and $N(F_0) = n$ when $H(F_0) \geq 2$. We prove the existence of iterative root of F, by assuming $H(F_0) \geq 2$.

Suppose $H(F_0) \geq 2$ with $N(F_0) = n$. Let $S(F_0) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ with $a' = \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \alpha_{n+1} = b'$. Therefore, Theorem 2.1.2 guarantees the the existence of type τ_1 iterative root of F_0. Also, the following lemma will be useful in the extension of iterative roots.

Lemma 2.4.6. *(Liu et al., 2012) If* $f_0 \in PM(I)$ *is an type* τ_1 *iterative root of order* $n = N(F_0)$ *of* F*, then* $N(f_0) = 1$ *and* $S(f_0)$ *is either* $\{\alpha_1\}$ *or* $\{\alpha_n\}$*.*

2.5 Extension of Iterative Roots

In this section, by assuming the existence of n^{th} iterative of F on the characteristic interval, we extend that iterative root on I.

Theorem 2.5.1. *Let* $F \in PM(I)$ *with* $H(F) \geq 2$*. Let* $F_0 := F|_{Ch_F}$ *be such that* $H(F_0) \geq 2$*,* $N(F_0) = n$ *and* $n < N(F)$*. Suppose*

(a) F_0 *has* n^{th} *iterative root* f_0 *of type* τ_1*.*

(b) $F([a, a']) \subseteq F([a', \alpha_1])$*,* $F([b', b]) \subseteq F[\alpha_n, b']$*.*

Then F *has an* n^{th} *iterative root* $f \in PM(I)$ *such that* $N(f) = N(F) - (n-1)$*.*

Proof. Since $f_0^n = F_0$, by Lemma 2.4.6, either $S(f_0) = \{\alpha_1\}$ or $S(f_0) = \{\alpha_n\}$. The function f_0 is of type τ_1, we see that, f_0 is strictly increasing on $[\alpha_1, b']$ and f_0 assumes minimum at α_1 when $S(f_0) = \{\alpha_1\}$ and f_0 is strictly increasing on $[a', \alpha_n]$ and f_0 assumes maximum at α_n when $S(f_0) = \{\alpha_1\}$. Therefore, by Remark 2.4.5, it is enough to prove this result by assuming $S(f_0) = \{\alpha_1\}$.

Suppose $S(f_0) = \{\alpha_1\}$, we see that from Theorem 2.4.3 that,

$$f_0(\alpha_i) = \alpha_{i-1} \text{ for } 2 \leq i \leq n. \tag{2.5.17}$$

Note that the function f_0 is injective on $[a', \alpha_1]$ and f_0 maps $[a', \alpha_1]$ into $[a', \alpha_1]$.

$$\text{i.e., } f_0([a', \alpha_1]) \subseteq [a', \alpha_1]. \tag{2.5.18}$$

Also, f_0 is injective on $[\alpha_1, b']$. Now, for $i \in \{1, 2, \ldots, n-1\}$, let

$$\phi_i : f_0^i([a', \alpha_1]) \to f_0^{i+1}([a', \alpha_1])$$

and

$$\psi_i : f_0^i([\alpha_n, b']) \to f_0^{i+1}([\alpha_n, b'])$$

be the homeomorphisms defined by

$$\phi_i(x) = f_0(x) \text{ for all } x \in f_0^i([a', \alpha_1])$$

and

$$\psi_i(x) = f_0(x) \text{ for all } x \in f_0^i([\alpha_n, b']).$$

Now, define the function $f : I = [a, b] \to I$ by

$$f(x) := \begin{cases} \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F(x), & \text{if } \ x \in [a, a') \\ f_0(x), & \text{if } \ x \in [a', b'] \\ \psi_1^{-1} \circ \psi_2^{-1} \circ \ldots \circ \psi_{n-1}^{-1} \circ F(x), & \text{if } \ x \in (b', b]. \end{cases}$$

By hypothesis (b),

$$F([a, a']) \subseteq F([a', \alpha_1]) = f_0^n([a', \alpha_1])$$

and

$$F([b',b]) \subseteq F([\alpha_n,b']) = f_0^n([\alpha_n,b']),$$

the function f is well-defined and $f^n(x) = F(x)$ for all $x \in [a',b']$.

Also, for $x \in [a,a')$,

$$\begin{aligned} f^n(x) &= f_0^{n-1} \circ \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F(x) \\ &= f_0^{n-1} \circ \underbrace{f_0^{-1} \circ f_0^{-1} \circ \ldots \circ f_0^{-1}}_{\text{(n-1) times}} \circ F(x) \\ &= F(x). \end{aligned}$$

Similarly, for $x \in (b',b]$,

$$\begin{aligned} f^n(x) &= f_0^{n-1} \circ \psi_1^{-1} \circ \psi_2^{-1} \circ \ldots \circ \psi_{n-1}^{-1} \circ F(x) \\ &= f_0^{n-1} \circ \underbrace{f_0^{-1} \circ f_0^{-1} \circ \ldots \circ f_0^{-1}}_{\text{(n-1) times}} \circ F(x) \\ &= F(x). \end{aligned}$$

Now, to prove f is continuous on $[a,b]$, it is enough to prove f is continuous at a' and b'. If $a = a'$ then the continuity follows immediately from the definition of f. Suppose $a' \in (a,b)$. Let $(x_n) \in [a,a')$ be a sequence such that $x_n \to a'$ as $n \to \infty$. Since F is continuous at a' we have

$$F(x_n) \to F(a') = F_0(a') \text{ as } n \to \infty.$$

By hypothesis (b), $F(x_n) \in f_0^n([a',\alpha_1])$ for all n and $\phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1}$ is continuous at $F_0(a') = f_0^n(a')$, we have

$$\phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F(x_n) \to \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1}(F_0(a')) = f_0(a') \text{ as } n \to \infty.$$

Thus f is continuous at a'. Similarly we can prove f is continuous at b' and hence f is continuous on $[a,b]$. As F, f_0 are PM functions, we have $f \in PM(I)$.

To prove $N(f) = N(F) - (n-1)$, it is enough if we prove $S(f) = S(F) \setminus \{\alpha_2, \alpha_3 \ldots \alpha_n\}$. Since $S(f) = S(F)$ on $I \setminus Ch_F$ and α_1 is a fort of f_0, it is enough to prove $a', b' \in S(f)$ whenever $a', b' \in S(F)$. Since a' is a fort for F we have for every $\epsilon > 0$, there exist $x_1 \in (a' - \epsilon, a')$ and $x_2 \in (a', a' + \epsilon)$ such that

$$\begin{aligned}
F(x_1) &= F(x_2) \\
\Rightarrow \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F(x_1) &= \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F(x_2), \\
\Rightarrow \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F(x_1) &= \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F_0(x_2), \\
\Rightarrow \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F(x_1) &= \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ f_0^n(x_2), \\
\Rightarrow \phi_1^{-1} \circ \phi_2^{-1} \circ \ldots \circ \phi_{n-1}^{-1} \circ F(x_1) &= f_0(x_2), \\
\Rightarrow f(x_1) &= f(x_2).
\end{aligned}$$

i.e., for every $\epsilon > 0$, there exist $x_1 \in (a' - \epsilon, a')$ and $x_2 \in (a', a' + \epsilon)$ such that $f(x_1) = f(x_2)$. Hence a' is a fort for f. Similarly we can show that b' is a fort for f.

□

Corollary 2.5.2. *Let $F \in PM(I)$ with $H(F) \geq 2$. Let $F_0 := F|_{Ch_F}$ with $S(F_0) = \{\alpha_1, \alpha_2\}$ and $H(F_0) \geq 2$. Suppose $F([a, a']) \subseteq F([a, \alpha_1])$, $F([b', b]) \subseteq F([\alpha_2, b'])$ and F satisfies any one of the following condition:*

(a) $F_0|_{[a', \alpha_1]}$ is a reversing correspondence and

$$F_0([a', \alpha_1]) \subseteq F_0([\alpha_1, \alpha_2]) \subseteq [a', \alpha_1].$$

(b) $F_0|_{[\alpha_2, b']}$ is a reversing correspondence and

$$F_0([\alpha_2, b']) \subseteq F_0([\alpha_1, \alpha_2]) \subseteq [\alpha_2, b'].$$

Then F has square iterative root f such that $N(f) = N(F) - 1$ and has no iterative roots of order $n \geq 3$.

Proof. Suppose that there exists $f \in PM(I)$ such that $f^n = F$. As $H(F) \geq 2$ we have $H(f) \geq 2$ and therefore

$$N(f) < N(f^2) < \ldots < N(f^n) = N(F),$$

which is a contradiction to $N(f) = N(F) - 1$ when $n \geq 3$.

If F satisfies either (a) or (b), by Theorem 2.1.2, there exists an type τ_1 iterative root f_0 on Ch_{F_0}. By Theorem 2.5.1, f_0 can be extended to a PM function f on I such that $f^2 = F$ and $N(f) = N(F) - 1$. $\square$

2.6 Illustrative Examples

Example 2.6.1. *Consider the function $F : [0, 1] \to [0, 1]$ defined as follows*

$$F(x) = \begin{cases} x, & \text{if} \quad x \in [0, \frac{1}{6}) \\ \frac{2}{6} - x, & \text{if} \quad x \in [\frac{1}{6}, \frac{2}{6}) \\ x - \frac{2}{6}, & \text{if} \quad x \in [\frac{2}{6}, \frac{3}{6}) \\ \frac{3}{2}x - \frac{7}{12}, & \text{if} \quad x \in [\frac{3}{6}, \frac{11}{18}) \\ \frac{9}{4}x - \frac{25}{24}, & \text{if} \quad x \in [\frac{11}{18}, \frac{4}{6}) \\ \frac{23}{24} - \frac{3}{4}x, & \text{if} \quad x \in [\frac{4}{6}, \frac{5}{6}) \\ \frac{3}{4}x - \frac{7}{24}, & \text{if} \quad x \in (\frac{5}{6}, 1]. \end{cases}$$

Here $S(F) = \{\frac{1}{6}, \frac{2}{6}, \frac{4}{6}, \frac{5}{6}\}$ and $H(F) = 2$, $Ch_F = [0, \frac{4}{6}]$ (See graph of the function F given in Figure. 2.4.1). Clearly $F_0 : Ch_F \to Ch_F$ is a PM function such that $H(F_0) \geq 2$, $S(F_0) = \{\frac{1}{6}, \frac{2}{6}\}$ and $f_0 : Ch_F \to Ch_F$ is defined by

$$f_0(x) = \begin{cases} \frac{1}{6} - x, & \text{if} \quad x \in [0, \frac{1}{6}) \\ x - \frac{1}{6}, & \text{if} \quad x \in [\frac{1}{6}, \frac{3}{6}) \\ \frac{3}{2}x - \frac{5}{12}, & \text{if} \quad x \in [\frac{3}{6}, \frac{4}{6}], \end{cases}$$

is a square iterative root of F_0. Now, by Theorem 2.5.1, define the homeomorphism $\phi : [\frac{1}{6}, \frac{4}{6}] \to [\frac{1}{6}, \frac{4}{6}]$ by $\phi(x) = f_0(x)$ and it is easy to calculate $\phi^{-1} \circ F = \frac{11}{12} - \frac{x}{2}$ on $[\frac{4}{6}, \frac{5}{6}]$ and $\phi^{-1} \circ F = \frac{x}{2} + \frac{1}{12}$ on $[\frac{5}{6}, 1]$ so that

$$f(x) = \begin{cases} f_0(x), & \text{if} \quad x \in [0, \frac{4}{6}] \\ \frac{11}{12} - \frac{x}{2}, & \text{if} \quad x \in (\frac{4}{6}, \frac{5}{6}) \\ \frac{x}{2} + \frac{1}{12}, & \text{if} \quad x \in [\frac{5}{6}, 1], \end{cases}$$

is a square iterative root of F with $S(f) = \{\frac{1}{6}, \frac{4}{6}, \frac{5}{6}\}$.

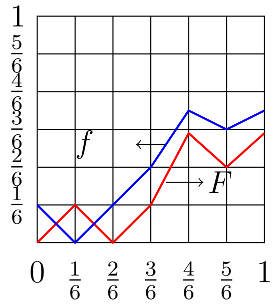

Figure. 2.4.1

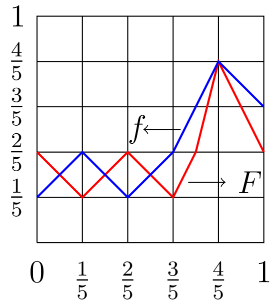

Figure. 2.4.2

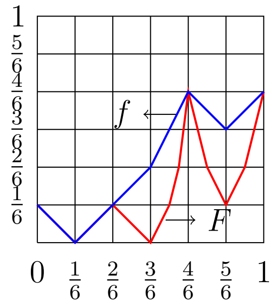

Figure. 2.4.3

Example 2.6.2. *Consider the function* $F : [0,1] \to [0,1]$ *defined by*

$$F(x) = \begin{cases} \frac{2}{5} - x, & \text{if} \quad x \in [0, \frac{1}{5}) \\ x, & \text{if} \quad x \in [\frac{1}{5}, \frac{2}{5}) \\ \frac{4}{5} - x, & \text{if} \quad x \in [\frac{2}{5}, \frac{3}{5}) \\ 2x - 1, & \text{if} \quad x \in [\frac{3}{5}, \frac{7}{10}) \\ 4x - \frac{12}{5}, & \text{if} \quad x \in [\frac{7}{10}, \frac{4}{5}) \\ \frac{12}{5} - 2x, & \text{if} \quad x \in [\frac{4}{5}, 1]. \end{cases}$$

Here $H(F) = \infty$, $S(F) = \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$ *and* $Ch_F = [\frac{1}{5}, \frac{4}{5}]$*(See graph of the function* F *given in Figure. 2.4.2). Therefore* $F_0 = F|_{Ch_F} : Ch_F \to Ch_F$ *is a self map such that* $S(F_0) = \{\frac{2}{5}, \frac{3}{5}\}$, $H(F_0) \geq 2$. *It is clear that,* F_0 *is reversing correspondence on* $[\frac{1}{5}, \frac{2}{5}]$ *and* $F_0([\frac{1}{5}, \frac{2}{5}]) = F_0([\frac{2}{5}, \frac{3}{5}]) = [\frac{1}{5}, \frac{2}{5}]$. *Therefore, by Corollary 2.5.2,* F_0 *has square iterative root on* $[\frac{1}{5}, \frac{4}{5}]$. *Indeed, it can be shown that the function* $f_0 : [\frac{1}{5}, \frac{4}{5}] \to [\frac{1}{5}, \frac{4}{5}]$ *defined by*

$$f_0(x) = \begin{cases} \frac{3}{5} - x, & \text{if} \quad x \in [\frac{1}{5}, \frac{2}{5}) \\ x - \frac{1}{5}, & \text{if} \quad x \in [\frac{2}{5}, \frac{3}{5}) \\ 2x - \frac{4}{5}, & \text{if} \quad x \in [\frac{3}{5}, \frac{4}{5}], \end{cases}$$

is a square iterative root of F_0. *Also,* $F([0, \frac{1}{5}] \subseteq F([\frac{1}{5}, \frac{2}{5}])$ *and* $F([\frac{4}{5}, 1] \subseteq F([\frac{2}{5}, \frac{4}{5}])$, *again by using Corollary 2.5.2, define* $\phi_1 : [\frac{1}{5}, \frac{2}{5}] \to [\frac{1}{5}, \frac{2}{5}]$ *by* $\phi_1(x) = x + \frac{1}{5}$ *and* $\phi_2 : [\frac{2}{5}, \frac{4}{5}] \to [\frac{1}{5}, \frac{2}{5}]$ *by*

$$\phi_2(x) = \begin{cases} x - \frac{1}{5}, & \text{if} \quad x \in [\frac{2}{5}, \frac{3}{5}) \\ 2x - \frac{4}{5}, & \text{if} \quad x \in [\frac{3}{5}, \frac{4}{5}]. \end{cases}$$

Now it is easy to calculate $\phi_1^{-1} \circ F = x + \frac{1}{5}$ *on* $[0, \frac{1}{5}]$ *and* $\phi_2^{-1} \circ F = \frac{8}{5} - x$ *on* $[\frac{4}{5}, 1]$ *so that* $f : [0,1] \to [0,1]$ *defined by*

$$f(x) = \begin{cases} x + \frac{1}{5}, & \text{if } \ x \in [0, \frac{1}{5}) \\ f_0(x), & \text{if } \ x \in [\frac{1}{5}, \frac{4}{5}] \\ \frac{8}{5} - x, & \text{if } \ x \in (\frac{4}{5}, 1], \end{cases}$$

is a square iterative root of F *with* $S(F) = \{\frac{1}{5}, \frac{2}{5}, \frac{4}{5}\}$.

Example 2.6.3. *Let* $F : [0,1] \to [0,1]$ *be defined by*

$$F(x) = \begin{cases} \frac{1}{6} - x, & \text{if } \ x \in [0, \frac{1}{6}) \\ x - \frac{1}{6}, & \text{if } \ x \in [\frac{1}{6}, \frac{2}{6}) \\ \frac{3}{6} - x, & \text{if } \ x \in [\frac{2}{6}, \frac{3}{6}) \\ 2x - 1, & \text{if } \ x \in [\frac{1}{6}, \frac{7}{12}) \\ 4x - \frac{13}{6}, & \text{if } \ x \in [\frac{7}{12}, \frac{15}{24}) \\ 8x - \frac{28}{6}, & \text{if } \ x \in [\frac{15}{24}, \frac{4}{6}) \\ \frac{20}{6} - 4x, & \text{if } \ x \in [\frac{4}{6}, \frac{9}{12}) \\ \frac{11}{6} - 2x, & \text{if } \ x \in [\frac{9}{12}, \frac{5}{6}) \\ 2x - \frac{9}{6}, & \text{if } \ x \in [\frac{5}{6}, \frac{11}{12}) \\ 4x - \frac{20}{6}, & \text{if } \ x \in (\frac{11}{12}, 1]. \end{cases}$$

Here $S(F) = \{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}\}$, $H(F) = \infty$ *and* $F([0,1]) = F([0, \frac{4}{6}]) = [0, \frac{4}{6}]$ *(See graph of the function* F *given in Figure. 2.4.3).*

Now $F_0 := F|_{[0,\frac{4}{6}]} : [0, \frac{4}{6}] \to [0, \frac{4}{6}]$ *such that* $S(F_0) = \{\frac{1}{6}, \frac{2}{6}, \frac{3}{6}\}$ *and* $f_0 : [0, \frac{4}{6}] \to [0, \frac{4}{6}]$ *defined by*

$$f_0(x) = \begin{cases} \frac{1}{6} - x, & \text{if } \ x \in [0, \frac{1}{6}) \\ x - \frac{1}{6}, & \text{if } \ x \in [\frac{1}{6}, \frac{3}{6}) \\ 2x - \frac{4}{6}, & \text{if } \ x \in [\frac{3}{6}, \frac{4}{6}], \end{cases}$$

is a cubic iterative root of F_0.

Define the homeomorphism $\phi : [\frac{1}{6}, \frac{4}{6}] \to [0, \frac{4}{6}]$ *by* $\phi(x) = f_0(x)$ *on* $[\frac{1}{6}, \frac{4}{6}]$ *as in Theorem 2.5.1. Now,* $\phi^{-1} \circ \phi^{-1} \circ F(x) = \frac{8}{6} - x$ *on* $[\frac{4}{6}, \frac{5}{6}]$ *and* $\phi^{-1} \circ \phi^{-1} \circ F(x) = x - \frac{2}{6}$

on $[\frac{5}{6}, 1]$*, so that*

$$f(x) = \begin{cases} f_0(x), & \text{if} \quad x \in [0, \frac{4}{6}] \\ \frac{8}{6} - x, & \text{if} \quad x \in (\frac{4}{6}, \frac{5}{6}) \\ x - \frac{2}{6}, & \text{if} \quad x \in [\frac{5}{6}, 1], \end{cases}$$

is a cubic iterative root of F *with* $S(f) = \{\frac{1}{6}, \frac{4}{6}, \frac{5}{6}\}$.

Summary of the Chapter

In this chapter, we introduced the concept of characteristic interval of any PM functions. Using the method of characteristic interval, we proved the following:

- The existence of iterative roots of PM functions of height greater than one on its characteristic interval.
- Extension of iterative roots from the characteristic interval to the whole interval.

We end this chapter with the following questions:

1. Does there exist τ_2 iterative roots of order n of PM functions F with $H(F) \geq 2$ and $n = N(F)$?
2. Does there exist iterative roots f of order n of PM functions F with $H(F) \geq 2$ and $N(f) < N(F) - (n-1)$?

Chapter 3

Iterative Roots of Non-PM Functions

3.1 Introduction

Lin et al., (Lin, 2014; Lin et al., 2017) discussed the iterative root problem for a particular class of non-PM continuous functions which are constant on some subinterval and piecewise monotone elsewhere.

Let $\alpha, \beta \in (0,1)$ such that $\alpha < \beta$. We say a continuous function $F : [0,1] \to [0,1]$ is in the class $\Omega_1([\alpha,\beta])$, if F satisfies the condition (C1). Similarly, we say F is in the class $\Omega_2([\alpha,\beta])$, if F satisfies the condition (C2), where,

(C1) F is constant on $[0,\alpha]$, and F is strictly decreasing on $[\alpha,\beta]$ but strictly increasing on $[\beta,1]$.

(C2) F is constant on $[0,\alpha]$, and F is strictly increasing on $[\alpha,\beta]$ but strictly decreasing on $[\beta,1]$.

Theorem 3.1.1. *(Lin, 2014) Suppose that $F \in \Omega_1([\alpha,\beta])$ with $F(\alpha) < \alpha$. Then F has infinitely many iterative roots of order $n \geq 2$.*

Theorem 3.1.2. *(Lin, 2014) Suppose that $F \in \Omega_2([\alpha,\beta])$ with $F([0,1]) \subseteq [\alpha,\beta]$ and either $F(\alpha) = \alpha$ or $F(1) > \alpha$. Then F has infinitely many iterative roots of order $n \geq 2$.*

For a detailed study on further results on the existence and nonexistence of iterative roots of the class of continuous functions $\Omega_1([\alpha,\beta])$ and $\Omega_2([\alpha,\beta])$, one can refer (Lin, 2014). Similar results on the existence of iterative roots of continuous functions $F : [0,1] \to [0,1]$ which is constant on the interval $[\alpha, \beta]$ and strictly monotone on $[0,1]\backslash[\alpha,\beta]$ can be found in (Lin et al., 2017).

3.2 Generalization of Forts and Characteristic Interval

Let $f : I \to I$ be any continuous function, recall from Definition 1.3.11 that, a point $\alpha \in int\ I$ is called a fort of f, if f is not strictly monotone in any neighborhood of α.

For continuous functions, the end points of the interval I may exhibit the same non-monotonic behavior like a fort which is an interior point of I. For example consider the functions

$$f_1(x) = \begin{cases} x \sin\frac{1}{x} & \text{if} \quad x \in [-1,1]\backslash\{0\} \\ 0 & \text{if} \qquad x = 0. \end{cases}$$

and

$$f_2(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0, \\ \frac{1}{2} + x^2 \sin(\frac{\pi}{x}), & \text{if } x \in (0,1]. \end{cases}$$

Note that, for each $\epsilon > 0$, both the functions f_1 and f_2 are not strictly monotone on the neighborhood $N_\epsilon(0) = \{x \in I \mid |x - 0| < \epsilon\}$. In other words, the point 0 exhibits the same non-monotonic behavior, for the functions f_1 and f_2, regardless of the interior point or end point of the interval.

By generalizing naturally, we define fort of a continuous functions as follows:

Definition 3.2.1. *Let $f : I \to I$ be any continuous function. A point $\alpha \in I$ is called a fort of f, if f is not strictly monotone in any neighborhood of α in I.*

This natural generalization of fort allows us to to define the characteristic interval for any continuous functions as follows:

Definition 3.2.2. *Let $f : I \to I$ be any continuous function. Then the smallest closed interval containing the range of f whose end points are either forts of f or the end points of $[a, b]$ is called the characteristic interval of f.*

Note that the Definition 3.2.2 is a generalization of Definition 1.3.17(cf. (Zhang, 1997)) for the class of PM functions of height less than two and Definition 2.3.1(cf. Chapter 2) for the class of all PM functions. We denote the characteristic interval of f by Ch_f. The existence and uniqueness of the characteristic interval of continuous functions is similar to the discussion in Chapter 2.

Example 3.2.3. *Let $f : [a, b] \to [a, b]$ be the constant function defined by $f(x) = \alpha$ for all $x \in [a, b]$ and $\alpha \in [a, b]$. Then the characteristic interval of f is the singleton set $\{\alpha\}$.*

Example 3.2.4. *Consider the function $f : [0, 1] \to [0, 1]$ defined as follows:*

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 0, \\ \frac{1}{2} + x^2 \sin(\frac{\pi}{x}), & \text{if } x \in (0, 1]. \end{cases}$$

Clearly, f is a continuous self-mapping on $[0, 1]$. The graph of the function f is given below.

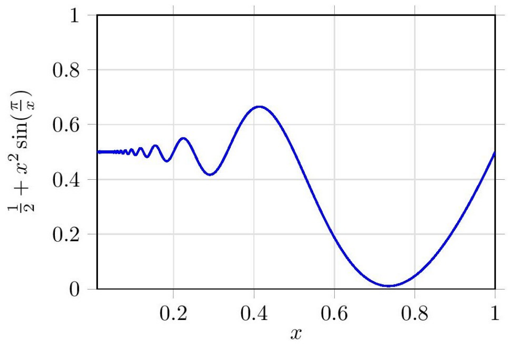

Figure. 3.2.1

Since the function f is not strictly monotone at 0, we see that 0 is a fort of f. The other forts of f are given by $\{x \in [0, 1] \mid x = \frac{\pi}{2} \cot(\frac{\pi}{x})\}$. Let $m = \inf\{f(x) \mid x \in [0, 1]\}$ and $M = \sup\{f(x) \mid x \in [0, 1]\}$. Put $\alpha = \sup\{x \in [0, m] \mid x = \frac{\pi}{2} \cot(\frac{\pi}{x})\}$ and $\beta = \inf\{x \in [M, 1] \mid x = \frac{\pi}{2} \cot(\frac{\pi}{x})\}$. Then, it is easy to see that the characteristic interval of f is $[\alpha, \beta]$.

Proposition 3.2.5. *Let $F : I \to I$ be a continuous function. Then*

(i) $Ch_F \supseteq Ch_{F^2} \supseteq \cdots \supseteq Ch_{F^n} \supseteq \cdots$*;*

(ii) If F is constant on Ch_F, then F^n is constant for all $n \geq 2$;

(iii) $Ch_F = I$ if and only if $S(F) \subseteq R(F)^0$, where $R(F)$ is the range of F;

Proof. (i) Let $I_n = [a_n, b_n]$ be the characteristic interval of F^n. Then $I_n \supseteq R(F^n)$, the range of F^n, and $a_n, b_n \in S(F^n) \cup \{a, b\}$. Since $R(F^n) \supseteq R(F^{n+1})$ and $S(F^{n+1}) \supseteq S(F^n)$ we have $I_n \supseteq R(F^{n+1})$ and $a_n, b_n \in S(F^{n+1}) \cup \{a, b\}$. Therefore the interval I_n contains the range of F^{n+1} and the end points of I_n are either forts of F^{n+1} or the end points of I. But I_{n+1} is the smallest interval having the above property, $I_n \supseteq I_{n+1}$.

(ii) If F is constant on Ch_F, then F is constant on $R(F)$. Now it is immediate that F^n is constant for all $n \in \mathbb{N}$.

(iii) Let $R(F) = [m, M]$. If $Ch_F = I$, then F has no forts on $(a, m]$ and $[M, b)$. Suppose there is a $t \in S(F)$ but $t \notin (m, M)$. If $t \in [a, m]$, then $t = a$ as $Ch_F = I$. Since a is a fort of F, F is not strictly monotone at a. Therefore every neighborhood of a has a fort of F other than a, which lead a contradiction as $Ch_F = I$. On the other hand if $t \in [M, b]$, then $t = b$. Again, as b is a fort of F, F is not strictly monotone at b. Therefore every neighborhood of b has a fort of F other than b, which leads to a contradiction as $Ch_F = I$. Hence $S(F) \subseteq R(F)^0 \cup \{a, b\}$.

Conversely, assume $S(F) \subseteq R(F)^0 \cup \{a, b\}$. Note that F has no forts on $(a, m]$ and $[M, b)$. Therefore, by definition of characteristic interval, $Ch_F = I$. □

3.3 Non-isolated Forts

In order to study the iterative root problem for the class of continuous functions having infinitely many forts (i.e., non-PM functions), we further classify the forts of continuous functions into two category as follows:

Definition 3.3.1. *Let $f : I \to I$ be a continuous function. We say a fort x^* of f is a non-isolated fort, if for each $\epsilon > 0$, the neighborhood $N_\epsilon(x^*) = \{x \in I \mid |x - x^*| < \epsilon\}$ contains a fort of f other than x^*. Otherwise, x^* is called an isolated fort of f.*

For any continuous function $f : I \to I$, we denote the set of all forts of f by $S(f)$ and the set of all non-isolated forts of f by $S^*(f)$. Let $N(F)$ and $N^*(f)$ denotes the number of forts and the number of non-isolated forts of f respectively.

Example 3.3.2. *Consider the function defined in Example 3.2.4, we see that, $S(f) = \{x \in [0,1] \mid x = \frac{\pi}{2}\cot(\frac{\pi}{x})\} \cup \{0\}$ and $S^*(f) = \{0\}$.*

For any continuous function $f : I \to I$, every non-isolated fort of f is a limit point of forts of f, moreover, if f is a PM function then every fort of f is an isolated fort.

Proposition 3.3.3. *The fort x^* is a non-isolated fort of f if and only if for each $\epsilon > 0$, there are distinct points $\alpha_1, \alpha_2, \alpha_3 \in N_\epsilon(x^*)$ such that $f(\alpha_1) = f(\alpha_2) = f(\alpha_3)$.*

Proof. Suppose that x^* is a non-isolated fort for f. If f is constant on a neighborhood of x^*, then the results is trivial. If not, then there is a sequence $\{x_m\}$ of distinct forts of f such that $\lim\limits_{m\to\infty} x_m \to x^*$. For sufficiently large m_i, choose four consecutive forts, say x_{m_i}, so that either $x_{m_i} \in (x^* - \epsilon, x^*)$ or $x_{m_i} \in (x^*, x^* + \epsilon)$ for $i = 0, 1, 2, 3$ with $x_{m_0} < x_{m_1} < x_{m_2} < x_{m_3}$. Since f is monotone on each of the subintervals $[x_{m_i}, x_{m_{i+1}}]$, either f is increasing on $[x_{m_0}, x_{m_1}]$ and $[x_{m_2}, x_{m_3}]$ and decreasing on $[x_{m_1}, x_{m_2}]$ and $[x_{m_3}, x_{m_4}]$ or f is decreasing on $[x_{m_0}, x_{m_1}]$ and $[x_{m_2}, x_{m_3}]$ and increasing on $[x_{m_1}, x_{m_2}]$ and $[x_{m_3}, x_{m_4}]$. Thus in either case, there exists

$$t \in f([x_{m_0}, x_{m_1}]) \cap f([x_{m_1}, x_{m_2}]) \cap f([x_{m_2}, x_{m_3}])$$

or

$$t \in f([x_{m_1}, x_{m_2}]) \cap f([x_{m_2}, x_{m_3}]) \cap f([x_{m_3}, x_{m_4}]).$$

Therefore, there are points $\alpha_1, \alpha_2, \alpha_3 \in (x_{m_0}, x^*)$ or $\alpha_1, \alpha_2, \alpha_3 \in (x^*, x_{m_3})$ such that

$$f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = t$$

depending on $x_{m_i} \in (x^* - \epsilon, x^*)$ or $x_{m_i} \in (x^*, x^* + \epsilon)$ for $i = 0, 1, 2, 3$.

Conversely, suppose for each positive integer $m \geq 1$, there exist

$$\alpha_{m_1}, \alpha_{m_2}, \alpha_{m_3} \in N_{\frac{1}{m}}(x^*)$$

such that $f(\alpha_{m_1}) = f(\alpha_{m_2}) = f(\alpha_{m_3})$. Note that at least two of the α_{m_i} lies in either $(x^* - \frac{1}{m}, x^*)$ or $(x^*, x^* + \frac{1}{m})$. Without loss of generality, assume $\alpha_{m_1}, \alpha_{m_2} \in (x^*, x^* + \frac{1}{m})$ such that $f(\alpha_{m_1}) = f(\alpha_{m_2})$. Therefore, there exists $x_m \in (\alpha_{m_1}, \alpha_{m_2})$ such that f assumes either local maximum or local minimum at x_m. Hence $x_m \in S(f)$ and $|x_m - x^*| < \frac{1}{m}$ for all m, so that $\lim_{m\to\infty} x_m \to x^*$. Therefore, x^* is an non-isolated fort of f. $\square$

Remark 3.3.4. *If x^* is a non-isolated fort of f, then for each $\epsilon > 0$ there are distinct points $\alpha_1, \alpha_2 \in N_\epsilon(x^*)$ such that $f(\alpha_1) = f(\alpha_2)$. But the converse is not necessarily true. For example, consider the tent map $T : [0, 1] \to [0, 1]$ defined by*

$$T(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 2(1 - x) & \text{if } x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly $\frac{1}{2}$ is a fort of T. In fact, for each $\epsilon > 0$ the points $\frac{1-\epsilon}{2}, \frac{1+\epsilon}{2} \in N_\epsilon(\frac{1}{2})$ such that $T(\frac{1-\epsilon}{2}) = T(\frac{1+\epsilon}{2})$. But $\frac{1}{2}$ is not a non-isolated fort of T.

Remark 3.3.5. *Suppose for each $\epsilon > 0$ there exist distinct $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in N_\epsilon(x^*)$ such that $f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = f(\alpha_4)$. Then x^* is a non-isolated fort of f. But the converse is not necessarily true. For example, consider the function $f : [0, 1] \to [0, 1]$ defined by*

$$f(x) = \begin{cases} 0 & \text{if } \quad x = 0, \\ 2x - \dfrac{3}{2^{n+2}} & \text{if } x \in (\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}}], n = 0, 1, 2, \ldots, \\ \dfrac{3}{2^{n+1}} - x & \text{if } x \in (\frac{3}{2^{n+2}}, \frac{1}{2^n}], n = 0, 1, 2, \ldots. \end{cases}$$

Then $S(f) = \{\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}} \mid n = 0, 1, 2, \ldots\} \cup \{0\}$. *Since, for each* $\epsilon > 0$, *the neighborhood* $N_\epsilon(0)$ *contains all but finitely many* $\frac{1}{2^n}$, *the point* 0 *is a non-isolated fort of* f. *See the graph of the function* f *given in Figure 3.3.1.*

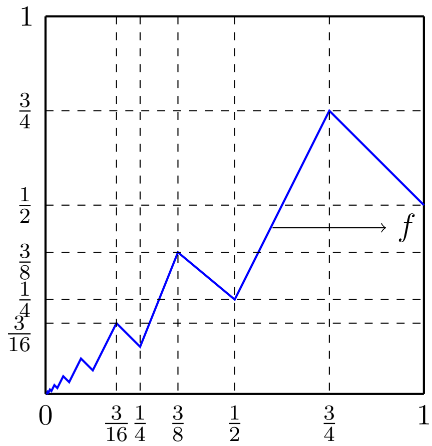

Figure. 3.3.1

Put $I_n = (\frac{3}{2^{n+2}}, \frac{1}{2^n}]$, $J_n = (\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}}]$ *and* $A_n = I_n \cup J_n$ *for all* $n = 0, 1, 2, \ldots$. *Let* m *and* n *are any non-negative integers such that* $m - n \geq 2$. *Then*

$$\begin{aligned} f(A_n) \cap f(A_m) &= f\left(\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right]\right) \cap f\left(\left(\frac{1}{2^{m+1}}, \frac{1}{2^m}\right]\right) \\ &= \left(\frac{1}{2^{n+2}}, \frac{3}{2^{n+2}}\right] \cap \left(\frac{1}{2^{m+2}}, \frac{3}{2^{m+2}}\right]. \end{aligned}$$

Since $m \geq n + 2$, *we have,*

$$\frac{1}{2^{m+2}} < \frac{3}{2^{m+2}} = \frac{3}{4}\frac{1}{2^m} < \frac{1}{2^{n+2}} < \frac{3}{2^{n+2}}.$$

Therefore

$$f(A_n) \cap f(A_m) = \emptyset \text{ for } m \geq n + 2. \tag{3.3.1}$$

Now, suppose for each $\epsilon > 0$ *there exist distinct points* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in N_\epsilon(0)$ *such that*

$$f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = f(\alpha_4). \tag{3.3.2}$$

Since the collection of sets $\{A_n \mid n = 0, 1, 2, \ldots\} \cup \{0\}$ *forms a partition of* $[0, 1]$, *the points* $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \cup_{n=0}^{\infty} A_n \cup \{0\}$. *Since the function* f *is strictly monotone* *on each I_n and J_n, the interval A_n can contain at most two α_i's. Without loss of generality, assume $\alpha_1, \alpha_2 \in A_n$ and $\alpha_3, \alpha_4 \in A_m$ for some n, m with $m \geq n+1$.*

If $m > n+1$, then equation (3.3.1) leads a contradiction to (3.3.2). Therefore, we must have $m = n+1$. But, in this case,

$$
\begin{aligned}
& f(A_{n+1}) \cap f(A_n) \\
&= \{f(J_{n+1}) \cup f(I_{n+1})\} \cap \{f(J_n) \cup f(I_n)\} \\
&= \left\{\left(\frac{1}{2^{n+3}}, \frac{3}{2^{n+3}}\right] \cup \left[\frac{1}{2^{n+2}}, \frac{3}{2^{n+3}}\right)\right\} \cap \left\{\left(\frac{1}{2^{n+2}}, \frac{3}{2^{n+2}}\right] \cup \left[\frac{1}{2^{n+1}}, \frac{3}{2^{n+2}}\right)\right\} \\
&= \left(\frac{1}{2^{n+3}}, \frac{3}{2^{n+3}}\right] \cap \left(\frac{1}{2^{n+2}}, \frac{3}{2^{n+2}}\right] \\
&= \emptyset \text{ for } n = 0, 1, 2, \ldots
\end{aligned}
$$

This leads a contradiction to equation (3.3.2). Therefore, if x^* is a non-isolated fort of f, then for each $\epsilon > 0$, it is not necessary that there exist distinct $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in N_\epsilon(x^*)$ such that $f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = f(\alpha_4)$.

For any continuous function f, we define

$$S_L^*(f) := \{x^* \in S^*(f) : x^* = \lim_{m\to\infty} x_m, \text{ where } x_m \in S(f) \cap (a, x^*)\} \tag{3.3.3}$$

and

$$S_R^*(f) := \{x^* \in S^*(f) : x^* = \lim_{m\to\infty} x_m, \text{ where } x_m \in S(f) \cap (x^*, b)\}. \tag{3.3.4}$$

Proposition 3.3.6. *If $x^* \in S^*(f)$, then the following statements are true:*

(i) $x^* \in S^*(f^n)$ *for all* $n \in \mathbb{N}$*;*

(ii) If $t \in f^{-(n-1)}(x^*) \cap S(f^{n-1})^c \cap I^0$, *then* $t \in S^*(f^n)$ *for all* $n \in \mathbb{N}$*;*

(iii) If $x^* \in S_L^*(f)$ *and* $t \in f^{-(n-1)}(x^*)$ *such that* f^{n-1} *attains local maximum at* t, *then* $t \in S^*(f^n)$*;*

(iv) If $x^* \in S_R^*(f)$ *and* $t \in f^{-(n-1)}(x^*)$ *such that* f^{n-1} *attains local minimum at* t, *then* $t \in S^*(f^n)$.

Proof. (i) is straight forward from Proposition 3.3.3.

To prove (ii), let $\epsilon > 0$ be given. Consider the neighborhood $N_\delta(f^{n-1}(t))$, by the continuity of f^{n-1} at t, we can find a neighborhood $N_{\epsilon'}(t)$ such that

$$f^{n-1}(N_{\epsilon'}(t)) \subseteq N_\delta(f^{n-1}(t))$$

and hence

$$f^{n-1}(N_r(t)) \subseteq N_\delta(f^{n-1}(t)),$$

where $r = \min\{\epsilon, \epsilon'\}$. Since $f^{n-1}(t)$ is an non-isolated fort of f, by Proposition 3.3.3, there are distinct points $y_1, y_2, y_3 \in f^{n-1}(N_r(t)) \cap N_\delta(f^{n-1}(t))$ such that

$$f(y_1) = f(y_2) = f(y_3).$$

Now, by the intermediate value theorem, choose three distinct points in $N_r(t)$ such that $f^{n-1}(x_i) = y_i$ for $i = 1, 2, 3$. Therefore, for any $\epsilon > 0$, there are three distinct points $x_1, x_2, x_3 \in N_\epsilon(t)$ such that

$$f^n(x_1) = f^n(x_2) = f^n(x_3)$$

and hence t is an non-isolated fort of f^n.

For (iii), choose $N_\delta(f^{n-1}(t)) = (f^{n-1}(t) - \delta, f^{n-1}(t))$ in the proof of (ii).

Similarly, for (iv), choose $N_\delta(f^{n-1}(t)) = (f^{n-1}(t), f^{n-1}(t) + \delta)$ in the proof of (ii). This completes the proof. □

Lemma 3.3.7. *If $f : I \to I$ is a continuous function, then $S^*(f^m) \subseteq S^*(f^{m+1})$ for all $m \in \mathbb{N}$.*

Proof. Let x^* be a non-isolated fort of f^m. Therefore there exist a sequence of forts $\{x_n\}$ of f^m such that $x_n \to x^*$ as $n \to \infty$. It is clear from the definition that $S(f^m) \subseteq S(f^{m+1})$ for all $m \in \mathbb{N}$. Therefore $\{x_n\}$ is a sequence of forts of f^{m+1} such that $x_n \to x^*$ as $n \to \infty$. Hence x^* is a non-isolated fort of f^{m+1}. □

Proposition 3.3.8. *Let $f : I \to I$ is a continuous function. Then* *(i) if $N^*(f) = 0$, then $N^*(f^m) = 0$ for all $m \in \mathbb{N}$.*

(ii) if $N^*(f^m) = N^*(f^{m+1})$, for some $m \in \mathbb{N}$ then $N^*(f^m) = N^*(f^{m+i})$ for all $i \in \mathbb{N}$.

Proof. (i) We prove the result using induction on m. If $N^*(f) = 0$ then $S^*(f) = \phi$. Since $S^*(f^2) = S^*(f) \cup \{x \in [a,b] \mid f(x) \in S^*(f)\}$, whence $S^*(f^2) = \phi$ and $N^*(f^2) = 0$. Assume $N^*(f^{m-1}) = 0$, then $S^*(f^{m-1}) = \phi$. Since

$$S^*(f^m) = S^*(f^{m-1}) \cup \{x \in [a,b] \mid f(x) \in S^*(f^{m-1})\},$$

we have $S^*(f^m) = \phi$ and $N^*(f^m) = 0$.

(ii) In view of Lemma 3.3.7, we have the following inequality.

$$N^*(f) \leq N^*(f^2) \leq N^*(f^3) \leq \cdots \leq N^*(f^n) \leq \cdots$$

If $N^*(f^m) = \infty$, then the result is trivial. Assume $N^*(f^m) < \infty$ and $N^*(f^m) = N^*(f^{m+1})$. Since

$$S^*(f^{m+1}) = S^*(f^m) \cup \{x \in [a,b] \mid f(x) \in S^*(f^m)\},$$

we have $\{x \in [a,b] \mid f(x) \in S^*(f^m)\} = \phi$. Now,

$$\begin{aligned} S^*(f^{m+2}) &= S^*(f^{m+1}) \cup \{x \in [a,b] \mid f(x) \in S^*(f^{m+1})\} \\ &= S^*(f^{m+1}) \cup \{x \in [a,b] \mid f(x) \in S^*(f^m)\} \\ &= S^*(f^m). \end{aligned}$$

Proceed similarly to prove $N^*(f^m) = N^*(f^{m+i})$ for all $i \in \mathbb{N}$. $\square$

3.4 Extension of Iterative Roots from the Characteristic Interval

Suppose that the given function behaves nicely in the sense that the function is piecewise monotone in the characteristic interval, then a series of results are available on the existence of iterative roots in the characteristic interval (see (Zhang, 1997; Liu et al., 2012; Li and Chen, 2014)).

The following theorem gives an extension of iterative roots of any continuous function $F : I = [a, b] \to I$, provided $F|_{Ch_F}$ has iterative root on Ch_F.

Theorem 3.4.1. *Let $F : I \to I$ be a continuous function and let $F_0 = F|_{Ch_F}$. Suppose there is a continuous function $f_0 : Ch_F = [a', b'] \to Ch_F$ such that $f_0^n = F_0$ on Ch_F with the following properties:*

(i) there exist $\alpha, \beta \in S(f_0)$ such that f_0 has no forts in the intervals (a', α) and (β, b') with $a < a' < \alpha < \beta < b' < b$,

(ii) $F([a, a']) \subseteq F_0([a', \alpha]) \subseteq f_0([a', \alpha]) \subseteq [a', \alpha]$ and $F([b', b]) \subseteq F_0([\beta, b']) \subseteq f_0([\beta, b']) \subseteq [\beta, b']$.

Then there exists a continuous function f on I such that $f|_{Ch_F} = f_0$ and $f^n = F$ on I.

Proof. For $i \in \{1, 2, \ldots, n-1\}$, let $\phi_i : f_0^{i-1}([a', \alpha]) \to f_0^i([a', \alpha])$ and $\psi_i : f_0^{i-1}([\beta, b']) \to f_0^i([\beta, b'])$ be the homeomorphisms defined respectively by

$$\phi_i(x) = f_0(x) \text{ for all } x \in f_0^{i-1}([a', \alpha])$$

and

$$\psi_i(x) = f_0(x) \text{ for all } x \in f_0^{i-1}([\beta, b']).$$

Now, define $f : I \to I$ as follows:

$$f(x) := \begin{cases} \phi \circ F(x), & \text{if } \ x \in [a, a'), \\ f_0(x), & \text{if } \ x \in [a', b'], \\ \psi \circ F(x), & \text{if } \ x \in (b', b], \end{cases}$$

where $\phi = \phi_1^{-1} \circ \phi_2^{-1} \circ \cdots \circ \phi_{n-1}^{-1}$ and $\psi = \psi_1^{-1} \circ \psi_2^{-1} \circ \cdots \circ \psi_{n-1}^{-1}$. By hypothesis (ii), the above definition is well-defined. Also, it is clear that, if $x \in [a', b']$, then

$f^n = {f_0}^n = F$. For any $x \in [a, a']$,

$$\begin{aligned} f^n(x) &= f_0^{n-1} \circ \phi \circ F(x) \\ &= f_0^{n-1} \circ \phi_1^{-1} \circ \phi_2^{-1} \circ \cdots \circ \phi_{n-1}^{-1} \circ F(x) \\ &= f_0^{n-1} \circ f_0^{-1} \circ f_0^{-1} \circ \cdots \circ f_0^{-1} \circ F(x) \\ &= F(x) \end{aligned}$$

and, for any $x \in [b', b]$,

$$\begin{aligned} f^n(x) &= f_0^{n-1} \circ \psi \circ F(x) \\ &= f_0^{n-1} \circ \psi_1^{-1} \circ \psi_2^{-1} \circ \cdots \circ \psi_{n-1}^{-1} \circ F(x) \\ &= f_0^{n-1} \circ f_0^{-1} \circ f_0^{-1} \circ \cdots \circ f_0^{-1} \circ F(x) \\ &= F(x). \end{aligned}$$

Therefore, f satisfies the functional equation $f^n = F$ on I. It remains to prove that f is continuous on I. To prove f is continuous, it is enough to prove that f is continuous at a' and b'.

Let $(x_n) \in [a, a')$ be a sequence such that $x_n \to a'$ as $n \to \infty$. Since F is continuous at a', it follows that $F(x_n) \to F(a')$ as $n \to \infty$. By hypothesis (ii) $F(x_n) \in F_0([a', \alpha])$ for all n. Since the function ϕ is continuous at $F(a) = F_0(a')$, we have

$$\phi(F(x_n)) \to \phi(F_0(a')) = f_0(a') \text{ as } n \to \infty.$$

$$\text{i.e., } f(x_n) \to f(a) \text{ as } n \to \infty.$$

Thus f is continuous at a'. Similarly, we can prove that f is continuous at b' and hence f is continuous on $[a, b]$. This completes the proof. □

In the view of Theorem 3.4.1, in order to study the iterative roots of continuous non-PM functions, it is sufficient to study the iterative roots on its characteristic interval. The problem of finding iterative roots becomes difficult only when F is not a PM function on its characteristic interval, i.e., if F has infinitely many forts in its characteristic interval. We address this problem partially in the next section.

Now, we present an example to illustrate Theorem 3.4.1.

Example 3.4.2. *Consider the continuous function $F : [0,1] \to [0,1]$ defined by*

$$F(x) = \begin{cases} \frac{1}{4}, & \text{if} \quad x = 0, \\ \frac{3}{4} - nx, & \text{if} \quad x \in (\frac{1}{2n+1}, \frac{1}{2n}] \ (n \geq 2), \\ nx - \frac{1}{4}, & \text{if} \quad x \in (\frac{1}{2n}, \frac{1}{2n-1}] \ (n \geq 3), \\ x, & \text{if} \quad x \in (\frac{1}{4}, \frac{3}{8}], \\ \frac{9}{8} - 2x, & \text{if} \quad x \in (\frac{3}{8}, \frac{7}{16}], \\ 4x - \frac{3}{2}, & \text{if} \quad x \in (\frac{7}{16}, \frac{9}{16}], \\ \frac{15}{8} - 2x, & \text{if} \quad x \in (\frac{9}{16}, \frac{5}{8}], \\ x, & \text{if} \quad x \in (\frac{5}{8}, \frac{3}{4}], \\ \frac{9}{8} - \frac{x}{2}, & \text{if} \quad x \in (\frac{3}{4}, 1]. \end{cases}$$

The graph of the function F is given below.

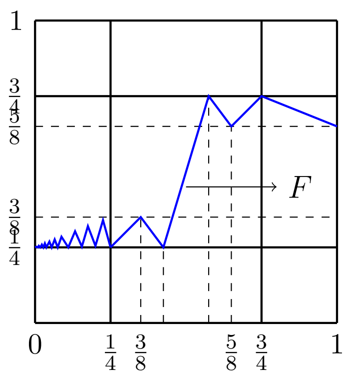

Figure. 3.4.1

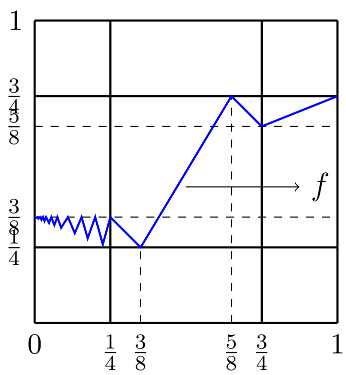

Figure. 3.4.2

Note that F is continuous on $[0,1]$ and $Ch_F = [\frac{1}{4}, \frac{3}{4}]$. Clearly, the function $f_0 : [\frac{1}{4}, \frac{3}{4}] \to [\frac{1}{4}, \frac{3}{4}]$ defined by

$$f_0(x) = \begin{cases} \frac{5}{8} - x, & \text{if} \quad x \in (\frac{1}{4}, \frac{3}{8}], \\ 2x - \frac{1}{2}, & \text{if} \quad x \in (\frac{3}{8}, \frac{5}{8}], \\ \frac{11}{8} - x, & \text{if} \quad x \in (\frac{5}{8}, \frac{3}{4}] \end{cases}$$

is a iterative root of order 2 of F on $[\frac{1}{4}, \frac{3}{4}]$, i.e., $f_0^2 = F_0$. Here $\alpha = \frac{3}{8}$, $\beta = \frac{5}{8}$ and

$$f_0([\tfrac{1}{4}, \tfrac{3}{8}]) = [\tfrac{1}{4}, \tfrac{3}{8}], \quad f_0([\tfrac{5}{8}, \tfrac{3}{4}]) = [\tfrac{5}{8}, \tfrac{3}{4}].$$

Further,

$$F([0,\tfrac{1}{4}]) \subseteq F_0([\tfrac{1}{4},\tfrac{3}{8}]) = [\tfrac{1}{4},\tfrac{3}{8}] \text{ and } F([\tfrac{3}{4},1]) \subseteq F_0([\tfrac{5}{8},\tfrac{3}{4}]) = [\tfrac{5}{8},\tfrac{3}{4}]$$

(see Figure. 3.4.1 and Figure. 3.4.2). Therefore, the function $f : [0,1] \to [0,1]$ *computed by using Theorem 3.4.1 as*

$$f(x) = \begin{cases} \frac{3}{8}, & \text{if} \quad x = 0 \\ nx - \frac{1}{8}, & \text{if} \quad x \in (\frac{1}{2n+1}, \frac{1}{2n}] \ (n \geq 2) \\ \frac{7}{8} - nx, & \text{if} \quad x \in (\frac{1}{2n}, \frac{1}{2n-1}] \ (n \geq 3) \\ f_0(x), & \text{if} \quad x \in (\frac{1}{4}, \frac{3}{4}] \\ \frac{1}{4} + \frac{x}{2}, & \text{if} \quad x \in (\frac{3}{4}, 1], \end{cases}$$

is actually continuous and satisfies $f^2(x) = F(x)$ *for all* $x \in [0,1]$.

3.5 Nonexistence of Iterative Roots

The following theorem gives the nonexistence of iterative roots of continuous functions having infinitely many forts:

Theorem 3.5.1. *If* $F \in C(I)$ *has only one non isolated fort, say* x^*, *on its range such that* $F(x^*) \neq x^*$ *and* $x^* \in S_L^*(F) \cap S_R^*(F)$, *then* F *has no iterative root of any order* $n \geq 2$.

Proof. Suppose that there exists $f \in C(I)$ such that $f^n = F$ on I. If f has no non-isolated forts in its range, then the function $F = f^n$ also have no non-isolated forts in its range, which is not possible. Therefore, f must have at least one non-isolated fort on its range. Further, this non-isolated fort cannot be a fixed point of f as F has no non-isolated fort which is also a fixed point. Hence, by Proposition 3.3.6 (ii), f^n have at least two non-isolated forts, which is a contradiction to our assumption. This completes the proof. □

Example 3.5.2. *Let* $F : [-1,1] \to [-1,1]$ *be the function defined by*

$$F(x) = \begin{cases} \frac{1}{4}, & \text{if} \quad x = 0, \\ \frac{1}{4} + x^2 \sin(\frac{\pi}{x}), & \text{if} \quad x \in [-1,1]\backslash\{0\}. \end{cases}$$

Clearly 0 is the only non-isolated fort which is not a fixed point of F. Therefore by Theorem 3.5.1, F has no iterative roots of order $n \geq 2$. The graph the function F is given below.

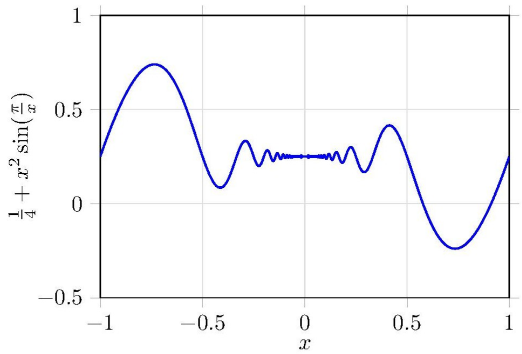

Figure. 3.5.1

Theorem 3.5.3. *Let Ω be the set of all $F \in C(I)$ which has only one non-isolated fort, say x^*, on its range such that $F(x^*) \neq x^*$ and $x^* \in S_L^*(F) \cap S_R^*(F)$. Then Ω is dense in $C(I)$.*

Proof. Let $\epsilon > 0$ be given. Since, any $f \in C(I)$ is uniformly continuous, we can find $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{4}$ whenever $|x - y| < \delta$. Choose a positive integer M such that $\frac{1}{M} < \delta$. Let $x_0 \in R(f)$ be any arbitrary point. Define

$$x_{-k} = x_0 - \frac{k}{M} \text{ for } 1 \leq k \leq m_1, \tag{3.5.5}$$

where m_1 is the least positive integer satisfies the inequality $(x_0 - \frac{m_1}{M}) - a < \frac{1}{M}$. Define

$$x_k = x_0 + \frac{k}{M} \text{ for } 1 \leq k \leq m_2, \tag{3.5.6}$$

where m_1 is the least positive integer satisfies the inequality $b - (x_0 + \frac{m_2}{M}) < \frac{1}{M}$. Therefore, equations (3.5.5) and (3.5.6), we have

$$a = x_{-(m_1+1)} < x_{-m_1} < \cdots < x_{-1} < x_0 < x_1 < \cdots < x_{m_2} < x_{m_2+1} = b.$$

Define the function $g : I \to I$ as follows:

$$g(x_k) = \begin{cases} f(x_k) + \frac{\epsilon}{8}, & \text{if } \quad k = 0, \\ f(x_k), & \text{if } \quad k \neq 0 \text{ and } -(m_1+1) \leq k \leq m_2 + 1, \end{cases}$$

and g is linear on each of the subintervals $[x_{-(k+1)}, x_{-k}]$ for $1 \leq k \leq m_1$ and linear on each of the subintervals $[x_k, x_{k+1}]$ for $1 \leq k \leq m_2$.

Now, consider the interval $[x_{-1}, x_1]$. Let m be the least positive integer such that $x_0 + \frac{1}{2^m} < x_1$ and $x_{-1} < x_0 - \frac{1}{2^m}$. First, we define g on the points $x_0 + \frac{1}{2^{m+k}}$ and $x_0 - \frac{1}{2^{m+k}}$ for $k = 0, 1, 2, \ldots$. Then extend this g linearly as before. Define

$$g\Big(x_0 - \frac{1}{2^{m+k}}\Big) = \Big(f(x_0 - \frac{1}{2^m}) - \frac{\epsilon}{4}\Big) + \Big(f(x_0) - \Big(f(x_0 - \frac{1}{2^m}) - \frac{3\epsilon}{8}\Big)\Big)\left(\frac{2^k - 1}{2^k}\right)$$

when $k \in \{0, 2, 4, \ldots\}$ and

$$g\Big(x_0 - \frac{1}{2^{m+k}}\Big) = f(x_{-1}) + \frac{(f(x_0) + \frac{\epsilon}{8}) - f(x_{-1})}{x_0 - x_{-1}}\Big((x_0 - \frac{1}{2^{m+k}}) - x_{-1}\Big)$$

when $k \in \{1, 3, 5, \cdots\}$. Similarly, define

$$g\Big(x_0 + \frac{1}{2^{m+k}}\Big) = \Big(f(x_0) + \frac{\epsilon}{8}\Big) - \Big(\Big(f(x_0) - \frac{\epsilon}{8}\Big) - f(x_0 + \frac{1}{2^m})\Big)\left(\frac{1}{2^k}\right)$$

when $k \in \{0, 2, 4, \cdots\}$ and

$$g\Big(x_0 + \frac{1}{2^{m+k}}\Big) = f(x_1) + \frac{(f(x_0) + \frac{\epsilon}{8}) - f(x_1)}{x_0 - x_1}\left((x_0 + \frac{1}{2^{m+k}}) - x_1\right)$$

when $k \in \{1, 3, 5, \cdots\}$. Now, define g is linear on each of the subintervals $[x_0 + \frac{1}{2^{m+k+1}}, x_0 + \frac{1}{2^{m+k}}]$ and $[x_0 - \frac{1}{2^{m+k}}, x_0 - \frac{1}{2^{m+k+1}}]$ for all $k = 0, 1, 2, \ldots$, also linear on $[x_{-1}, x_0 - \frac{1}{2^m}]$ and $[x_0 + \frac{1}{2^m}, x_1]$.

Note that g is continuous on I and having only one non-isolated fort on the range of g, namely, x_0 (see Figure. 3.4.2). By choice, $x_0 \in S_L^*(g) \cap S_R^*(g)$ and, if $g(x_0) = f(x_0) + \frac{\epsilon}{8} = x_0$, then we replace $\frac{\epsilon}{8}$ by $\frac{\epsilon}{16}$ in the definition of $g(x_0)$ so that we may assume $g(x_0) \neq x_0$. Therefore $g \in \Omega$.

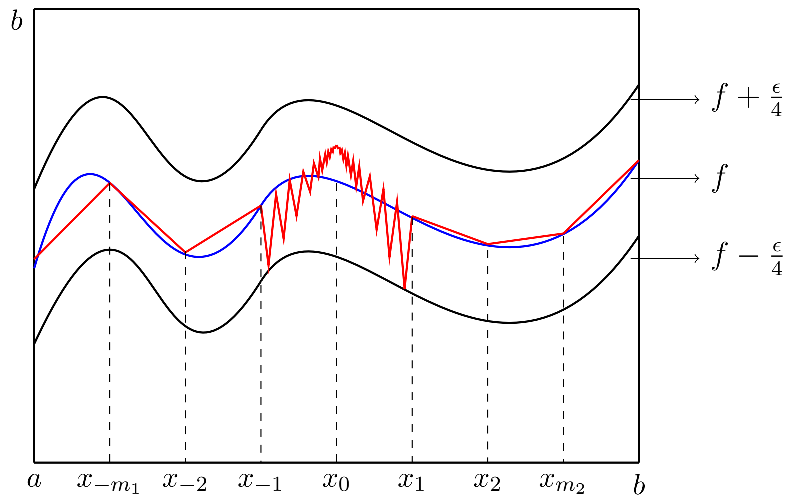

Figure.3.4.2

Also, whenever $x \in [x_k, x_{k+1}]$ for some $-(m_1+1) \leq k \leq m_2$, then $g(x)$ lies between $f(x_k) \pm \frac{\epsilon}{4}$ and $f(x_{k+1}) \pm \frac{\epsilon}{4}$. Moreover, as $|x_k - x| \leq \frac{1}{M} < \delta$, we have

$$|f(x_k) - f(x)| < \frac{\epsilon}{4}, \quad |g(x) - f(x_k)| \leq |f(x_{k+1}) - f(x_k)| + \frac{\epsilon}{2} < \frac{3\epsilon}{4}.$$

Therefore $|f(x) - g(x)| < \epsilon$ and hence $||f - g||_\infty = \sup_{x \in I} |f(x) - g(x)| < \epsilon$. This completes the proof. $\square$

We now prove an another theorem on nonexistence of iterative roots of continuous functions having finitely many non-isolated forts by generalizing the concepts of height given in (Zhang, 1997).

From Proposition 3.3.8, we see that $\{N^*(f^k)\}_{k=1}^{\infty}$ is a non-decreasing sequence of non-negative integers. Let $H^*(f)$ denotes the least positive integer m such that $N^*(f^m) = N^*(f^{m+1})$, if it exists, otherwise $H^*(f) = \infty$.

Note that, $H^*(F) \leq 1$ if and only if F has no non-isolated forts in its characteristic interval. Therefore, if $H^*(F) \leq 1$, then the existence of iterative roots of F can be given based on the assumptions of Theorem 3.4.1. On the other hand, if $H^*(F) > 1$ the following theorem establishes the nonexistence of iterative roots.

Theorem 3.5.4. *Let $F : I \to I$ be a continuous function such that $H^*(F) > 1$ then F has no iterative roots of order $n > N^*(F)$.*

Proof. Suppose f be an iterative root of order $n > N^*(F)$ of F. Since $H^*(F) > 1$, we have

$$N^*(f^n) = N^*(F) < N^*(F^2) = N^*(f^{2n})$$

and hence $H^*(f) > n$. Therefore

$$N^*(f) < N^*(f^2) < N^*(f^3) < \cdots < N^*(f^n).$$

This implies that $N^*(f^n) = N^*(F) \geq n$, a contradiction. □

Theorem 3.5.5. *Let $G : I \to I$ be a continuous function. Then for any $\epsilon > 0$ and any $n \in \mathbb{N}$, there exists a continuous function $F : I \to I$ with $N^*(F) = n$ and $H^*(F) > 1$ such that $||F - G||_\infty < \epsilon$.*

Proof. Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be any points in $[a, b]$ such that at least one of the α_i is a fixed point of G and $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. For $1 \leq i \leq n$, let $[a_i, b_i]$ be a closed neighborhood of α_i such that $[a_i, b_i] \cap [a_j, b_j] = \phi$ for all $i \neq j$. Define the functions $G_i : [a_i, b_i] \to \mathbb{R}$ by,

$$G_i(x) = \begin{cases} (x - \alpha_i) \sin \frac{1}{(x-\alpha_i)}, & \text{if } \quad x \neq \alpha_i \\ 0, & \text{if } \quad x = \alpha_i, \end{cases}$$

where $1 \leq i \leq n$. Choose a positive real number β_i such that

$$\beta_i G_i : [a_i, b_i] \to [G(\alpha_i) - \frac{\epsilon}{2}, G(\alpha_i) + \frac{\epsilon}{2}].$$

Let $\Lambda_0 = [a, a_1]$ and $\Lambda_i = [b_i, a_{i+1}]$ for $1 \leq i \leq n - 1$ and $\Lambda_n = [b_n, b]$. Choose continuous functions $H_i : \Lambda_i \to G(\Lambda_i)$ such that

$$H_i(x) \in (G(x) - \frac{\epsilon}{2}, G(x) + \frac{\epsilon}{2}) \text{ for all } x \in \Lambda_i,$$

in particular $H_0(a) = G(a)$, and $H_i(a_{i+1}) = G_i(a_{i+1})$, $H_i(b_i) = G_i(b_i)$ for $1 \leq i \leq n - 1$ and $H_n(b_n) = G_n(b_n)$, $H_n(b) = G(b)$. Now, define $G : I \to I$ by

$$F(x) = \begin{cases} H_i(x), & \text{if } \quad x \in \Lambda_i \\ \beta_i G_i(x), & \text{if } \quad x \in [a_i, b_i]. \end{cases}$$

By construction, F is continuous on I, $S^*(F) = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $H^*(F) > 1$ and $||G - F||_\infty < \epsilon$. $\square$

From Theorem 3.5.5, we can conclude that, for each positive integer n if $C_n = \{F \in C(I) \mid H^*(F) > 1 \text{ and } N^*(F) = n\}$ then C_n is dense in $C(I)$ where each member in C_n does not possess iterative root of order $k > n$.

Summary of the Chapter

In this chapter we studied the iterative root problem for the class of non-PM functions using characteristic interval. We proved the following:

- An extension theorem of iterative roots from characteristic interval.
- As an application of nonexistence theorem we constructed a class of functions which are dense in $C(I)$.

We also observe the following problems for future work.

1. Let $F : I \to I$ be any continuous function with finitely many non-isolated forts and $H^*(F) > 1$. Does there exist iterative roots of order $n \leq N^*(F)$?

Chapter 4

Series-Like Iterative Functional Equation for PM Functions

In this chapter, we study the existence of solutions of the series-like iterative functional equation

$$\sum_{n=1}^{\infty} \lambda_n f^n(x) = F(x) \text{ for all } x \in I, \tag{4.0.1}$$

where $\{\lambda_n\}$ is a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} \lambda_n = 1$ and $F \in C(I)$. The equation (4.0.1) is obviously a generalization of the functional equation

$$f^n(x) = F(x) \text{ for all } x \in I, \tag{4.0.2}$$

discussed in Chapter 2. It has been proved that the iterative functional equation (4.0.1) has solution provided F is strictly increasing. We extend this problem for the class of piecewise monotone functions.

4.1 Preliminaries

The existence of continuous and differentiable solutions of the polynomial-like iterative functional equation

$$\sum_{i=1}^{n} \lambda_i f^i(x) = F(x) \text{ for all } x \in I \tag{4.1.3}$$

has been studied by W. Zhang (Zhang, 1988, 1990). For given real numbers $M \geq 0$ and $m > 0$, we define

$$\mathcal{R}(I) = \{f \in C(I,I) \ : \ f(a) = a, \ f(b) = b\},$$

$$\mathcal{R}(I,M) = \{f \in \mathcal{R}(I) \ : \ 0 \le f(x) - f(y) \le M(x-y) \ \forall \ x,y \in I \text{ with } x > y\}$$

and

$$\mathcal{R}(I,m,M) = \{f \in \mathcal{R}(I,M) \ : \ m(x-y) \le f(x) - f(y) \ \forall \ x,y \in I \text{ with } x > y\}.$$

Theorem 4.1.1. *(Zhang, 1988) Let m, M and λ_1 be positive real numbers and $\lambda_i \ge 0$ for $2 \le i \le n$ with $\sum_{i=1}^{n} \lambda_i = 1$. Then for each $F \in \mathcal{R}(I, \lambda_1 M)$ there is a function $f \in \mathcal{R}(I,m,M)$ such that f satisfies the polynomial-like iterative functional equation (4.1.3). Moreover, if $\lambda_1 \ge 1 - \frac{m}{\sum_{i=1}^{n-1} M^i}$, then the solution is unique and stable.*

Further, Zhang (Zhang, 1990) proved the existence of differentiable solutions of the polynomial-like functional equation (4.1.3).

For given constants $M \ge 0$, $M^* \ge 0$, and $m > 0$, we define the families of functions

$$\mathcal{R}^1(I,M) = \{f \in C^1(I,I) \ : \ f(a) = a, \ f(b) = b, \ 0 \le f'(x) \le M \ \forall \ x \in I\},$$

$$\mathcal{R}^1(I,M,M^*) = \{f \in \mathcal{R}^1(I,M) \ : \ |f'(x) - f'(y)| \le M^*|x-y| \ \forall \ x,y \in I\}$$

and

$$\mathcal{R}^1(I,m,M,M^*) = \{f \in \mathcal{R}^1(I,M,M^*) \ : \ m \le f'(x) \ \forall \ x \in I\}.$$

Theorem 4.1.2. *(Zhang, 1990) Let m, M and M^* be positive real numbers. Suppose that $M > 1$ and $\lambda_1 > K_0 M^2$ where $K_0 = \frac{1}{M-1} \sum_{i=1}^{n-1} \lambda_{i+1} M^{i-1}(M^i - 1)$ and $\lambda_1 > 0$, $\lambda_i \ge 0$, $i = 2,3,\ldots,n$ with $\sum_{i=1}^{n-1} \lambda_i = 1$. Then for each $F \in \mathcal{R}^1(I,m,\lambda_1 M, M^*)$ there is a function $f \in \mathcal{R}^1(I,M,M')$ such that f satisfies the polynomial-like functional equation (4.1.3), where $M' = \frac{M^*}{(\lambda_1 - K_0 M^2)}$.*

The existence of solutions of the equation (4.1.3) with variable coefficients can be found in (Zhang and Baker, 2000; Li and Deng, 2005). Moreover, the existence of continuous and differentiable solutions and stability of the solution of a more general equation of the form (4.0.1) has been studied in (Jarczyk, 1997; Murugan and Subrahmanyam, 2005, 2009).

For any sequence of real numbers $\{a_i\}$, define the support of the sequence $\{a_i\}$ by $Supp\ \{a_i\} = \{i \in \mathbb{N}\ :\ a_i \neq 0\}$. Further, for any non-empty, non-zero subset of integers I, the greatest common divisor of I is defined to be the maximal number $p \in \mathbb{N}$ such that $I \subseteq p\mathbb{Z}$.

Theorem 4.1.3. *(Jarczyk, 1997) Let $\{a_i\}$ be a non-zero sequence of non-negative real numbers such that the greatest common divisor of $Supp\{a_i\}$ equals 1. If either $D \subseteq (-\infty, 0)$ or $D \subseteq (0, \infty)$ and $f : D \to D$ satisfies the equation*

$$\sum_{i=1}^{\infty} a_i f^i(x) = x, \tag{4.1.4}$$

then there is exactly one positive real root of the equation $\sum_{i=1}^{\infty} a_i \lambda^i = 1$ and $f(x) = cx$ for all $x \in D$.

Theorem 4.1.4. *(Murugan and Subrahmanyam, 2005) Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of non-negative numbers and $\alpha_n \leq \lambda_n \leq \beta_n$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Suppose $0 < m < 1$, $M > 1$ and*

(i) $\sum_{n=1}^{\infty} \beta_n M^n < \infty$ *(ii)* $K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} > 0$.

Then for any $F \in \mathcal{R}(I, K_1 m, K_0 M)$ the series-like functional equation (4.0.1) has a solution $f \in \mathcal{R}(I, m, M)$, where $K_1 = \sum_{n=1}^{\infty} \beta_n M^{n-1}$.

We remark here that the solutions of all the above functional equations has studied exclusively for the class of continuous strictly increasing functions. Recently Liu et al., (Liu and Gong, 2017), proved the existence of solutions of polynomial-like functional equations for PM functions of height less than two. In the rest of this chapter we study the continuous solutions of the series-like functional equation (4.0.1) for the class of PM functions of height less than or equal to one using characteristic interval.

4.2 A Topological Result

In order to study the existence of solutions of series-like iterative functional equation (4.0.1) for the class of PM functions, at first we study the existence of solutions in the characteristic interval and then we extend that iterative root to the whole interval.

Let $I = [a, b]$ be any closed and bounded interval in the real line and $a', b' \in I$ such that $a' < b'$ and m, M be positive real numbers. Define,

$$\mathbb{S}([a', b'], m, M) = \{f \in PM(I) \ : \ Ch_f = [a', b'], \ f|_{Ch_f} \in \mathcal{R}([a', b'], m, M)\}.$$

Since any $f \in \mathbb{S}([a', b'], m, M)$ is strictly increasing in Ch_f, the set $\mathbb{S}([a', b'], m, M)$ consists only a collection of PM functions of height less than or equal to one whose characteristic interval is $[a', b']$.

Proposition 4.2.1. *The set* $\mathbb{S}([a', b'], m, M)$ *is a compact, convex subset of* $C(I, \mathbb{R})$.

Proof. First we prove $\mathbb{S}$ is a convex set. Let $f, g \in \mathbb{S}$. For each $t \in (0, 1)$, put $h_t = tf + (1-t)g$. Clearly $h_t(a') = a'$ and $h_t(b') = b'$. Now we prove $Ch_{h_t} = [a', b']$. For any $x \in I$, since $f(x), g(x) \in [a', b']$ and every interval is convex we have, $h_t(x) \in [a', b']$. Consequently the range of h_t is contained in $[a', b']$. As h_t is a order preserving homeomorphism on $[a', b']$, to prove $Ch_{h_t} = [a', b']$, it is enough if we prove a' and b' are forts of h_t. For each $\epsilon > 0$, if $\alpha \in (a' - \epsilon, a')$ we have,

$$h_t(\alpha) = tf(\alpha) + (1-t)g(\alpha) \geq tf(a') + (1-t)g(a') = h_t(a')$$

as $f(\alpha) \geq f(a')$ and $g(\alpha) \geq g(a')$. Therefore for every $\epsilon > 0$, h_t is monotonically decreasing on $(a' - \epsilon, a')$. Since h_t is monotonically increasing on $[a', b']$, it follows that a' is a fort of h_t. Similarly we can prove b' is a fort of h_t.
Also, for any $x, y \in [a', b']$ with $x > y$, an easy calculation shows that

$$m(x - y) \leq h_t(x) - h_t(y) \leq M(x - y)$$

as $m(x-y) \leq f(x) - f(y) \leq M(x-y)$ and $m(x-y) \leq g(x) - g(y) \leq M(x-y)$. This proves that $\mathbb{S}$ is convex.

To prove $\mathbb{S}$ is compact, first we prove $\mathbb{S}$ is closed. Let $\{f_n\}$ be a sequence in $\mathbb{S}$ such that f_n converges to f uniformly. Then $f(a') = \lim_{n\to\infty} f_n(a') = a'$ and $f(b') = \lim_{n\to\infty} f_n(b') = b'$. Also, it is easy to verify that

$$m(x-y) \leq f(x) - f(y) \leq M(x-y) \ \forall\ x, y \in [a', b'] \text{ with } x > y.$$

Note that range of f is contained in $[a', b']$ and f is monotone on $[a', b']$. Therefore to prove $Ch_f = [a', b']$, it is enough to prove for each $\epsilon > 0$, the function f is monotonically decreasing on $(a' - \epsilon, a')$ and $(b', b' + \epsilon)$. Let $\alpha \in (a' - \epsilon, a')$ and $\beta \in (b', b' + \epsilon)$. Since $f_n(\alpha) \geq f_n(a')$ and $f_n(b') \leq f_n(\beta)$ we have,

$$f(\alpha) \geq f(a') \text{ and } f(b') \leq f(\beta).$$

This shows that $f \in \mathbb{S}$. i.e., $\mathbb{S}$ is closed. Also, for any $f \in \mathbb{S}$, $|f(x)| \leq \max\{|a|, |b|\}$ and hence $\mathbb{S}$ is uniformly bounded. Now it follows from Arzela-Ascoli's theorem that $\mathbb{S}$ is compact. $\square$

4.3 Existence of Solutions on the Characteristic Interval

Let $F \in PM(I)$ with $H(F) \leq 1$ and $Ch_F = [a', b']$. Let $F_0 = F|_{[a',b']}$. Then F_0 is strictly monotone on $[a', b']$. Therefore, by applying Theorem 4.1.4, the functional equation (4.0.1) has a solution for F_0 on $[a', b']$. The following Lemma establishes a solution to the functional equation (4.0.1) for the class of functions in $\mathcal{R}([a', b'], m, M)$. The proof of the lemma given below follows from Theorem 4.1.4.

Lemma 4.3.1. *Let $F \in PM(I)$ with $H(F) \leq 1$, $Ch_F = [a', b']$ and $F_0 = F|_{[a',b']}$. Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of non-negative numbers and $\alpha_n \leq \lambda_n \leq \beta_n$*

such that $\sum_{n=1}^{\infty} \lambda_n = 1$. *Suppose* $0 < m < 1$, $M > 1$ *and*

(i) $\sum_{n=1}^{\infty} \beta_n M^n < \infty$ $\quad$ (ii) $K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} > 0$.

If $F_0 \in \mathcal{R}([a', b'], K_1 m,\ K_0 M)$, *then there exists a function* $f_0 \in \mathcal{R}([a', b'], m, M)$ *such that*

$$\sum_{n=1}^{\infty} \lambda_n f_0^n(x) = F_0(x) \textit{ for all } x \in [a', b'],$$

where $K_1 = \sum_{n=1}^{\infty} \beta_n M^{n-1}$.

For each $F \in \mathbb{S}([a', b'], m, M)$, Lemma 4.3.1 guarantees the solution of the series-like iterative functional equation (4.0.1) on the characteristic interval of F. The following lemma will be useful in extending the above solution to the whole interval I.

Lemma 4.3.2. *Let* $\{\lambda_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequence of non-negative numbers and* $\alpha_n \leq \lambda_n \leq \beta_n$ *such that* $\sum_{n=1}^{\infty} \lambda_n = 1$. *Suppose* $0 < m < 1$, $M > 1$ *and*

(i) $\sum_{n=1}^{\infty} \beta_n M^n < \infty$ $\quad$ (ii) $K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} > 0$.

For each $f \in \mathcal{R}([a', b'], K_0, K_1)$, *let* $L_f : [a', b'] \to [a', b']$ *be a function defined by*

$$L_f(x) = \sum_{n=1}^{\infty} \lambda_n f^{n-1}(x) \quad \textit{for all } \ x \in [a', b']. \tag{4.3.5}$$

Then $L_f \in \mathcal{R}([a', b'], K_0, K_1)$ *and the function* L_f *is invertible, in particular* $L_f^{-1} \in \mathcal{R}([a', b'], \frac{1}{K_0}, \frac{1}{K_1})$.

Proof. The proof follows from Lemma 3.2 of (Murugan and Subrahmanyam, 2009).

□

4.4 Extension of Solutions from the Characteristic Interval

Suppose that $F \in PM(I)$ with $H(F) \leq 1$. Then $F|_{[a',b']}$ is strictly monotone, where $Ch_F = [a', b']$. Therefore, by using the hypothesis of Lemma 4.3.1, any $F|_{Ch_F} \in \mathcal{R}([a', b'], K_1 m, K_0 M)$ the series-like functional equation (4.0.1) has a solution in $\mathcal{R}([a', b'], m, M)$. By extending this solution, we prove that, any function $F \in \mathbb{S}([a', b'], m, M)$ and for any non-negative sequence $\{\lambda_n\}$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$, the functional equation (4.0.1) has a solution.

Theorem 4.4.1. *Let $\{\lambda_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence of non-negative numbers and $\alpha_n \leq \lambda_n \leq \beta_n$ such that $\sum_{n=1}^{\infty} \lambda_n = 1$. Suppose $0 < m < 1$, $M > 1$ and*

(i) $\sum_{n=1}^{\infty} \beta_n M^n < \infty$, *(ii)* $K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} > 0$, *(iii)* $F(I) \subseteq F([a', b'])$.

Then for any $F \in \mathbb{S}([a', b'], K_1 m, K_0 M)$ the series-like functional equation (4.0.1) has a solution $f \in \mathbb{S}([a', b'], m, M)$, where $K_1 = \sum_{n=1}^{\infty} \beta_n M^{n-1}$.

Proof. Put $F_0 = F|_{[a',b']}$. Then $F|_{[a',b']} \in \mathcal{R}([a', b'], K_1 m, K_0 M)$. Therefore, by Lemma 4.3.1, there exists $f_0 \in \mathcal{R}([a', b'], m, M)$ such that

$$\sum_{n=1}^{\infty} \lambda_n f_0^n(x) = F_0(x) \text{ for all } x \in [a', b].$$

We now extend this f_0 to the whole interval I. For this, we define the function $L_{f_0} : [a', b'] \to [a', b']$ by

$$L_{f_0}(x) = \sum_{n=1}^{\infty} \lambda_n f_0^{n-1}(x) \text{ for all } x \in [a', b'].$$

Then, by Lemma 4.3.2, $L_{f_0} \in \mathcal{R}([a', b'], K_0, K_1)$ and L_{f_0} is invertible. Moreover $L_{f_0}^{-1} \in \mathcal{R}([a', b'], \frac{1}{K_1}, \frac{1}{K_0})$.

Now we define $f : I \to I$ as follows:

$$f(x) := \begin{cases} L_{f_0}^{-1} \circ F(x) & \text{if} \quad a \le x < a' \\ f_0(x) & \text{if} \quad a' \le x \le b' \\ L_{f_0}^{-1} \circ F(x) & \text{if} \quad b' < x \le b. \end{cases}$$

The function f is well defined by condition (iii). Consequently, for each $x \in [a', b']$ we have $\sum_{n=1}^{\infty} \lambda_n f^n(x) = \sum_{n=1}^{\infty} \lambda_n f_0^n(x) = F_0(x) = F(x)$.
On other hand, if $x \in I \backslash [a', b]$ then

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n f^n(x) &= \lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) + \cdots \\ &= \lambda_1 L_{f_0}^{-1}(F(x)) + \lambda_2 f_0(L_{f_0}^{-1}(F(x))) + \cdots + \lambda_n f_0^{n-1}(L_{f_0}^{-1}(F(x))) + \cdots \\ &= \sum_{n=1}^{\infty} \lambda_n f_0^{n-1}(L_{f_0}^{-1}(F(x))) \\ &= L_{f_0}(L_{f_0}^{-1}(F(x))) \\ &= F(x). \end{aligned}$$

Hence f satisfies the functional equation (4.0.1) for all $x \in I$. To prove f is continuous on I it is enough if we prove f is continuous at a' and b'. Let $\{x_n\}$ be a sequence in $[a, a')$ such that $x_n \to a'$ as $n \to \infty$. Now,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} L_{f_0}^{-1}(F(x_n)) = L_{f_0}^{-1}(F(a')) = L_{f_0}^{-1}(a') = f(a').$$

Therefore f is continuous at a'. Similarly, we can prove that f is continuous at b'. Since $Ch_f = [a', b']$ and $f_0 \in \mathcal{R}([a', b'], m, M)$, we have $f \in \mathbb{S}([a', b'], m, M)$. $\square$

Example 4.4.2. *Let $F : [0, 1] \to [0, 1]$ be a function defined by*

$$F(x) = \begin{cases} x & \textit{if} \quad x \in [0, \frac{1}{2}] \\ 1 - x & \textit{if} \quad x \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly $F \in PM([0, 1])$ with $H(F) = 1$ and $N(F) = 1$. Now, consider the series-like iterative functional equation (4.0.1) with $\lambda_1 = \frac{2}{3}$ and $\lambda_n = \frac{1}{4^{n-1}}$ for all $n \ge 2$. Choose $\alpha_n = \beta_n = \lambda_n$ for all $n \in \mathbb{N}$ and $m = \frac{1}{4}$, $M = 2$. Note that

$$\sum_{n=1}^{\infty} \beta_n M^n = \frac{4}{3} + 2\sum_{n=2}^{\infty} \frac{1}{2^{n-1}} < \infty,$$

and

$$K_0 = \sum_{n=1}^{\infty} \alpha_n m^{n-1} = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{4^{2n-2}} = \frac{11}{15},$$

also

$$K_1 = \sum_{n=1}^{\infty} \beta_n M^{n-1} = \frac{2}{3} + \sum_{n=2}^{\infty} \frac{1}{2^{n-2}} = \frac{8}{3}.$$

Clearly $F \in \mathbb{S}([0, \frac{1}{2}], \frac{2}{3}, \frac{22}{15})$. Therefore, by Theorem 4.4.1, the functional equation (4.0.1) has a solution $f \in \mathbb{S}([0, \frac{1}{2}], \frac{1}{4}, 2)$.

Summary of the Chapter

In this chapter, we discussed the solutions of series-like iterative functional equation for a class of PM functions. More specifically we proved the following.

- For every $F \in \mathbb{S}([a', b'], K_1 m, K_0 M)$, the series-like iterative functional equation (4.0.1) has a solution.

Observations and future work:

1. Under what condition the series-like iterative functional equation (4.0.1) has a solution for a PM function having height greater than one?

Chapter 5

Uniqueness of Iterative Roots

It is known that the iterative roots of continuous functions are not necessarily unique, if exists. In this chapter, we discuss the uniqueness of iterative roots of order preserving homeomorphisms. We prove an order preserving homeomorphism from an interval onto itself do not possess different iterative roots which are subcommuting or comparable using the points of coincidence of functions.

5.1 Introduction

Let us recall Theorem 1.3.6 for the case of an order preserving homeomorphism.

Theorem 5.1.1. *(Kuczma et al., 1990) Let $I \subseteq \mathbb{R}$ be any interval. Then every strictly increasing continuous function F from I into itself possesses strictly increasing continuous iterative roots of order $n \in \mathbb{N}$.*

Theorem 5.1.1 guarantees the existence of strictly increasing continuous iterative roots of a strictly increasing continuous functions. Moreover, this strictly increasing continuous n^{th} order iterative root depends on arbitrary strictly increasing homeomorphisms (see Theorem 1.3.6), and hence its iterative roots are not unique. In fact, every strictly increasing continuous function, other than identity, possesses infinitely many strictly increasing continuous n^{th} order iterative roots.

The uniqueness of iterative roots of a special class of monotonic functions was conjectured by Bödewadt (Bödewadt, 1944) and answered in negative by Smajdor (Smajdor, 1973). Motivated by Bödewadt, suppose f and g are two iterative roots of order n of a strictly increasing homeomorphism F (i.e. $f^n = g^n = F$), it is reasonable to ask under what condition f and g are identically equal?

Zdun (Zdun, 1988) gave an affirmative answer to the above question.

Theorem 5.1.2. *(Zdun, 1988) If f and g are strictly order preserving homeomorphisms from I onto itself such that $f \circ g = g \circ f$ and $f^n = g^n$ for some $n \in \mathbb{N}$, then $f = g$.*

In this chapter, our aim is to provide weaker condition than commutativity to get the uniqueness of iterative roots of order preserving homeomorphisms. One of the weaker condition of commutativity given by Sessa (Sessa, 1982) is as follows: The functions $f, g : X \to X$ are called *weakly commuting*, if $d(fg(x), gf(x)) \leq d(f(x), g(x))$ for all $x \in X$. It is clear from definition that every pair of commuting functions are weakly commuting, but not conversely.

Remark 5.1.3. *If f, g are order preserving homeomorphisms such that f, g are weakly commuting but not commuting, then there is at least one $x \in I$ such that $d(fg(x), gf(x)) > 0$. Therefore, $d(f(x), g(x)) \geq d(fg(x), gf(x)) > 0$. This immediately implies that $f \neq g$ on I. Thus, the condition of weakly commuting is helpless in the study of uniqueness of iterative roots of order preserving homeomorphisms.*

We further investigate this uniqueness problem. Indeed, we prove Theorem 5.1.2 with a weaker condition. In due course, we also provide some sufficient conditions on the uniqueness of iterative roots order preserving homeomorphisms, using the points of coincidence of functions.

Throughout this chapter we fix $I = (a, b)$, where $-\infty \leq a \leq b \leq \infty$, and let $\mathscr{H}(I)$ denote the set of all order preserving homeomorphisms from I onto itself. Here after we always assume all the functions are in the class $\mathscr{H}(I)$ unless otherwise stated.

5.2 Set of Points of Coincidence

Let f and g be two order preserving homeomorphisms from the interval I onto $J \subseteq I$. We say f and g are *comparable*, if either $f(x) \leq g(x)$ or $g(x) \leq f(x)$ for all $x \in I$, and if the inequalities are strict then we say f and g are *strictly comparable.*

Proposition 5.2.1. *If f and g are two strictly comparable order preserving homeomorphisms from I onto $J \subseteq I$ then f^n and g^n are strictly comparable order preserving homeomorphisms, for all $n \in \mathbb{N}$. In addition to that, if $J = I$ then f^{-n} and g^{-n} are also strictly comparable order preserving homeomorphisms, for all $n \in \mathbb{N}$.*

Proof. First we prove the result for positive integers. Assume $f(x) < g(x)$ for all $x \in I$. Since f is strictly increasing, by applying f on the above inequality, we have

$$f^2(x) < f(g(x)) < g^2(x) \text{ for all } x \in I. \tag{5.2.1}$$

Now, by applying f on the inequality (5.2.1) repeatedly we get,

$$f^n(x) < g^n(x) \text{ for all } x \in I \ (n \in \mathbb{N}). \tag{5.2.2}$$

Now, we prove the result for negative integers by assuming $J = I$. First we prove if $f(x) < g(x)$, then $g^{-1}(x) < f^{-1}(x)$ for all $x \in I$. Suppose there is a $t \in I$ such that $g^{-1}(t) \geq f^{-1}(t)$. Then

$$t = g(g^{-1}(t)) \geq g(f^{-1}(t)) > f(f^{-1}(t)) = t.$$

This is a contradiction. Hence $g^{-1}(x) < f^{-1}(x)$ for all $x \in I$. Therefore, by proceeding similar to (5.2.1) and (5.2.2), we get

$$g^{-n}(x) < f^{-n}(x) \text{ for all } x \in I \ (n \in \mathbb{N}). \tag{5.2.3}$$

This completes the proof.

□

For any two functions f and g, we denote the *set of points of coincidence* of f and g by $Z(f,g)$ and is defined by $Z(f,g) = \{x \in I \mid f(x) = g(x)\}$.

Theorem 5.2.2. *If $Z(f,g)$ is a finite set, then $f^n \neq g^n$ for all $n \in \mathbb{Z}\backslash\{0\}$.*

Proof. If $Z(f,g)$ is empty, then either $f(x) < g(x)$ or $g(x) < f(x)$ for all $x \in I$. Therefore, by Proposition 5.2.1, $g^n(x) \neq f^n(x)$ for all $x \in I$ and for all $n \in \mathbb{Z}\backslash\{0\}$. On the other hand, if $Z(f,g)$ is non empty, we proceed as follows:

If f and g does not have a common fixed point, then the set $\{x \in I \mid f(x) = g(x) = x\}$ must be empty. On the one hand, if at least one of them have a fixed point say $f(t) = t$ for some $t \in I$, then $g(t) \neq t$. Without loss of generality, let $g(t) < t$. Hence $g^n(t) < t$ but $f^n(t) = t$, which in turn implies $f^n \neq g^n$ for all $n \in \mathbb{Z}$.

On the other hand, if none of them have a fixed point, then either $f(x) < x$ and $g(x) < x$ for all $x \in I$ or $x < f(x)$ and $x < g(x)$ for all $x \in I$.

In the first case, choose $\alpha \in Z(f,g)$ such that $t \notin Z(f,g)$ for all $t \in (a,\alpha)$. Then the functions f and g are order preserving homeomorphisms from (a,α) onto $(f(a), f(\alpha)) \subseteq (a,\alpha)$, moreover, f and g are comparable on (a,α). Therefore, by Proposition 5.2.1, $f^n \neq g^n$ for all $n \in \mathbb{Z}$.

In the later case, choose $\alpha \in Z(f,g)$ such that $t \notin Z(f,g)$ for all $t \in (\alpha,b)$. Then the functions f and g are order preserving homeomorphisms from (α,b) onto $(f(\alpha), f(b)) \subseteq (\alpha,b)$, moreover, f and g are comparable on (α,b), again by Proposition 5.2.1, $f^n \neq g^n$ for all $n \in \mathbb{Z}$.

If f and g have common fixed points, then choose $\alpha \in \{x \in I \mid f(x) = g(x) = x\}$ such that $t \notin \{x \in I \mid f(x) = g(x) = x\}$ for all $t \in (a,\alpha)$. Hence f and g are self maps on (a,α) and none of the function f and g have a common fixed point on (a,α). Therefore, by above argument $f^n \neq g^n$ on $(a,\alpha) \subseteq I$ for all $n \in \mathbb{Z}$. □

Lemma 5.2.3. *If $fg = gf$, then $f^n g^m = g^m f^n$ for all $n, m \in \mathbb{Z}$.*

Proof. First we prove $f^n g = g f^n$ for all $n \in \mathbb{N}$ using induction on n. Clearly

$f^2g = f(fg) = f(gf) = (gf)f = gf^2$. Assume

$$f^k g = gf^k \text{ for all } 1 \leq k \leq n-1. \tag{5.2.4}$$

Hence, $f^n g = f(f^{n-1}g) = f(gf^{n-1}) = gf^n$. Therefore

$$f^n g = gf^n \text{ for all } n \in \mathbb{N}. \tag{5.2.5}$$

Pre and post multiplying by f^{-1} on $fg = gf$, we get $f^{-1}g = gf^{-1}$. Hence by repeating process as in (5.2.4) and (5.2.5), we get $f^{-n}g = gf^{-n}$ for all $n \in \mathbb{N}$. Therefore,

$$f^n g = gf^n \text{ for all } n \in \mathbb{Z}. \tag{5.2.6}$$

Since $f^n g = gf^n$ for each $n \in \mathbb{Z}$, again by above argument, we have $f^n g^m = g^m f^n$ for all $m \in \mathbb{Z}$. □

Proposition 5.2.4. *If $x \in Z(f,g)$ and $fg = gf$, then $f^n(x), g^n(x) \in Z(f,g)$ for all $n \in \mathbb{Z}$.*

Proof. For $x \in Z(f,g)$, $f(f(x)) = f(g(x)) = g(f(x))$, therefore $f(x) \in Z(f,g)$. Hence $f^n(x) \in Z(f,g)$ for all $n \in \mathbb{N}$. Since

$$f(f^{-1}(x)) = f^{-1}(f(x)) = f^{-1}(g(x)) = g(f^{-1}(x)), \tag{5.2.7}$$

we must have, $f^{-1}(x) \in Z(f,g)$, here the last equality in (5.2.7) holds by Lemma 5.2.3. Therefore, by above argument, $f^{-n}(x) \in Z(f,g)$ for all $n \in \mathbb{N}$. Similarly $g^n(x) \in Z(f,g)$ for all $n \in \mathbb{Z}$. □

Theorem 5.2.5. *If $fg = gf$, then $Z(f,g) = Z(f^n, g^n)$ for all $n \in \mathbb{Z}\backslash\{0\}$*

Proof. **Step: 1** We prove $Z(f,g) = Z(f^n, g^n)$ for all $n \in \mathbb{N}$ using induction on n. First we prove $Z(f,g) = Z(f^2, g^2)$. By Proposition 5.2.4, we have $Z(f,g) \subseteq Z(f^2, g^2)$. Let $x \in Z(f^2, g^2)$. If $f(x) \neq g(x)$, without loss of generality, say $f(x) < g(x)$ then

$$f^2(x) < f(g(x)) = g(f(x)) < g^2(x),$$

which is not possible. Therefore $Z(f,g) = Z(f^2, g^2)$.

Assume $Z(f,g) = Z(f^k, g^k)$ for $2 \leq k \leq n-1$. For $x \in Z(f,g)$, we have

$$\begin{aligned}
f^n(x) &= f^{n-1}(f(x)) \\
&= f^{n-1}(g(x)) \\
&= f(g^{n-1}(x)) \quad \text{(since } f^n(x) = f^{n-1}(g(x)) = f(g^{n-1}(x)) = f^n(x)) \\
&= g^{n-1}(f(x)) \quad \text{(by Lemma 5.2.3)} \\
&= g^{n-1}(g(x)) \\
&= g^n(x).
\end{aligned}$$

Therefore $Z(f,g) \subseteq Z(f^n, g^n)$. If $x \in Z(f^n, g^n)$ with $f(x) < g(x)$, then

$$\begin{aligned}
f^2(x) < f(g(x)) &= g(f(x)) < g^2(x) \\
\Longrightarrow f^3(x) < f^2(g(x)) &= g(f^2(x)) < g^3(x) \\
&\vdots \\
\Longrightarrow f^n(x) < f^{n-1}(g(x)) &= g(f^{n-1}(x)) < g^n(x)
\end{aligned}$$

which is not possible. Therefore, $Z(f,g) = Z(f^n, g^n)$ for all $n \in \mathbb{N}$.

Step: 2 We prove $Z(f,g) = Z(f^{-n}, g^{-n})$ for all $n \in \mathbb{N}$.

It is clear from Step: 1 that, $Z(f^{-1}, g^{-1}) = Z(f^{-n}, g^{-n})$ for all $n \in \mathbb{N}$. Therefore to prove Step: 2, it is enough to prove $Z(f,g) = Z(f^{-1}, g^{-1})$.

For $x \in Z(f,g)$, if $f^{-1}(x) < g^{-1}(x)$ then $x < f(g^{-1}(x))$. But,

$$\begin{aligned}
f(g^{-1}(x)) &= g^{-1}(f(x)) \quad \text{(by Lemma 5.2.3)} \\
&= g^{-1}(g(x)) \quad \text{(as } f(x) = g(x)) \\
&= x,
\end{aligned}$$

which is a contradiction to the fact that $x < f(g^{-1}(x))$. On the other hand, if $g^{-1}(x) < f^{-1}(x)$ then $x < g(f^{-1}(x))$. But,

$$\begin{aligned}
g(f^{-1}(x)) &= f^{-1}(g(x)) \quad \text{(by Lemma 5.2.3)} \\
&= f^{-1}(f(x)) \quad \text{(as } f(x) = g(x)) \\
&= x,
\end{aligned}$$

again a contradiction to $x < g(f^{-1}(x))$. Therefore $f^{-1}(x) = g^{-1}(x)$ whenever $f(x) = g(x)$, i.e. $Z(f,g) \subseteq Z(f^{-1}, g^{-1})$. Now by replacing f and g by f^{-1} and g^{-1} respectively, we get $Z(f,g) = Z(f^{-1}, g^{-1})$. □

Now, it is easy to observe that the Theorem 5.1.2 is a straight forward application of the above theorem.

Theorem 5.2.6. *Let $f, g \in \mathscr{H}(I)$ with out fixed points such that $fg = gf$. Suppose $Z(f^n, g^n)$ is an interval for some $n \in \mathbb{Z}$. Then $f = g$ on I.*

Proof. Since $fg = gf$, by Theorem 5.2.5, $Z(f,g) = Z(f^n, g^n)$. Without loss of generality, let $\alpha \in Z(f,g)$ such that $\alpha < f(\alpha)$. Also by Proposition 5.2.4, $f(\alpha) \in Z(f,g)$. Since $f^m(\alpha) \to b$ and $f^{-m}(\alpha) \to a$ as $m \to \infty$. Therefore,

$$I = (a,b) = \cup_{m \in \mathbb{Z}} [f^m(\alpha), f^{m+1}(\alpha)].$$

Let $y \in [f^m(\alpha), f^{m+1}(\alpha)]$ be arbitrary. Then there is an element $x \in [\alpha, f(\alpha)]$ such that $y = f^n(x)$. Since $f = g$ on $[\alpha, f(\alpha)]$, we have $y = f^m(x) = g^m(x)$. Therefore, by Lemma 5.2.3,

$$f(y) = f(g^m(x)) = g^m(f(x)) = g^m(g(x)) = g(g^m(x)) = g(y).$$

This completes the proof. □

5.3 Subcommuting and Comparable Iterative Roots

Definition 5.3.1. *(Głazowska and Matkowski, 2016) Let f and g be order preserving homeomorphisms on I. We say f subcommutes with g, if $fg(x) \leq gf(x)$ for all $x \in I$ and g subcommutes with f, if $gf(x) \leq fg(x)$ for all $x \in I$.*

Note that every commuting functions are subcommuting, but the converse is not necessarily true.

Example 5.3.2. *Let $f, g : (0,\infty) \to (0,\infty)$ be two functions defined by $f(x) = 2x$ and $g(x) = x^2$ for all $x \in (0,\infty)$. Clearly, f subcommutes with g as $f(g(x)) = 2x^2 \leq g(f(x)) = 4x^2$ for all $x \in (0,\infty)$. But f and g do not commute with each other, as $f(g(x)) = 2x^2 \neq g(f(x)) = 4x^2$ for all $x \in (0,\infty)$.*

Let $F : I \to I$ be an order preserving homeomorphism. We prove that it is not possible to have different iterative roots of F which are either comparable or subcommuting.

Theorem 5.3.3. *Let $F \in \mathscr{H}(I)$. Suppose $f, g \in \mathscr{H}(I)$ satisfies $f^n = g^n = F$ for some $n \in \mathbb{Z}$, then the following are equivalent.*

(i) f subcommutes with g.

(ii) f and g are comparable.

(iii) $f = g$.

Proof. (iii) implies (i) and (ii) are trivial.
($(i) \Rightarrow (iii)$) In view of Theorem 5.2.5, it is enough if we prove $fg = gf$ on I.
Suppose $fg(x) < gf(x)$ for some x. Then

$$\begin{aligned} g^{n+1}(x) &= g^n(g(x)) \\ &= f^n(g(x)) \\ &= f^{n-1}(f(g(x))) \\ &< f^{n-1}(g(f(x))) \\ &\leq f^{n-2}(g(f^2(x))) \\ &\vdots \\ &\leq g(f^n(x)) \\ &= g^{n+1}(x). \end{aligned}$$

i.e, $g^{n+1}(x) < g^{n+1}(x)$, a contradiction. Hence $fg = gf$. Therefore, by Theorem 5.2.5, $f = g$ on I.

$((ii) \Rightarrow (iii))$ Assume

$$f \leq g. \tag{5.3.8}$$

If possible, let $f(t) \neq g(t)$ for some $t \in I$. Therefore $f(t) < g(t)$. Since $f^n = g^n$, we have

$$g^n(t) = f^n(t) < f^{n-1}(g(t)) \leq g(f^{n-2}(g(t))), \tag{5.3.9}$$

where the last inequality in (5.3.9) holds by (5.3.8). But then $g^{n-1}(t) < f^{n-2}(g(t))$ as g^{-1} is an order-preserving homeomorphisms. Therefore,

$$g^{n-1}(t) < f^{n-2}(g(t)) \leq g(f^{n-3}(g(t))), \tag{5.3.10}$$

here the last inequality in (5.3.10) holds by (5.3.8). Since g^{-1} is an order-preserving homeomorphisms, the inequality (5.3.10) becomes, $g^{n-2}(t) < f^{n-3}(g(t))$. Continuing this process up to $(n-2)$ times we get

$$g(g(t)) < f(g(t)),$$

a contradiction to our assumption. Therefore $f = g$ on I. $\square$

Part of a theorem due to McShane (McShane, 1961) is observed below.

Corollary 5.3.4. *(McShane, 1961) The only order preserving iterative root of any order of the identity function on $\mathbb{R}$ is the identity function.*

Proof. Clearly, identity function is an iterative root of any order of the identity function, it follows from Theorem 5.3.3, that any order preserving homeomorphism whose iteration is identity becomes identity, as the identity function subcommutes (also commutes, so Theorem 5.2.5 also applicable) with any function. $\square$

Further, if $f \in \mathscr{H}(I)$ such that $f^n(x) = x$ for all $x \in I$ but f not identity, then there exists an interval (α, β) such that either $f(x) < x$ or $f(x) > x$ for all $x \in (\alpha, \beta)$ and $f((\alpha, \beta)) = (\alpha, \beta)$. Since $f^n(x) = x$ for all $x \in (\alpha, \beta)$ and f is comparable with identity, by Theorem 5.3.3, $f(x) = x$ on (α, β), which is a contradiction. This forces that identity is the only order preserving homeomorphism of the identity function.

From Theorem 5.3.3, we can conclude that the non-commuting, non-comparable iterative roots of an order preserving homeomorphism are all different. We provide an illustrative example. The construction given in this example is based on Theorem 1.3.6.

Example 5.3.5. *Consider the order preserving homeomorphism $F : [0,1] \to [0,1]$ defined by*

$$F(x) = \begin{cases} 4x, & \text{if } \ x \in [0, \frac{1}{8}) \\ \frac{4}{3}x + \frac{1}{3}, & \text{if } \ x \in [\frac{1}{8}, \frac{1}{4}) \\ \frac{4}{9}x + \frac{5}{9}, & \text{if } \ x \in [\frac{1}{4}, 1]. \end{cases}$$

In order to construct iterative roots of this function, first we define a sequence of disjoint intervals whose union is $[0,1]$ and on each interval we define homeomorphism which serves as a iterative root of order 2 of F.

To start with, let $x_0 = \frac{1}{8}$ and $x_1 = \frac{1}{4}$. Define

$$x_{2k} := F(x_{2k-2}),\ x_{2k+1} := F(x_{2k-1}) \text{ for all } k \in \mathbb{N}$$

and

$$x_{-(2k+1)} := F^{-1}(x_{-(2k-1)}),\ x_{-2k} := F^{-1}(x_{-(2k-2)}) \text{ for all } k \in \mathbb{N} \cup \{0\}.$$

Note that $x_2 = F(x_0) = \frac{1}{2}$; $x_3 = F(x_1) = \frac{2}{3}$; $x_4 = F(x_2) = \frac{1}{2}(\frac{4}{9}) + \frac{5}{9}$; $x_5 = F(x_3) = \frac{2}{3}(\frac{4}{9}) + \frac{5}{9}$. In general,

$$x_{2k} = \frac{1}{2}\left(\frac{4}{9}\right)^{k-1} + \frac{5}{9}\sum_{i=0}^{k-2}\left(\frac{4}{9}\right)^{i},\quad x_{2k+1} = \frac{2}{3}\left(\frac{4}{9}\right)^{k-1} + \frac{5}{9}\sum_{i=0}^{k-2}\left(\frac{4}{9}\right)^{i}\ \ \forall\ k \in \mathbb{N}.$$

Also, $x_{-1} = F^{-1}(x_1) = \frac{1}{4}(\frac{1}{4})$; $x_{-2} = F^{-1}(x_0) = \frac{1}{8}(\frac{1}{4})$; $x_{-3} = F^{-1}(x_{-1}) = \frac{1}{4}(\frac{1}{4})^2$; $x_{-4} = F^{-1}(x_{-2}) = \frac{1}{8}(\frac{1}{4})^2$. In general,

$$x_{-(2k+1)} = \frac{1}{4}\left(\frac{1}{4}\right)^{k+1},\quad x_{-2k} = \frac{1}{8}\left(\frac{1}{4}\right)^{k}\ \ \forall\ k \in \mathbb{N} \cup \{0\}.$$

Define $I_k = [x_k, x_{k+1}]$ for $k \in \mathbb{Z}$. Since $x_{2k} \to 1,\ x_{2k+1} \to 1,\ x_{-2k} \to 0,\ x_{-(2k+1)} \to 0$ as $k \to \infty$ we have $\cup_{k\in\mathbb{Z}} I_k = [0,1]$. Let $\phi_0 : I_0 \to I_1$ be the homeomorphism *defined by* $\phi_0(x) = 2x$ *for all* $x \in I_0$. *Now, define* $\phi_k : I_k \to I_{k+1}$ *by*

$$\phi_k(x) = F \circ \phi_{k-1}^{-1}(x) \text{ for all } x \in I_k \text{ and } k \in \mathbb{N}.$$

Also define $\phi_{-k} : I_{-k} \to I_{-(k-1)}$ *by*

$$\phi_{-k}(x) = \phi_{-(k-1)}^{-1} \circ F(x) \text{ for all } x \in I_k \text{ and } k \in \mathbb{N}.$$

Consider the homeomorphism $f : [0,1] \to [0,1]$ *defined by* $f(x) = \phi_k(x)$ *if* $x \in I_k$ *for all* $k \in \mathbb{Z}$. *By calculation we can show that*

$$f(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{4}) \\ \frac{2}{3}x + \frac{1}{3}, & \text{if } x \in [\frac{1}{4}, 1] \end{cases}$$

and $f^2(x) = F(x)$ *for all* $x \in [0,1]$.

Now we construct another order preserving homeomorphism g *which does not sub-commute and not comparable with* f *but* $g^2 = F$.

For this, let $\psi_0 : I_0 \to I_1$ *be the homeomorphism defined by*

$$\psi_0(x) = \begin{cases} x + \frac{1}{8}, & \text{if } x \in [\frac{1}{8}, \frac{3}{16}) \\ 3x - \frac{1}{4}, & \text{if } x \in [\frac{3}{16}, \frac{1}{4}]. \end{cases}$$

Now, define $\psi_k : I_k \to I_{k+1}$ *by*

$$\psi_k(x) = F \circ \psi_{k-1}^{-1}(x) \text{ for all } x \in I_k \text{ and } k \in \mathbb{N}.$$

Also, define $\psi_{-k} : I_{-k} \to I_{-(k-1)}$ *by*

$$\psi_{-k}(x) = \psi_{-(k-1)}^{-1} \circ F(x) \text{ for all } x \in I_k \text{ and } k \in \mathbb{N}.$$

Then the homeomorphism $g : [0,1] \to [0,1]$ *defined by*

$$g(x) = \psi_k(x) \text{ if } x \in I_k, \text{ for all } k \in \mathbb{Z}$$

satisfies $g^2(x) = F(x)$ *for all* $x \in [0,1]$. *Since,*

$$\psi_1(x) = F \circ \psi_0^{-1}(x) = \begin{cases} \frac{4}{3}x + \frac{1}{6}, & \text{if } x \in [\frac{1}{4}, \frac{5}{16}) \\ \frac{4}{9}x + \frac{4}{9}, & \text{if } x \in [\frac{5}{16}, \frac{1}{2}], \end{cases}$$

and

$$\psi_2(x) = F \circ \psi_1^{-1}(x) = \begin{cases} \frac{1}{3}x + \frac{1}{2}, & \textit{if} \quad x \in [\frac{1}{2}, \frac{7}{12}) \\ x + \frac{1}{9}, & \textit{if} \quad x \in [\frac{7}{12}, \frac{2}{3}], \end{cases}$$

we observe that

$$f\left(g\left(\frac{3}{16}\right)\right) = f\left(\psi_0\left(\frac{3}{16}\right)\right) = f\left(\frac{5}{16}\right) = \frac{13}{24} < g\left(f\left(\frac{3}{16}\right)\right) = \psi_1\left(\frac{3}{8}\right) = \frac{11}{18},$$

and

$$g\left(f\left(\frac{13}{32}\right)\right) = \psi_2\left(\frac{29}{48}\right) = \frac{103}{144} < f\left(g\left(\frac{13}{32}\right)\right) = f\left(\psi_1\left(\frac{13}{32}\right)\right) = f\left(\frac{45}{72}\right) = \frac{27}{36}.$$

Moreover, $g(\frac{3}{16}) = \frac{5}{16} < f(\frac{3}{16}) = \frac{3}{8}$ *and* $f(\frac{5}{16}) = \frac{13}{24} < g(\frac{5}{16}) = \frac{7}{12}$. *Thus we have two order preserving homeomorphisms* f *and* g *such that they are neither comparable nor subcommuting but* $f^2 = g^2 = F$ *and* $f \neq g$.

Summary of the chapter

In this chapter we discussed the conditions for which the iterative roots of an order preserving roots are equal using the set of points of coincidence. We proved the following results:

- Suppose that the n^{th} iterate of two commuting order preserving homeomorphisms are equal in a subinterval. Then the functions are equal on the whole interval.
- The set of points of coincidence of two commuting order preserving homeomorphisms are preserved under iteration.

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