HIGHER ORDER ASYMPTOTICS AND VISCOSITY METHOD TO BURGERS SOLUTIONS

Thesis

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To my family and my teachers

DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the thesis entitled **"HIGHER ORDER ASYMPTOTICS AND VISCOSITY METHOD TO BURGERS SOLUTIONS"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to **certify** that the thesis entitled **"HIGHER ORDER ASYMPTOTICS AND VISCOSITY METHOD TO BURGERS SOLUTIONS"** submitted by **MANASA M**, (Reg. No. 155015 MA15P01) as the record of the research work carried out by her, is *accepted as the thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. Satyanarayana Engu.
Guide

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ABSTRACT

The viscous Burgers equation $u_t + uu_x = \nu u_{xx}$ is a nonlinear partial differential equation, named after the great physicist *Johannes Martinus Burgers* (1895-1981). We focused on the study of the large time asymptotic for solutions to the viscous Burgers equation and also to the adhesion model via heat equation. Using generalization of the truncated moment problem to a complex measure space, we construct asymptotic N-wave approximate solution to the heat equation subject to the initial data whose moments exist up to the order $2n+m$ and i-th order moment vanishes, for $i = 0, 1, 2 \ldots m-1$. We provide a different proof for a theorem given by Duoandikoetxea and Zuazua (1992), which plays a crucial role in error estimations. In addition to this we describe a simple way to construct an initial data in Schwartz class whose m moments are equal to the m moments of given initial data.

Secondly, we focus on the Riemann problem for de-coupled system and obtain the weak solutions explicitly. It is to be noted here that real valued solution for the system exists in the case of rarefaction wave and the weak solution consist of δ- measures in the case of raising the speed of characteristics. Eventually, we consider inviscid Burgers equation with a forcing term, this is in fact the first equation in the de-coupled system, but with a general initial function $u_0(x) = o(|x|), \quad \text{as} \quad |x| \to \infty$. We then pick up an explicit solution from Satyanarayana et al. (2017) for the parabolic approximation of the hyperbolic partial differential equation using vanishing viscosity method, we construct weak solutions for the considered hyperbolic partial differential equation.

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Chapter 1

General Introduction

Differential equations usually describe the change in the behavior of every material object in the nature with respect to time and space variables. It can be change in single variable for which one can use the concept of Ordinary Differential Equation(ODE). Otherwise one can describe the change in the behavior of the object for several variables through Partial Differential Equation(PDE).

A PDE is an equation involving two or more independent variables, an unknown function and its partial derivatives with respect to the independent variables up to certain order.

1.1 Preliminaries

We use the following notations for spaces in the thesis.

1. $C(\mathbb{R})$ denotes the space of all continuous functions $f : \mathbb{R} \to \mathbb{R}$.

2. $C^\infty(\mathbb{R})$ denotes the space of infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$.

3. $L^p(\mathbb{R})$ for $1 \leqslant p < \infty$, denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$\|f\|_{L^p(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

4. $L^\infty(\mathbb{R})$ denotes the space of all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy
$$\|f\|_{L^\infty(\mathbb{R})} := \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| < \infty.$$

Definition 1.1.1. *(Evans (1998)).* ***Big-oh notation:*** *We write* $f = O(g)$ *as* $x \to x_0$*, provided that there exists a constant* C *such that*

$$|f(x)| \leqslant C|g(x)|,$$

for all x *sufficiently close to* x_0*.*

Definition 1.1.2. *(Evans (1998)).* ***Little-oh notation:*** *We write* $f = o(g)$ *as* $x \to x_0$*, provided*

$$\lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = 0.$$

Definition 1.1.3. *(Kesavan (1989)).* ***Support of a function:*** *Let* ϕ *be a real (or complex) valued continuous function defined on an open set in* $\mathbb{R}^n$*. The support of* ϕ*, written as supp*(ϕ)*, is defined as the closure of the set on which* ϕ *is non-zero.*

Definition 1.1.4. *(Kesavan (1989)).* ***Test functions:*** *The set of all infinitely differentiable functions defined on* $\mathbb{R}^n$ *with compact support is called test functions.*

Definition 1.1.5. ***Weak Solution:*** *(Stavroulakis and Tersian (2004)). Assume that* $u_0(x) \in L^1_{loc}(\mathbb{R})$*. A function* $u(x,t) \in L^2_{loc}(\mathbb{R} \times [0,\infty))$ *is a weak solution of*

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, t > 0, \tag{1.1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R} \tag{1.1.2}$$

if and only if

$$\int_0^\infty \int_{-\infty}^\infty \left(u\rho_t + \frac{u^2}{2}\rho_x \right) dxdt + \int_{-\infty}^\infty u_0(x)\rho(x,0)dx = 0, \tag{1.1.3}$$

for every test function $\rho \in C_c^\infty(\mathbb{R} \times [0,\infty))$*.*

Definition 1.1.6. *(Evans (1998)).* ***Minkowski's Inequality:***
Assume $1 \leq p \leq \infty$ *and* $u, v \in L^p(U)$*,* U *is open in* $\mathbb{R}^n$*. Then*

$$\| u + v \|_{L^p(U)} \leq \| u \|_{L^p(U)} + \| v \|_{L^p(U)} .$$

Definition 1.1.7. *(Kesavan, 1989).* ***The Schwartz Space,*** $\mathbb{S}$***:*** *The Schwartz Space, or the space of rapidly decreasing functions,* $\mathbb{S}$*, is given by*

$$\mathbb{S} = \{ f \in C^\infty(\mathbb{R}^n) / \lim_{|x| \to \infty} |x^\beta D^\alpha f(x)| = 0, \quad \text{for all multi-indicies } \alpha \text{ and } \beta \}.$$

Definition 1.1.8 (Kim (2011)). *Let a doubly indexed complex sequence* $\alpha_{ij} \in \mathbb{C}$ *satisfy* $\alpha_{ij} = \overline{\alpha}_{ji}$. *Then the full* K*-moment problem related to a set* $K \subset \mathbb{C}$ *and the sequence* $\{\alpha_{ij}\}$ *is to find a positive Borel measure* μ *that is supported on* K *and satisfies*

$$\alpha_{ij} = \int \overline{z}^i z^j d\mu, \qquad i, j \geq 0. \tag{1.1.4}$$

Depending on the choices of K, the problem is called with the names Stieltjes ($K = \mathbb{R}^+$), Hamburger ($K = \mathbb{R}$), Hausdorff ($K = [a, b]$), and Toeplitz ($K = \mathbb{T}$)(see Akhiezer (1965), Atzmon (1975)). If $K \subset \mathbb{R}$, then $\alpha_{ij} = \int x^{i+j} d\mu = \overline{\alpha}_{ji}$. Therefore the doubly indexed sequence α_{ij} is actually a singly indexed one with real values and we write

$$\alpha_k = \int x^k d\mu, \qquad k \geq 0.$$

Definition 1.1.9. *(Kim (2011)). The truncated* K*-moment problem related to a set* $K \subset \mathbb{R}$ *and a sequence* $\{\alpha_k\}$ *is to find a positive Borel measure* μ *such that*

$$\alpha_k = \int x^k d\mu, \qquad 0 \leq k < n, \tag{1.1.5}$$

and

$$supp(\mu) \subseteq K.$$

Usually, a measure in an atomic representation $d\mu = \sum_{i=1}^{n} \rho_i \delta(x - c_i) dx$ is considered, where $\delta_{c_i}(x) = \delta(x - c_i)$ is the Dirac measure centered at c_i. Therefore, the truncated moment problem in (1.1.5) is to find $2n$ unknowns $\rho_i \geq 0$ and $c_i \in \mathbb{R}$ that satisfy

$$\alpha_k = \sum_{i=1}^{n} \rho_i c_i^k, \quad 0 \leq k < 2n, \tag{1.1.6}$$

provided $supp(\mu) \subset \mathbb{R}$.

We start our discussion with the heat equation.

Heat Equation

As we know, one of the application of the heat equation is to study the heat conduction. In this case, the heat equation provides an information about temperature at a given location in a metal bar as time changes. To determine the

temperature in the bar at any given time, we need to solve the heat equation subject to the initial and boundary conditions.
The one dimensional heat equation is given by

$$u_t = k u_{xx}, \tag{1.1.7}$$

where k is the thermal conductivity. Then the function

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{\frac{-x^2}{4kt}}$$

is called the fundamental solution of the heat equation and is known as heat kernel.

We consider the initial value problem for heat equation on the whole real line as following:

$$\begin{cases} u_t = k u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = \phi(x), & x \in \mathbb{R}. \end{cases} \tag{1.1.8}$$

Then for $\phi(x) \in L^1(\mathbb{R})$ or $\phi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$,

$$u(x,t) = \int_{\mathbb{R}} \frac{\phi(y)}{\sqrt{4\pi kt}} e^{\frac{-(x-y)^2}{4kt}} dy \tag{1.1.9}$$

is the solution for (1.1.8) such that

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = \phi(x_0).$$

One can observe that, the heat equation is a second order linear parabolic partial differential equation. The solution (1.1.9) is the convolution of initial data with the heat kernel.

Our main work in the thesis is on the study of Burgers solutions via heat equation. Hereby we begin discussing about the Burgers equation.

1.1.1 Burgers Equation

Burgers equation is a second order non-linear parabolic partial differential equation, which is of the form

$$u_t + u u_x = \epsilon u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1.10}$$

where ϵ is a coefficient of viscosity and $u = u(x,t)$ is the velocity of fluid. This equation consists of both non-linearity convection and diffusion terms. It is one of the PDE occurring in various field of applied mathematics such as fluid mechanics, traffic flow etc.

This equation was first discussed by Bateman (1915), due to extensive work of Burgers (1948), it is known as Burgers equation. Later, this equation has got much attention and studied by Hopf (1950), Cole (1951) and many others beginning from 1948.
If $\epsilon = 0$, then (1.1.10) becomes

$$u_t + uu_x = 0, \;\; x \in \mathbb{R}, \;\; t > 0. \tag{1.1.11}$$

The equation (1.1.11) is called a inviscid Burgers equation and it is a hyperbolic partial differential equation. As an application, the Burgers equation (1.1.10) can be derived from Navier Strokes equations.

Navier Strokes Equations

Navier Strokes equations describes the motion of viscous fluid and the solution to these equations gives the flow velocity.
Consider the Navier Strokes equations for incompressible flow

$$\begin{cases} \bigtriangledown \cdot v = 0, \\ (\rho v)_t + \bigtriangledown \cdot (\rho v v) + \bigtriangledown p - \mu \bigtriangledown^2 v = 0, \end{cases} \tag{1.1.12}$$

where ρ is the density of the fluid, p is the pressure, v is the velocity of the fluid and μ is the viscosity of the fluid.
Simplification of second equation in (1.1.12) with respect to x component, we get

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial x} + \rho v^y \frac{\partial v^x}{\partial y} + \rho v^z \frac{\partial v^x}{\partial z} + \frac{\partial p}{\partial x} - \mu \left(\frac{\partial^2 v^x}{\partial x^2} + \frac{\partial^2 v^x}{\partial y^2} + \frac{\partial^2 v^x}{\partial z^2} \right) = 0.$$

If we take one dimensional problem with zero pressure gradient, then the above equation is reduced to

$$\rho \frac{\partial v^x}{\partial t} + \rho v^x \frac{\partial v^x}{\partial x} - \mu \frac{\partial^2 v^x}{\partial x^2} = 0.$$

Let us take $v^x = u$ and kinematic viscosity $\epsilon = \frac{\mu}{\rho}$. Then we get the viscid Burgers equation

$$u_t + uu_x = \epsilon u_{xx}.$$

1.1.2 Solution to viscous Burgers equation

Consider the initial value problem for viscous Burgers equation.

$$\begin{cases} u_t + uu_x = \epsilon u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.1.13}$$

Hopf (1950) and Cole (1951) considered a method to solve the above initial value problem by introducing a transformation, later known as Cole-Hopf transformation by reducing (1.1.13) to a linear problem. This transformation is given by

$$\phi(x,t) = \exp\left\{ -\frac{1}{2\epsilon} \int_{-\infty}^{x} u(y,t)dy \right\}.$$

Note that, the lower limit for the integral above can be any real number also. In fact, Hopf (1950) studied with lower limit zero.
Then the initial value problem (1.1.13) was reduced to

$$\begin{cases} \phi_t = \epsilon \phi_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ \phi(x,0) =: \phi_0(x) = \exp\{-\frac{1}{2\epsilon}\int_{-\infty}^{x} u_0(y)dy\}, & x \in \mathbb{R}. \end{cases} \tag{1.1.14}$$

The solution to the above initial value problem is given by

$$u(x,t) = \int_{\mathbb{R}} \frac{\phi_0(y)}{\sqrt{4\pi\epsilon t}} e^{\frac{-(x-y)^2}{4\epsilon t}} dy.$$

From this Hopf (1950) derived an explicit solution for the initial value problem (1.1.13), which is given by

$$\begin{aligned} u(x,t) &= -2\epsilon \frac{\phi_x}{\phi} \\ &= -2\epsilon \frac{\int_{-\infty}^{\infty} \frac{x-y}{t} \exp\{-\frac{(x-y)^2}{4\epsilon t}\}\phi_0(y)dy}{\int_{-\infty}^{\infty} \exp\{-\frac{(x-y)^2}{4\epsilon t}\}\phi_0(y)dy}. \end{aligned}$$

Though the explicit solution exists, it is very difficult to evaluate both the integrals in the numerator and denominator for many of the initial data. This is the main motivation to study the behavior of Burgers solution via asymptotic analysis. This difficulty may be resolved by finding a simple approximate solution or one has to try evaluating the integrals using numerical methods.

Initially, Hopf (1950) studied the behavior of the solution of Burgers equation to obtain the higher order asymptotic in the following cases.

- The behavior of the solution as $t \to \infty$ while keeping viscosity constant.
- The behavior of the solution as $\epsilon \to 0$ while x and t are fixed.

The following consists of brief review of work done for the higher order asymptotic for Burgers solutions.

Duoandikoetxea and Zuazua (1992) has taken

$$\psi_{2n}(x,t) = \sum_{i=0}^{2n-1} \frac{(-1)^i \gamma_i}{i!\sqrt{4\pi t}} \partial_x^i (e^{\frac{-x^2}{4t}})$$

as an approximate solution to the heat equation and they obtained ψ_{2n} approaches to the solution u of heat equation with a convergence order $O\left(t^{(\frac{1}{2p}-\frac{2n+1}{2})}\right)$ as $t \to \infty$. Yanagisawa (2007) constructed an approximate solution to Viscous Burgers equation, which is of the form

$$\chi^k(x,t) = -2\frac{\phi^k(x,t)}{1+\int_{-\infty}^{x} \phi^k(y,t)dy}$$

where

$$\phi^k(x,t) = \sum_{j=1}^{k-1} (-1)^j \frac{M_j(H[u_0]^{'})}{j!} \Big(\frac{\partial}{\partial x}\Big)^j G_t(x) + (-1)^k \frac{M_k(H[u_0]^{'})}{k!} \Big(\frac{\partial}{\partial x}\Big)^k G_{(t+(t_k)_+)}(x-\gamma_k)$$

is the solution for heat equation, $M_j(H[u_0]^{'})$ is the j^{th} moment of $H[u_0]^{'}$, $j \geq 0$, $(t_k)_+$ is the time shift, γ_k is the space shift and $G_t(x)$ is the one dimensional heat kernel. He estimated that the approximation differs from the true solution by an error whose L^p- norm is of order $O(t^{-(\frac{4+k}{2})+\frac{1}{2p}})$ as $t \to \infty$, by assuming that the initial data satisfies $(1+|x|^{k+3+\epsilon}) \in L^1(\mathbb{R})$, where $1 \leq p \leq \infty$.

Kim and Ni (2009) considered an initial value problem for heat equation

$$\begin{cases} u_t - u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}. \end{cases}$$

They studied the large time behavior of the solution of the heat equation using the moments of the solution by imposing conditions on initial data. They assumed that

$$x^{2n}u_0 \in L^1(\mathbb{R}) \text{ and } u_0 \text{ is bounded,}$$

which shows that the moments of the initial data exist and is given by

$$\gamma_k = \int_{\mathbb{R}} x^k u_0(x)dx \text{ for } k = 0, 1, ..., 2n-1.$$

Kim and Ni (2009) has taken a linear combination of n heat kernels

$$\phi_n(x,t) = \sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi t}} e^{-(x-c_i)^2/4t}.$$

as an approximate solution for heat equation, where these ρ_i's and c_i's are chosen in such a way that, the moments

$$\gamma_k = \lim_{t\to 0} \int_{\mathbb{R}} x^k \phi_n(x,t)dx,$$

agree with those of true solution of heat equation. The above relation turns out to a truncated moment problem and the problem has a unique solution if $u_0(x)$ is a non-negative function (Curto and Fialkow, 1991). Hence, there exists unique ρ_i's and c_i's which gives the positive approximate solution. Thus, the constructed solution converges to the exact solution in the order $t^{-(\frac{2n+1}{2})+\frac{1}{2p}}$ as $t \to \infty$.

Further, Kim (2011) removed the condition of non-negative initial data and generalized the truncated moment problem in Kim and Ni (2009) to a complex measure space. He considered a complex sequence which is of the form,

$$m_k := \alpha_k + i\beta_k, \quad 0 \le k \le 2n-1, \quad \text{where } \beta_k = \int x^k q_0(x)dx.$$

Now the truncated moment problem related to this complex sequence is to find a complex measure μ such that

$$m_k := \int z^k d\mu, \quad supp(\mu) \subseteq \mathbb{C}, \quad 0 \le k \le 2n-1,$$

i.e, it is reduced to find complex solutions ρ_i's and c_i's that satisfy,

$$\sum_{i=1}^{n} \rho_i c_i^k = m_k \tag{1.1.15}$$

and he proved that there exists $\rho_i, c_i \in \mathbb{C}$ such that (1.1.15) is satisfied. Thus, for any sequence $\alpha_k \in \mathbb{R}$, there exist ρ_i's and c_i's such that

$$\alpha_k = Re\left(\sum_{i=1}^{n} \rho_i c_i^k\right), \quad \rho_i, c_i \in \mathbb{C}, \quad 0 \le k < 2n.$$

Thus the approximate solution is

$$\phi_n(x,t) = Re\left(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi t}} e^{-(x-c_i)^2/4t}\right),$$

where $\rho_i, c_i \in \mathbb{C}$.

Now we introduce briefly the works related to N-wave initial data, a special case of zero mass initial data.

It is seen that (Sachdev, 1987), the N-waves move out from a planar source in the form

$$u(x,t_i) = \begin{cases} x, & |x| < 1, \\ 0, & otherwise. \end{cases} \tag{1.1.16}$$

To find the evolution of (1.1.16) under the planar Burgers equation, we consider the following solution of heat equation:

$$\phi(x,t) = 1 + \sqrt{\left(\frac{t_0}{t}\right)} \exp\left\{-\frac{x^2}{2\delta t}\right\}, \tag{1.1.17}$$

where t_0 is a constant. Using Cole-Hopf transformation, $u = -\delta \frac{\phi_x}{\phi}$, we obtain

$$u(x,t) = \frac{x}{t[1 + \sqrt{\frac{t_0}{t}}\ e^{-\frac{x^2}{2\delta t}}]}. \tag{1.1.18}$$

Now define the Reynolds number as follows:

$$\begin{aligned} R &= \frac{1}{\delta}\int_0^\infty u(x,t)dx \\ &= \log[\phi(0,t)] \\ &= \log\left[1 + \sqrt{\frac{t_0}{t}}\right]. \end{aligned} \tag{1.1.19}$$

Hence, (1.1.18) can be expressed as

$$u(x,t) = \frac{x}{t[1 + \frac{1}{e^R - 1} e^{-\frac{x^2}{2\delta t}}]}. \tag{1.1.20}$$

It is to be noted that the above N -wave solution is not self-similar, unlike single hump solutions for viscous Burgers equation.

Sachdev and Joseph (1994) showed that the true solution of

$$u_t + uu_x = \frac{\delta}{2}u_{xx} \tag{1.1.21}$$

with

$$u(x,0) = \begin{cases} x, & |x| < l_0, \\ 0, & otherwise, \end{cases} \tag{1.1.22}$$

can be expressed as

$$u(x,t) = \frac{x}{t[1 + \frac{\sqrt{t}}{C_0}e^{\frac{x^2}{2\delta t}}]} + O\left(\frac{1}{t}\right), \tag{1.1.23}$$

where

$$C_0 = \frac{2}{\pi}e^{\frac{l_0^2}{2\delta}}\int_0^{\frac{l_0^2}{\sqrt{2\delta}}} e^{-\xi^2}d\xi - \sqrt{\frac{2}{\pi\delta}}l_0.$$

It is well known that most of the generalizations of Burgers equation can't be linearized by Hopf like transformation.

Let us now consider the following generalized Burgers equations, namely, non planar Burgers equation

$$u_t + uu_x + \frac{ju}{2t} = \frac{\delta}{2}u_{xx}, \quad \text{for } \ j \geq 0. \tag{1.1.24}$$

While studying the propagation of weakly non-linear longitudinal waves in liquids, the nonplanar Burgers equation (1.1.24) was obtained by Lighthill (1956) and Leibovich and Seebass (1974). Here $j = 1$ corresponds to the case when the source is cylindrical symmetric and $j = 2$ for the case when the source is spherical symmetric.

Sachdev et al. (1999) considered (1.1.24) subject to

$$u(x,t_i) = \begin{cases} (\frac{1-j}{2})\frac{x}{t_i} & \text{if } \ x < d_0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.25}$$

Here d_0 is the length of one lobe of the initial profile (1.1.25). They introduced the similarity variables

$$t, \quad \eta = xt^a, \quad u = t^c v(\eta, t)$$

and reduced (1.1.24) to

$$\left(\frac{c+j}{2}\right) v + a\eta v_\eta + t v_t = \frac{\delta}{2} t^{(2a+1)} v_{\eta\eta} - t^{a+c+1} v v_\eta. \tag{1.1.26}$$

The above equations play an important role in analyzing the solutions in the following three regions.

- Convection dominated region.
- Diffusion dominated region.
- Convection-diffusion balanced region

Sachdev et al. (1999) first considered the convection dominated region [that is $a = -1, \quad c = 0$] and obtained the following old-age solution:

$$u(x,t) = \left(\frac{1-j}{2}\right) \frac{x}{t} + O(1), \qquad j \neq 2 \ \text{ as } \ t \to \infty.$$

In fact this behavior of the solution agrees with that of the work done by Crighton and Scott (1979). They then considered the diffusion dominated region, i.e. the case when

$$a = -\frac{1}{2} \ \text{ and } \ c < -\frac{1}{2}.$$

Motivated by the old-age solution

$$u(x,t) = c_1 \frac{x}{t^{(3+j)/2}} e^{-\frac{x^2}{2\delta t}}$$

of (1.1.24), they took

$$c = -\frac{1+j}{2}.$$

This, in turns, changes (1.1.26) to

$$-v - \frac{\eta}{2} v_\eta + t v_t = \frac{\delta}{2} v_{\eta\eta} - t^{-(j+1)/2} v v_\eta. \tag{1.1.27}$$

Assuming

$$v(\eta, t) = v_0(\eta) + o(1) \quad \text{as} \quad t \to \infty,$$

they obtained the old-age behavior as

$$v_0(\eta) = A\eta e^{-\eta^2/2\delta}. \tag{1.1.28}$$

It is to be noted that they happen to depend on numerical scheme to find the old-age constant A in (1.1.28).

To incorporate the behavior of solution at far in back time from old-age behavior, they assumed the following form for v:

$$v = v_0(\eta) + \epsilon(\eta, t) \quad \text{as} \quad t \to \infty. \tag{1.1.29}$$

Substituting (1.1.29) into (1.1.27) leads to a PDE. To reduce the resulting PDE to ODE, they took

$$\epsilon(\eta, t) = t^{-(j+1)/2} f(\eta) \quad \text{as} \quad t \to \infty. \tag{1.1.30}$$

With the help of confluent hyper-geometric equation, the variation of parameters and the antisymmetricity of the solution u, they arrived at

$$u(x,t) = t^{-(1+j/2)} \left[\frac{Ax \exp\left(-\dfrac{x^2}{2\delta t} \right)}{t^{1/2} + t^{-(j+1)/2} f(\frac{x}{t^{1/2}}) + O(t^{-(j+1)})} \right]. \tag{1.1.31}$$

It was then showed that the constructed solution agrees with the N-wave solution of viscous Burgers equation up to the error of order $O(t^{-5/2})$ when $j = 0$. They then obtained an asymptotic expression for Reynolds number . Finally they analyzed the N-wave solution for generalized Burgers equation (1.1.24) under the assumption that

$$j = \frac{m}{n} \quad , \quad 0 < j < 2$$

with m and n are positive integers with no common factors.

Sachdev and Srinivasa Rao (2000) constructed N-wave solutions for the following generalized Burgers equation

$$u_t + u^n u_x = \frac{\delta}{2} u_{xx} \tag{1.1.32}$$

with

$$u(x,0) = \begin{cases} -x, & |x| \le 1 \\ 0, & |x| > 1. \end{cases} \tag{1.1.33}$$

It is known that

$$u(x,t) = c\frac{x}{t^{3/2}}e^{-x^2/2\delta t} \tag{1.1.34}$$

is the antisymmetric profile satisfying the equation (1.1.32) in the diffusion dominated region. However, numerical study of (1.1.32)-(1.1.33) suggests that the node will be shifting from y-axis when (1.1.33) evolves under generalized Burgers equation (1.1.32). It suggested to assume the old-age solution for (1.1.32)-(1.1.33) as follows:

$$u(x,t) = c\frac{(x-x_0)}{t^{3/2}}e^{-\frac{(x-x_0)^2}{2\delta t}}.$$

They introduced the similarity variables

$$\xi = \frac{(x-x_0)}{\sqrt{2\delta t}}, \qquad \tau = t^{1/2}, \qquad u = \frac{(2\delta)^{1/2}\xi}{v^{1/n}}. \tag{1.1.35}$$

Using these variables, (1.1.32) leads to

$$2nvv_\xi + \xi[nvv_\xi - 2n^2v^2 - 2n\tau vv_\tau] + 2n\xi^2vv_\xi + 4n^2(2\delta)^{(n-1)/2}\tau\xi^n v \\ -4n(2\delta)^{(n-1)/2}\tau\xi^{n+1}v_\xi - (n+1)\xi v_\xi^2 = 0. \tag{1.1.36}$$

Inspired by the exact N-wave solution for viscous Burgers equation, they sought the solution in the following form:

$$u = \frac{\sqrt{2\delta}\xi}{v^{1/n}}$$

with

$$v = \sum_{i=0}^{\infty} f_i(\tau)\frac{\xi^i}{i!}.$$

Substituting the above expression of v into (1.1.36), they found that

$$\begin{aligned} f_1(\tau) &= 0, \\ f_3(\tau) &= -4\sqrt{2\delta}\tau, \quad n=2, \\ &= 0, \quad n>2. \end{aligned}$$

They then sought the expression f_0,f_2 and f_4 in the following form:

$$f_0(\tau) = \tau^{2n}\sum_{k=0}^{p} a_k\tau^{-k}, \quad f_2(\tau) = \tau^{2n}\sum_{k=0}^{p} b_k\tau^{-k}, \quad f_4(\tau) = \tau^{2n}\sum_{k=0}^{p} c_k\tau^{-k}. \tag{1.1.37}$$

Though passing the limit $p \to \infty$ in (1.1.37) would give better approximation, it was only possible explicitly to evaluate f_0,f_2 and f_4 for the cases $p = 2, 3, 4$.

They defined the Reynolds number for the N-wave solution of (1.1.32) to be

$$R(t) = \frac{1}{\delta} \int_{-\infty}^{x_0} u(x,t)dx,$$

then

$$R(t) = \begin{cases} R(t_0) + \frac{1}{c_2} log\left[\frac{c_1 + c_2/t^{1/2}}{c_1 + c_2/t_0^{1/2}}\right], & n = p, \\ R(t_0) + \frac{1}{c_2}\frac{n}{n-p}(h(t) - h(t_0)), & n \neq p, \end{cases} \tag{1.1.38}$$

where

$$h(s) = \left(c_1 + \frac{c_2}{s^{1/2}}\right)^{(n-p)/n}. \tag{1.1.39}$$

They finally compared Reynolds number (1.1.38)-(1.1.39) obtained by improving the old-age solution with the Reynolds numbers obtained via numerical scheme for some specific cases and found very good agreement.

Enflo and Rudenko (1994) studied N-wave solutions for a Generalized Burgers equation

$$u_t + uu_x = \frac{\epsilon}{\sqrt{2t}} u_{xx} \tag{1.1.40}$$

subject to the N-wave initial condition

$$u(x,t) = \begin{cases} x, & \text{if } |x| < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1.41}$$

In the case as $\epsilon \to 0$, they obtained the behavior of the outer solution of (1.1.40)-(1.1.41) as follows:

$$u(x,t) = \begin{cases} \dfrac{x}{t} + O(\epsilon^n), & \text{if } |x| < \sqrt{t}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1.42}$$

Srinivasa Rao and Satyanarayana (2008), first, constructed large time N-wave solution, namely,

$$\begin{aligned} u(x,t) &= t^{-(1+\frac{j}{2})}\Bigg[A\xi e^{-\frac{\xi^2}{2\delta}} + t^{-(\frac{j+1}{2})} e^{-\xi^2/\delta} \\ &\times\left(- A^2\xi + \frac{A^2 j}{6\delta}\xi^3 + A^2\frac{(8j - j^2)}{120\delta^2}\xi^5 + \ldots\right) + \ldots\Bigg] \end{aligned} \tag{1.1.43}$$

for the nonplanar Burgers equation

$$u_t + uu_x + \frac{ju}{2t} = \frac{\delta}{2} u_{xx}, \quad j \geq 0, \quad \delta \geq 0. \tag{1.1.44}$$

For which, they assumed that the leading order behavior of solution is

$$u(x,t) = c_1 t^{-(3+j)/2} e^{-\frac{x^2}{2\delta t}}. \tag{1.1.45}$$

Then they incorporated the effect of non-linear term by adding a correction term $\epsilon(\xi, t)$ to old-age solution.

Inspired by the form of (1.1.43), they arrived at

$$u(x,t) = t^{-(1+\frac{j}{2})} \Bigg[A\xi e^{-\frac{\xi^2}{2\delta}} + t^{-k} e^{-\frac{\xi^2}{\delta}} \Big(- A^2\xi + \frac{A^2 j}{6\delta}\xi^3 + A^2 \frac{(8j - j^2)}{120\delta^2}\xi^5 + \ldots + A_{2n+3}\xi^{2n+3} + \ldots \Big)$$
$$+ t^{-2k} e^{-\frac{3\xi^2}{2\delta}} \Big(A^2\xi - \frac{A^3 j}{3\delta}\xi^3 + \frac{A^3 j(j-5)}{30\delta^2}\xi^5 + \ldots + C_{2n+1}\xi^{2n+1} + ... \Big) + \ldots \Bigg] \tag{1.1.46}$$

with the help of N-wave solution of viscous Burgers equation. It is to be noticed that the old-age constant A in (1.1.46) was found from numerical solution. Then they picked up approximate N-wave solutions from Parker (1981) for (1.1.44) subject to two different zero mass initial datas:

1. $u(x, t_0) = xe^{-x^2}$

2. $u(x, t_0) = x^3 e^{-x^2}$

and found the agreement of these approximate solutions with that of (1.1.46) for large t. They noticed that Sachdev et al. (1999)'s N wave solution depends on only one unknown function f_0 in the following form :

$$\begin{aligned} u(x,t) &= \frac{\sqrt{2\delta\zeta}}{V(\eta, T)}, \\ V(\eta, T) &= \sum_{i=0}^{\infty} f_i(T) \frac{\eta^i}{i!}. \end{aligned}$$

That is once f_0 is found, remaining all f_i, $i \geq 1$ are found. This f_0 was found by Srinivasa Rao and Satyanarayana (2008) from (1.1.46) via the relation

$$u_x(0,t) = \frac{1}{\sqrt{t} f_0(\sqrt{t})}.$$

They found a drawback of N-wave solution in Sachdev et al. (1999)'s work that it is not possible to truncate the series in the denominator as every term is of the order $O(t^{\frac{2+j}{2}})$. To rectify this , they proposed N-wave solution in the following form

$$\begin{aligned} u(x,t) &= \frac{\sqrt{2\delta\zeta}}{V(\eta,T)}, \\ V(\eta,T) &= T^{2+j}U(\eta,T), \\ U(\eta,T) &= g_0(\eta) + T^{-k_1}g_1(\eta) + T^{-2k_1}g_2(\eta) + \ldots, \end{aligned}$$

where $k_1 = j + 1$. This, eventually, led to closed form like solution which is convenient for computational techniques.

1.2 Organization of the thesis

This thesis is organized as follows:

Chapter 2 deals with the study of large time asymptotic of solutions to the viscous Burgers equation and adhesion model subject to a class of zero mass initial data. This is done by transforming the Burgers equation to heat equation using Cole-Hopf transformation. Initially, we transform Burgers equation to heat equation via Cole-Hopf transformation and construct an N-wave approximation using truncated moment problem. Then we obtain the error estimates between the exact and approximate solutions of heat equation as well as Burgers equation. This chapter ends with a proposition which helps us to construct an initial data in Schwartz class whose m moments matches with the moments of given initial data.

In Chapter 3, we consider a Riemann problem for a de-coupled system and obtain an explicit solutions. Later, we consider a forced Burgers equation with the forcing term $\frac{k}{(2\beta t+1)^{3/2}}$, $\beta > 0$ and k is a non-zero constant, subject to the initial condition $u_0(x) = o(|x|)$ as $|x| \to \infty$. From Satyanarayana et al. (2017), we find the exact solution to the forced Burgers equation. Further we will prove that the obtained solution of forced Burgers equation will approach to the generalized solution of $u_t + uu_x = \frac{k}{(2\beta t+1)^{3/2}}$ as viscosity tends to zero. Finally, Chapter 4 sets forth the conclusions of the thesis and future work.

Chapter 2

Higher order asymptotic for Burgers equation and Adhesion model

2.1 Introduction

The aim of this chapter is to study higher order asymptotic of Burgers equation,

$$u_t + uu_x = \mu u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \tag{2.1.1}$$

with initial datum

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R} \tag{2.1.2}$$

and de-coupled system

$$\begin{aligned} u_t + uu_x &= \mu u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \\ \rho_t + (u\rho)_x &= \mu \rho_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \end{aligned} \tag{2.1.3}$$

with initial data

$$(u(x,0), \rho(x,0)) = (u_0(x), \rho_0(x)), \qquad x \in \mathbb{R},$$

where μ is the viscosity coefficient. Burgers equation (2.1.1) was introduced as the simplest model for the differential equations of fluid flow (Hopf, 1950). The de-coupled system (2.1.3) is the one dimensional adhesion model for large scale structure formation of universe, see, Gurbatov and Saichev (1993). For more literature on this decoupled equation we cite Joseph (2010); Sahoo (2015); Oberguggenberger (1992) and references therein.

Cole-Hopf transformation gives the explicit solution of (2.1.1)- (2.1.2) as

$$u(x,t) = -2\mu \frac{v(x,t)}{1 + \int_{-\infty}^{x} v(y,t)dy}. \tag{2.1.4}$$

Here $v(x,t)$ has the integral representation

$$v(x,t) = \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{\infty} v_0(y) e^{\frac{-(x-y)^2}{4\mu t}} dy \tag{2.1.5}$$

satisfying the heat equation

$$v_t = \mu v_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{2.1.6}$$

$$v(x,0) = -\frac{1}{2} u_0(x) \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_0(s)ds\right) =: v_0(x), \quad x \in \mathbb{R}. \tag{2.1.7}$$

Though the solution (2.1.5) in the explicit form is available (Hopf, 1950) for (2.1.6)-(2.1.7), it is a tedious work to evaluate the integral exactly in right hand side of (2.1.5) for several initial functions v_0. Then one can imagine the difficulty in evaluating the integral exactly in the denominator of right hand side of (2.1.4). So, one may resort to evaluate the integrals numerically. As an alternative (Chern and Liu, 1987; Jaywan et al., 2010; Miller and Bernoff, 2003; Witelski and Bernoff, 1998; Yanagisawa, 2007), we go for finding an approximate solution in a simpler form which has fine asymptotic order similar to the solution.

By scaling time and space variables, we can reduce (2.1.1) and (2.1.3) to

$$u_t + uu_x = u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0 \tag{2.1.8}$$

and

$$u_t + uu_x = u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{2.1.9}$$

$$\rho_t + (u\rho)_x = \rho_{xx}, \qquad x \in \mathbb{R}, \quad t > 0. \tag{2.1.10}$$

Therefore, without loss of generality, one can consider (2.1.8) with the initial condition

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R} \tag{2.1.11}$$

and the de-coupled system (2.1.9)-(2.1.10) with initial condition

$$(u(x,0),\ \rho(x,0)) = (u_0(x),\ \rho_0(x)), \qquad x \in \mathbb{R}. \tag{2.1.12}$$

We first study the large time behavior of solutions to the Cauchy problem (2.1.8) and (2.1.11) by imposing conditions on the initial function u_0 that $u_0,\ x^{2n+1}u_0 \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} u_0(x)dx = 0$. Set $M := \int_{-\infty}^{\infty} u_0(x)dx$. If M is a non zero real number, the solution of (2.1.8) and (2.1.11) approaches Gaussian form as $t \to \infty$, which is a solution of (2.1.8) with the initial profile $M\delta_0(x)$. Here δ_0 is the Dirac measure giving unit mass to the point 0. In case of zero mass initial profiles i.e., $M = 0$, the solution of (2.1.8) and (2.1.11) approaches to N-wave solutions (see Whitham (1974)) of (2.1.8) for sufficiently large t. For which, we construct asymptotic N-wave approximate solution to the concerned heat equation using generalization of truncated moment problem. This approximation is made as a spatial derivative of linear combination of n heat kernels. We also provide a different proof from the existing one that the constructed solution differs from the true solution by an error of order $O(t^{-(m+1)+\frac{1}{2p}})$ in L^p-norm, where $1 \leq p \leq \infty$, if m moments of initial data vanish and m-th order moment exists. This result is used to get higher order asymptotic for Burgers equation.

We then generalize the above results to heat equation having initial data whose i-th order moment vanishes, for $i = 0, 1, 2 \ldots m-1$. This approximation is made as a m-th order spatial derivative of linear combination of n heat kernels. Further, using a generalized Hopf-Cole transformation, we provide higher order asymptotics for the adhesion model (2.1.9)-(2.1.10) by imposing conditions on the concerned initial data. In addition to this we describe a simple way to construct an initial data in Schwartz class whose m moments are equal to those of given initial data. This in turn gives an error estimate of order $O(t^{-(\frac{1+m}{2})+\frac{1}{2p}})$ in L^p-norm for heat solutions if m-th order moment exists.

We now describe three remarks of our N-wave approximation over the Gaussian approximation [Jaywan et al. (2010); Kim (2011)].

1. The $2n$-th moment of N-wave approximation for the heat solution matches with the $2n$-th moment of true solution (see Remark 2.2.2).

2. The N-wave approximation for the Burgers solution consists of a simpler form without any integrals to be evaluated (see equation (2.1.14) and Remark 2.3.1).

3. The rate of convergence is higher when compared to Jaywan et al. (2010); Kim (2011) (see Remark 2.3.1).

The main results are stated herewith:

Theorem 2.1.1. *Let $u(x,t)$ be a solution to the Burgers equation (2.1.8) subject to (2.1.11) with the zero mass initial data $u_0(x)$ satisfying $u_0,\ x^{2n+1}u_0 \in L^1(\mathbb{R})$. Then, for any $t_0 > 0$, there exist $b_i, c_i \in \mathbb{C}$ and $T > 0$ such that*

$$||u(.,t) - u_n(.,t)||_p = O(t^{-(n+1)+1/2p}), \quad t \to \infty, \tag{2.1.13}$$

where $1 \leq p \leq \infty$ and

$$u_n(x,t) := -2\frac{Re\left(\sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi(t+t_0)}}\partial_x\left(e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right)\right)}{1 + Re\left(\sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi(t+t_0)}}e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right)} \tag{2.1.14}$$

is well defined for $t \geq T$.

Theorem 2.1.2. *Let $u(x,t),\ \rho(x,t)$ be solutions to the de-coupled system (2.1.9)-(2.1.10) with the initial condition (2.1.12) satisfying the following conditions:*

$$V_0,\ x^{2n+m}V_0,\ C_0,\ x^{2n+m}C_0 \in L^1(\mathbb{R}),$$
$$\int_{-\infty}^{\infty} x^k V_0(x)dx = \int_{-\infty}^{\infty} x^k C_0(x)dx = 0,\ 0 \leq k < m$$

with

$$C_0(x) = -\frac{1}{2}\int_{-\infty}^{x} \rho_0(s)ds \quad exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_0(s)ds\right), \tag{2.1.15}$$

$$V_0(x) = \exp\Big(-\frac{1}{2}\int_{-\infty}^{x} u_0(y)dy - 1\Big). \tag{2.1.16}$$

Then, there exist $b_i, c_i, \tilde{b}_i, \tilde{c}_i \in \mathbb{C}$ and $T > 0$ such that

$$\| u(.,t) - u_n(.,t) \|_p = O(t^{-\frac{2n+1+m}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{2.1.17}$$

$$\| \rho(.,t) - \rho_n(.,t) \|_p = O(t^{-\frac{2n+2+m}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{2.1.18}$$

where $u_n(x,t)$ *is given by (2.1.14) and*

$$\rho_n(x,t) := \frac{\left[1+\int_{-\infty}^{x} v_n(y,t)dy\right]\partial_x C_n(x,t) - v_n(x,t)C_n(x,t)}{\left[1+\int_{-\infty}^{x} v_n(y,t)dy\right]^2}$$

is well defined for $t \geq T$ *with*

$$C_n(x,t) = Re\left(\sum_{i=1}^{n} \frac{\tilde{b}_i}{\sqrt{4\pi t}}\partial_x^m\left(e^{-\frac{(x-\tilde{c}_i)^2}{4t}}\right)\right). \tag{2.1.19}$$

Proposition 2.1.3. *If* $n+1$ *moments, namely,*

$$\int_{-\infty}^{\infty} x^j f(x)dx = c_j, \quad j = 0,1,2,\ldots n. \tag{2.1.20}$$

exist for a given function f*, then there exists a Schwartz class function* g *such that the moments of* g *agree with that of* f*. Further if* $x^{n+1}f \in L^1(\mathbb{R})$*, the solution of heat equation with initial profile* $f-g$ *is of order* $O(t^{-\frac{n+2}{2}+\frac{1}{2p}})$ *as* $t \to \infty$ *in* L^p*-norm,* $1 \leq p \leq \infty$*.*

Miller and Bernoff (2003) investigated the asymptotic self-similar behavior of solutions to (2.1.1) with non negative initial data. They estimated that their asymptotic self-similar approximation differs from the true solution by an error whose L^p-norm is of order $O(t^{-2+\frac{1}{2p}})$ as $t \to \infty$, where $1 \leq p \leq \infty$. For which, they incorporated total mass, space shift and time shift into the approximate heat kernel while studying asymptotic behavior of solutions to the concerned heat equation. Their work is an improvement over the work of Chern and Liu (1987) by a factor of $\frac{1}{t}$. Duoandikoetxea and Zuazua (1992) considered the linear combination of derivatives of atmost order $2n-1$ to the heat kernel as approximate solutions for heat equation. They then gave the L^p-norm rates of convergence to the true solutions of heat equation as of order $O(t^{-\frac{2n+1}{2}+\frac{1}{2p}})$ when $t \to \infty$, where $1 \leq p \leq \infty$. Motivated by the works of Duoandikoetxea and Zuazua (1992) and Miller and Bernoff (2003), Yanagisawa (2007) constructed higher order approximate solutions to the viscous Burgers equation (2.1.8). Kim (2011) successfully generalized truncated moment problem to a complex measure space and subsequently dealt with sign changing initial data to the heat equation. Kim and Ni

(2009) introduced linear combination of n heat kernels as an approximation to the solutions of relevant heat equation and showed that the kth order moments of the true and approximate solutions of (2.1.1) were contracting with an error whose L^p-norm is of order $O((\sqrt{t})^{k-2n-1+\frac{1}{p}})$ as $t \to \infty$. For an interesting study of large time asymptotics to the solutions of heat equation and Porous media equation in the context of introducing total mass, space shift and time shift, we refer to the work of Witelski and Bernoff (1998). For the existence and decay rate of solutions to generalizations of viscous Burgers equation, we refer to Lu and Jäger (2001) and the references therein.

The scheme of this chapter is as follows. Section 2.2 deals with the construction of asymptotic N-wave solution, $v_n(x,t)$, to the relevant heat equation and then brings out error estimates to the solutions of heat equation. Section 2.3 gives the N-wave asymptotic approximation, $u_n(x,t)$, to the solution $u(x,t)$ of viscous Burgers equation (2.1.8). The decay order of $u(x,t) - u_n(x,t)$ for sufficiently large t is also derived in the L^p-norm, where $1 \leq p \leq \infty$. In Section 2.4, we also present higher order asymptotics to the solutions of adhesion model. In addition to this we construct a smooth function in Schwartz class whose m moments are equal to the m moments of any given initial data.

2.2 On the moments and asymptotics of heat solutions

In this section, we reduce the initial value problem (2.1.8) and (2.1.11) to an initial value problem for heat equation via Cole-Hopf transformation and then study the asymptotic behavior of solutions to the relevant heat equation by introducing a suitable approximate solution.

Assume that the initial data u_0 is of zero mass, i.e., $\int_{-\infty}^{\infty} u_0(x)dx = 0$ and

$$(1 + x^{2n})u_0 \in L^1(\mathbb{R}). \tag{2.2.21}$$

We employ Cole -Hopf transformation

$$H(u) = \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u(s,t)ds\right) - 1 =: V(x,t)$$

to the initial value problem (2.1.8), (2.1.11) and then obtain

$$V_t = V_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{2.2.22}$$

$$V(x,0) = \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_0(s)ds\right) - 1, \quad x \in \mathbb{R}. \tag{2.2.23}$$

In general, the initial data (2.2.23) is not summable on $\mathbb{R}$ and as we deal with moments of initial data, we consider space derivative of $V(x,t)$;

$$\partial_x V(x,t) = -\frac{1}{2}u(x,t)\ \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u(s,t)ds\right) =: v(x,t).$$

Thus, the Cauchy problem (2.2.22)-(2.2.23) leads to

$$v_t = v_{xx}, \quad x \in \mathbb{R},\ t > 0 \tag{2.2.24}$$

$$v(x,0) = -\frac{1}{2}u_0(x)\ \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_0(s)ds\right) =: v_0(x), \quad x \in \mathbb{R}. \tag{2.2.25}$$

It can be seen that v_0 is of zero mass if and only if u_0 is of zero mass. We define the k-th order moment of the function $v(x,t)$ as

$$\alpha_k(t) := \int_{-\infty}^{\infty} x^k v(x,t)dx, \quad k = 0, 1, 2, \ldots.$$

For $t = 0$, the above moments are well defined in view of the condition (2.2.21) imposed on u_0. It is known that the initial mass $\alpha_0(t)$ and the center of mass $\alpha_1(t)$ are conserved for all $t \geq 0$. However, it is not true for the moments of order two or higher. The moments of the solution to (2.2.24)-(2.2.25) satisfy the following algebraic relations Kim (2011) for any time $t \geq 0$:

$$\alpha_{2k}(t) = \sum_{l=0}^{k} \frac{(2k)!}{(k-l)!(2l)!} t^{k-l} \alpha_{2l}(0), \tag{2.2.26}$$

$$\alpha_{2k+1}(t) = \sum_{l=0}^{k} \frac{(2k+1)!}{(k-l)!(2l+1)!} t^{k-l} \alpha_{2l+1}(0). \tag{2.2.27}$$

We define the backward moment $\alpha_k(-t_0)$ of the function $v(x,t)$ at a backward time $t_0 > 0$ by making use of summations in (2.2.26)-(2.2.27). One may notice that the backward moments of $v(x,t)$ are also moments of $v(x,t)$ for $t > -\tau$, where τ is the age Philip (1968) of the heat distribution $v_0(x)$.

Suppose that $v(x,t)$ and $\tilde{v}(x,t)$ are solutions of the heat equation (2.2.24) with initial profiles $v_0(x)$ and $\tilde{v}_0(x)$ respectively. If v and $\tilde{v}$ share the moments upto k-th order at any specific value of t, then they share the moments upto the same order for all $t \in \mathbb{R}$ in view of (2.2.26)-(2.2.27).

2.2.1 Contraction of moments

Lemma 2.2.1. *Assume $v(x,t)$ is the solution of the heat equation (2.2.24)-(2.2.25) with zero mass initial data $v_0(x)$ such that $(1+x^{2n})v_0(x) \in L^1(\mathbb{R})$. Then, for any given $t_0 > 0$, there exist $b_i,\ c_i \in \mathbb{C}$ such that*

$$\int_{-\infty}^{\infty} x^k v(x,t)dx = \int_{-\infty}^{\infty} x^k v_n(x,t)dx, \quad 0 \le k \le 2n, \tag{2.2.28}$$

where

$$v_n(x,t) := Re\left(\sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi(t+t_0)}}\partial_x\left(e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right)\right).$$

Proof. Consider a linear combination of the spatial derivative of heat kernels, namely,

$$w_n(x,t) := \sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi(t+t_0)}}\partial_x\left(e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right), \tag{2.2.29}$$

where x and t are real variables with $t > -t_0$, b_i's and c_i's are complex parameters and t_0 is any positive constant. Further, consider the real part of the complex function $w_n(x,t)$, denoted by

$$v_n(x,t) := \mathrm{Re}\left(w_n(x,t)\right). \tag{2.2.30}$$

We now make the first $2n+1$ moments of $v(x,t)$ and $v_n(x,t)$ equal by suitably assigning values to $2n$ parameters b_i's, $\quad c_i$'s, $\ i = 1, 2, \ldots, n$. Since

$$\lim_{t\to -t_0}\int_{-\infty}^{\infty} x^k v_n(x,t)dx = -k\ Re\left(\sum_{i=1}^{n} b_i c_i^{k-1}\right),\ 1 \le k \le 2n,$$

one obtains

$$\alpha_k(-t_0) = \begin{cases} 0, & \text{for } k = 0 \\ Re\left(-k\sum_{i=1}^{n} b_i c_i^{k-1}\right), & \text{for } k = 1, 2, \ldots, 2n, \end{cases} \tag{2.2.31}$$

by equating the first $2n+1$ moments of $v(x,t)$ and $v_n(x,t)$ when $t \to -t_0$. Since the initial mass under the heat equation is conserved and we picked up the zero mass initial data (2.2.25), $k = 0$ case in (2.2.31) is obtained. One can observe that, if we take $t_0 = 0$, the initial data v_0 is approximated by real part of linear combination of dipole distributions. So, taking into consideration of backward time helps in approximating the initial data v_0 by the function $v_n(x,0)$.

Introducing

$$\tilde{\alpha}_{k-1} := -\frac{\alpha_k(-t_0)}{k} \text{ for } \ k = 1, 2, \ldots, 2n, \tag{2.2.32}$$

equations (2.2.31) reduce to

$$\tilde{\alpha_k} = \mathrm{Re}\left(\sum_{i=1}^{n} b_i c_i^k\right), \quad 0 \le k < 2n. \tag{2.2.33}$$

It is known that, if $\tilde{\alpha_k}$'s are moments of a non-negative function, then the existence of $b_i, c_i \in \mathbb{R}$ such that

$$\tilde{\alpha_k} = \sum_{i=1}^{n} b_i c_i^k, \quad 0 \le k < 2n \tag{2.2.34}$$

is guaranteed by truncated moment problem. However, the truncated moment problem (2.2.34) need not be solvable for arbitrary $\tilde{\alpha_k}$'s. Using the generalization of truncated moment problem to a complex measure space developed by Kim (2011), we solve (2.2.33) for the complex values b_i's and c_i's.

Pick up a nonnegative function $q_0(x)$ satisfying the following properties:

I It decays fast enough as $|x| \to \infty$ so that its k-*th* order moments, β_k, are well defined for $0 \le k < 2n$. Denote $m_k := \tilde{\alpha}_k + i\beta_k,\ 0 \le k < 2n$.

II The auxiliary n-*th* degree complex polynomial $g_n(z) := z^n - \sum_{i=0}^{n-1} \psi_i z^i$ has n- distinct zeros, where the column vector $(\psi_0, \psi_1, \ldots, \psi_{n-1})^t$ is the unique solution of the system,

$$\begin{pmatrix} m_0 & m_1 & \ldots & m_{n-1} \\ m_1 & m_2 & \ldots & m_n \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ m_{n-1} & m_n & \ldots & m_{2n-2} \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ . \\ . \\ . \\ \psi_{n-1} \end{pmatrix} = \begin{pmatrix} m_n \\ m_{n+1} \\ . \\ . \\ . \\ m_{2n-1} \end{pmatrix}$$

It is to be noted that the coefficient matrix in the above system is Hankel Matrix. Then the Generalized truncated moment problem,

$$\sum_{i=1}^{n} b_i c_i^k = m_k, \quad 0 \le k < 2n \tag{2.2.35}$$

has a solution set $b_i, c_i \in \mathbb{C}$ which is unique upto reordering. Here, c_i's are none other than zeros of the auxiliary complex polynomial g_n. Thus, the zeros depend not only on $\tilde{\alpha}_k$'s, but also on nonnegative function q_0. Further, b_i's, $1 \le i \le n$, are obtained by solving the Augmented system:

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ c_1 & c_2 & \dots & c_n \\ . & . & \dots & . \\ . & . & \dots & . \\ . & . & \dots & . \\ c_1^{n-1} & c_2^{n-1} & \dots & c_n^{n-1} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ . \\ . \\ . \\ m_{n-1} \end{pmatrix}.$$

It is to be noted that these complex values c_i, b_i also satisfy the rest n equations in (2.2.35). Equating the real parts on both sides of (2.2.35), we get

$$\mathrm{Re}\left(\sum_{i=1}^{n} b_i c_i^k\right) = \tilde{\alpha}_k, \quad 0 \le k < 2n$$

concluding the proof. □

Remark 2.2.2. *Under the same hypothesis of Lemma 2.2.1, we can find a Gaussian approximation $\tilde{v}_n$ from Kim (2011) as follows:*
For any given $t_0 > 0$, there exist ρ_i, $c_i \in \mathbb{C}$ such that

$$\int_{-\infty}^{\infty} x^k v(x,t)dx = \int_{-\infty}^{\infty} x^k \tilde{v}_n(x,t)dx, \quad k = 0, 1, \dots, 2n-1, \tag{2.2.36}$$

where

$$\tilde{v}_n(x,t) := Re\left(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi(t+t_0)}} e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right).$$

In consideration of (2.2.28) and (2.2.36), we can say that $2n+1$ moments of N-wave approximation v_n for heat solution agree with $2n+1$ moments of v whereas only $2n$ moments of the Gaussian approximation $\tilde{v}_n$ agree with those of v. It is *to be noticed that we imposed an extra condition of zero mass when compared to the class of initial functions considered in Kim (2011). So the vanishing of the* 0*-th moment of* v *makes the* $2n$*-th moment of* v_n *to agree with* $2n$*-th moment of* v*. Illustrating this contraction of moments for* $n = 2$*, an example is discussed in Section 2.2.2.*

2.2.2 An example of contracting moments

Consider the initial value problem (2.2.24)-(2.2.25) with a discontinuous function

$$v_0(x) = \begin{cases} -\dfrac{1}{5}, & -1 < x < 1, \\ \dfrac{1}{x^6}, & \text{otherwise.} \end{cases} \tag{2.2.37}$$

It is easy to see that v_0 has moments upto 4-th order only. These are;

$$\alpha_0(0) = 0,\ \alpha_1(0) = 0,\ \alpha_2(0) = 8/15,\ \alpha_3(0) = 0,\ \alpha_4(0) = 48/25.$$

Then the backward moments at $t_0 = 1$ are

$$\alpha_0(-1) = 0,\ \alpha_1(-1) = 0,\ \alpha_2(-1) = \frac{8}{15},\ \alpha_3(-1) = 0,\ \alpha_4(-1) = -\frac{112}{25},$$

where the algebraic relations (2.2.26)-(2.2.27) were used. Then (2.2.32) gives

$$\tilde{\alpha}_0 = 0,\ \tilde{\alpha}_1 = -\frac{4}{15},\ \tilde{\alpha}_2 = 0,\ \tilde{\alpha}_3 = \frac{28}{25}.$$

Pick up a nonnegative function $q_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ and introduce $m_k = \tilde{\alpha}_k + i\beta_k$ with $\beta_k = \int_{-\infty}^{\infty} x^k q_0(x)dx$ for $k = 0, 1, 2, 3$. Then, we have

$$m_0 = i, m_1 = -\frac{4}{15},\ m_2 = \frac{i}{2},\ m_3 = \frac{28}{25}.$$

Solving the Hankel system,

$$\begin{pmatrix} m_0 & m_1 \\ m_1 & m_2 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} m_2 \\ m_3 \end{pmatrix},$$

we have

$$\psi_0 = -\frac{219}{2570},\ \psi_1 = -\frac{564}{257} i. \tag{2.2.38}$$

Then the auxiliary polynomial $g_2(z) = z^2 - \psi_0 - \psi_1 z$ has two distinct complex zeros, namely,

$$z = \frac{-2820i + \sqrt{8515230}i}{2570} =: c_1, \; z = \frac{-2820 - \sqrt{8515230}}{2570}i =: c_2. \qquad (2.2.39)$$

Now, solving the Augmented system,

$$\begin{pmatrix} 1 & 1 \\ c_1 & c_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} i \\ -4/5 \end{pmatrix},$$

we have

$$b_1 = \frac{10516 + 3\sqrt{8515230}}{6\sqrt{8515230}}i, \; b_2 = \frac{i}{2} - \frac{5258}{12772845}i. \qquad (2.2.40)$$

Hence, the N-wave approximation for (2.2.24)-(2.2.25) with (2.2.37) is given by

$$v_2(x,t) = \frac{1}{\sqrt{4\pi(t+1))}} Re\left[\partial_x\left(b_1 e^{-\frac{(x-c_1)^2}{4(t+1)}} + b_2 e^{-\frac{(x-c_2)^2}{4(t+1)}}\right)\right],$$

where c_1, c_2, b_1, b_2 are given by (2.2.39)-(2.2.40) satisfying

$$\int_{-\infty}^{\infty} x^k v(x,t)dx = \int_{-\infty}^{\infty} x^k v_2(x,t)dx, \quad 0 \le k \le 4. \qquad (2.2.41)$$

Let us now obtain a result which reveals the relation between moments and asymptotic behavior of solution of heat equation.

Theorem 2.2.3. *Let $E(x,t)$ be a solution to the Cauchy problem*

$$E_t(x,t) = E_{xx}(x,t), \quad x \in \mathbb{R}, \quad t > 0, \qquad (2.2.42)$$

$$E(x,0) = E_0(x), \quad x \in \mathbb{R} \qquad (2.2.43)$$

satisfying

$$\int_{-\infty}^{\infty} x^k E_0(x)dx = 0, \quad 0 \le k < m \qquad (2.2.44)$$

and $x^m E_0 \in L^1(\mathbb{R})$. Then, for $1 \le p \le \infty$,

$$||E(.,t)||_p = O(t^{-\frac{(m+1)}{2}+\frac{1}{2p}}), \quad t \to \infty. \qquad (2.2.45)$$

Remark 2.2.4. *In fact, Duoandikoetxea and Zuazua (1992) gave the proof of Theorem 2.2.3. They defined the sequence of functions recursively;*

$$E_k(x) = \int_{\infty}^{x} E_{k-1}(x)dx \quad k = 1, 2, \ldots, m.$$

They then proved that 0-th moment of each E_k vanishes and E_k's approach to zero as $|x| \to \infty$ by applying integration by parts and making use of inductive arguments. The following proof makes use of Taylor's series expansion and Minkowski inequality and is of our independent interest only.

Proof of Theorem 2.2.3. The solution of heat equation (2.2.42)-(2.2.43) is given by

$$E(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} E_0(\xi) e^{\frac{-(x-\xi)^2}{4t}} d\xi. \tag{2.2.46}$$

Changing the variable as $(r,s) = \left(\frac{x}{\sqrt{4t}}, \frac{\xi}{\sqrt{4t}}\right)$ and applying (2.2.44), the above equation (2.2.46) reduces to

$$\begin{aligned} E(x,t) =& \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} E_0(2\sqrt{t}s) e^{-(r-s)^2} ds \\ =& \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} E_0(2\sqrt{t}s) \left[\sum_{j=m}^{\infty} p_j(r) \frac{(-s)^j}{j!}\right] ds. \end{aligned}$$

Here we used the following Taylor expansion,

$$e^{-(r-s)^2} = \sum_{j=0}^{\infty} p_j(r) \frac{(-s)^j}{j!} \text{ with } p_j(r) = \frac{d^j}{ds^j} e^{-(r-s)^2}\Big|_{s=0}.$$

and the conditions (2.2.44). We now introduce the notation,

$$T_m(r,y) = \sum_{j=m}^{\infty} p_j(r) \frac{(-y)^j}{j!}.$$

Then $e^{-(r-s)^2} = \sum_{j=0}^{m-1} p_j(r) \frac{(-s)^j}{j!} + T_m(r,y)$. This implies

$$\partial_s^m \left[e^{-(r-s)^2}\right] = \partial_s^m T_m(r,s). \tag{2.2.47}$$

Now since $\partial_s^i(T_m(r,s))\Big|_{s=0} = 0$, for $i = 0,1,2...m-1$, we have

$$\partial_s^i(T_m(r,s)) = \int_0^s \partial_s^{i+1}(T_m(r,s'))(r,s')ds'.$$

Applying L^p norm on both sides and using Minkowski inequality we get:

$$\|\partial_s^i(T_m(r,s))\|_p \leq |s| \|\partial_s^{i+1}(T_m(r,s))\|_p, \qquad \text{for } i = 0,1,\ldots,m.$$

Applying the above inequality m-times, starting with $i = 0$, we have

$$\|(T_m(r,s))\|_p \leq |s|^m \|\partial_s^m (T_m(r,s))\|_p = |s|^m \left\| \partial_s^m \left(e^{-(r-s)^2} \right) \right\|_p = O(|s|^m). \tag{2.2.48}$$

In the above derivation we have used (2.2.47).

For $1 \leq p < \infty$,

$$\begin{aligned}
\|E(.,t)\|_p &= \frac{(2\sqrt{t})^{1/p}}{\sqrt{\pi}} \left(\int_{r=-\infty}^{\infty} \left| \int_{s=-\infty}^{\infty} E_0(2\sqrt{t}s) T_m(r,s) ds \right|^p dr \right)^{1/p} \\
&\leq \frac{(2\sqrt{t})^{1/p}}{\sqrt{\pi}} \int_{s=-\infty}^{\infty} \left(\int_{r=-\infty}^{\infty} \left| E_0(2\sqrt{t}s) T_m(r,s) \right|^p dr \right)^{1/p} ds \\
&\leq \frac{(2\sqrt{t})^{1/p}}{\sqrt{\pi}} \int_{-\infty}^{\infty} |E_0(2\sqrt{t}s)| \; \|T_m(.,s)\|_p ds, \\
&= O(1)(\sqrt{t})^{1/p} \int_{-\infty}^{\infty} |E_0(2\sqrt{t}s)| \; |s|^m ds \\
&= O(1)(\sqrt{t})^{-(m+1)+\frac{1}{p}} \int_{-\infty}^{\infty} |E_0(\xi)| \; |\xi|^m d\xi \\
&= O(t^{-\frac{(m+1)}{2}+\frac{1}{2p}}).
\end{aligned}$$

In the above derivation, we used Minkowski inequality and inequality (2.2.48). An analogous result can be obtained for the case $p = \infty$. □

Lemma 2.2.5. *Let $v(x,t)$ be a solution to the heat equation (2.2.24)-(2.2.25) with the zero mass initial data $v_0(x)$ satisfying $v_0, x^{2n+1}v_0 \in L^1(\mathbb{R})$. Then, for any $t_0 > 0$, there exist $b_i, c_i \in \mathbb{C}$ with*

$$v_n(x,t) := Re \left(\sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi(t+t_0)}} \partial_x \left(e^{-\frac{(x-c_i)^2}{4(t+t_0)}} \right) \right) \tag{2.2.49}$$

such that

$$||v(.,t) - v_n(.,t)||_p = O(t^{-(n+1)+1/2p}), \quad t \to \infty, \tag{2.2.50}$$

where $1 \leq p \leq \infty$.

Proof. Making use of Lemma 2.2.1, we can construct v_n as if $2n+1$ moments of $v_n(x,t)$ agrees with $2n+1$ moments of $v(x,t)$. Having the agreement of moments for $v(x,t)$ and $v_n(x,t)$, we now define the difference function as follows:

$$E(x,t) := v(x,t) - v_n(x,t) \tag{2.2.51}$$

$$E(x,0) = v_0(x) - v_n(x,0) =: E_0(x), \quad x \in \mathbb{R}. \tag{2.2.52}$$

Then $E(x,t)$ satisfies the hypothesis of Theorem 2.2.3 with $m = 2n+1$ and hence the conclusion is derived. $\square$

We can now generalize the above Lemma 2.2.5 and proof is omitted as it is almost a repetition.

Lemma 2.2.6. *Let $v(x,t)$ be a solution to the heat equation (2.2.24)-(2.2.25) with the measurable function $v_0(x)$ satisfying $\int_{-\infty}^{\infty} x^k v_0(x)dx = 0,\ 0 \le k < m$ and $v_0,\ x^{2n+m}v_0 \in L^1(\mathbb{R})$. Then, for any $t_0 > 0$, there exist $b_i, c_i \in \mathbb{C}$ such that*

$$||v(.,t) - v_n(.,t)||_p = O(t^{-(\frac{2n+1+m}{2})+\frac{1}{2p}}), \quad t \to \infty,$$

where $1 \le p \le \infty$, with

$$v_n(x,t) := Re\left(\sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi(t+t_0)}}\partial_x^m\left(e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right)\right).$$

2.3 Asymptotics for Burgers solutions

In this section, we introduce the asymptotic N-wave approximation, $u_n(x,t)$, to the solution $u(x,t)$ of (2.1.8) and then obtain the error estimates. It is to be noted that the moments of the true solution and approximate solution of the heat equation (2.2.24) are made equal for attaining higher order error estimates. The same higher order error estimates between the true solution and the approximate solution of the Burgers equation (2.1.8) are obtained even though the concerned moments of the solutions of (2.1.8) are not necessarily agreeing.

The inverse Cole-Hopf transformation H^{-1} gives us

$$u(x,t) = -\frac{2v(x,t)}{1+\int_{-\infty}^{x} v(s,t)ds}.$$

We now prove Theorem 2.1.1. It is to be noted that the L^p contraction is mainly due to the agreement of moments, not due to the specific form of v_n.

Proof of Theorem 2.1.1. We prove this theorem in 4 steps.

Step 1: We prove that there exists a $T > 0$ such that the solution $u_n(x,t)$, given in (2.1.14), of (2.1.8) is well defined for all $x \in \mathbb{R}$ and $t \ge T$.

In order to prove that (2.1.14) is well defined for all $(x,t) \in \mathbb{R} \times [T, \infty)$, it suffices to prove that the denominator in right hand side of (2.1.14) is positive for all $(x,t) \in \mathbb{R} \times [T, \infty)$. A calculation on the denominator in right hand side of (2.1.14) gives

$$\begin{aligned} 1 + \int_{-\infty}^{x} v_n(s,t)ds =& 1 + \int_{-\infty}^{x} Re\left[\left(\sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi(t+t_0)}} \partial_x e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right)\right] dx \\ =& 1 + \mathrm{Re}\left[\sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi(t+t_0)}} e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right]. \end{aligned}$$

Hence, the above expression converges to 1 as $t \to \infty$ uniformly on $\mathbb{R}$. Thus, there exists a $T > 0$ such that

$$1 + \int_{-\infty}^{x} v_n(s,t)ds \geq \delta > 0, \quad \forall (x,t) \in \mathbb{R} \times [T, \infty) \tag{2.3.53}$$

for any sufficiently small δ.

Step 2: We prove that $1 + V(x,t)$ is bounded below by a positive constant for all $t > 0$ and $x \in \mathbb{R}$.

Set $B := \sup_{x \in \mathbb{R}} \int_{-\infty}^{x} u_0(s)ds$. Since u_0 is L^1-function, one can say that B exists and so $1 + V(x,0)$ in (2.2.23) is bounded below by $e^{-B/2}$ for all $x \in \mathbb{R}$. Thus, as $V(x,t)$ is a solution of (2.2.22)-(2.2.23), it is not hard to see that $1 + V(x,t)$ is also bounded below by $e^{-B/2}$ $\forall (x,t) \in \mathbb{R} \times (0, \infty)$.

Step 3: We now get the L^∞-norm estimate for $\int_{-\infty}^{x} E(s,t)ds$ with respect to the space variable x.

For $1 \leq k \leq 2n$, integration by parts gives us

$$\int_{-\infty}^{\infty} x^k E_0(x)dx = -k \int_{-\infty}^{\infty} x^{k-1} \left(\int_{-\infty}^{x} E_0(s)ds\right) dx + \left[x^k \int_{-\infty}^{x} E_0(s)ds\right]_{x=-\infty}^{\infty}.$$

Thus, for $0 \leq r < 2n$, we obtain

$$\int_{-\infty}^{\infty} x^r \left(\int_{-\infty}^{x} E_0(s)ds\right) dx = 0,$$

as $x^{2n}E_0(x) \in L^1(\mathbb{R})$ and $\int_{-\infty}^{\infty} x^k E_0(x)dx = 0$ with $0 \leq k \leq 2n$. Further, a calculation shows that

$$\int_{-\infty}^{\infty} \left| x^{2n} \int_{-\infty}^{x} E_0(s)ds \right| dx \leq \frac{1}{2n+1} \int_{-\infty}^{\infty} \left| x^{2n+1} E_0(x) \right| dx.$$

Therefore, $x^{2n}\int_{-\infty}^{x} E_0(s)ds \in L^1(\mathbb{R})$. Hence, in view of Theorem 2.2.3 and its Remark 2.2.4, we arrive at

$$\left\|\int_{-\infty}^{x} E(s,t)ds\right\|_{\infty} = O(t^{-(n+1/2)}) \quad \text{as} \quad t \to \infty. \tag{2.3.54}$$

Step 4: We now obtain the L^p-norm estimate for $u(x,t)-u_n(x,t)$ with respect to the space variable x.

Consider

$$|u(x,t)-u_n(x,t)| = 2\left|\frac{v(x,t)}{1+\int_{-\infty}^{x} v(s,t)ds} - \frac{v_n(x,t)}{1+\int_{-\infty}^{x} v_n(s,t)ds}\right|$$
$$=2\left|\frac{E(x,t)(1+\int_{-\infty}^{x} v_n(s,t)ds) - v_n(x,t)\int_{-\infty}^{x} E(s,t)ds}{(1+V(x,t))(1+\int_{-\infty}^{x} v_n(s,t)ds)}\right|.$$

By virtue of *Step 2* and the inequality (2.3.53) of Step 1, there exists a positive real number k_1 such that

$$|u(x,t)-u_n(x,t)| \le k_1\left|E(x,t)\left(1+\int_{-\infty}^{x} v_n(s,t)ds\right) - v_n(x,t)\int_{-\infty}^{x} E(s,t)ds\right|$$
$$\forall (x,t) \in \mathbb{R}\times(T,\infty). \tag{2.3.55}$$

We now get the L^p-norm estimates for the two terms in the right hand side of the inequality (2.3.55). Since $v_n(x,t)$ is an L^1-function, there exists a positive real number k_2 such that

$$\left\|E(x,t)\left(1+\int_{-\infty}^{x} v_n(s,t)ds\right)\right\|_p \le k_2||E(x,t)||_p.$$

Then by (2.2.45), we have

$$\left\|E(x,t)\left(1+\int_{-\infty}^{x} v_n(s,t)ds\right)\right\|_p = O(t^{-(n+1)+1/2p}) \quad \text{as } t\to\infty. \tag{2.3.56}$$

In view of (2.3.54) and Minkowski inequality, there exists a constant $k_3 > 0$ so that

$$\left\|v_n(x,t)\int_{-\infty}^{x} E(s,t)ds\right\|_p \le k_3 t^{-(n+\frac{1}{2})}\sum_{i=1}^{n}\left\|\mathrm{Re}\left(\frac{b_i}{\sqrt{4\pi(t+t_0)}}\partial_x\left(e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right)\right)\right\|_p$$
$$=O(t^{-(n+\frac{3}{2})+\frac{1}{2p}}) \quad \text{as} \quad t\to\infty. \tag{2.3.57}$$

Taking L^p-norm to (2.3.55) and then making use of the results (2.3.56)-(2.3.57), we obtain, for each $1 \leq p \leq \infty$,

$$||u(x,t) - u_n(x,t)||_p = O(t^{-(n+1)+\frac{1}{2p}}) \quad \text{as} \quad t \to \infty.$$

$\square$

Remark 2.3.1. *Under the same hypothesis as in Theorem 2.1.1, one can obtain an approximation in terms of Gaussian approximation (Jaywan et al., 2010) as follows:*

For any $t_0 > 0$, there exist $\rho_i, c_i \in \mathbb{C}$ and $T > 0$ such that

$$||u(.,t) - \tilde{u}_n(.,t)||_p = O(t^{-\frac{2n+1}{2}+\frac{1}{2p}}), \quad t \to \infty, \tag{2.3.58}$$

where $1 \leq p \leq \infty$ and

$$\tilde{u}_n(x,t) := -2\frac{\tilde{v}_n(x,t)}{1 + \int_{-\infty}^{x} \tilde{v}_n(y,t)dy} \tag{2.3.59}$$

is well defined for $t \geq T$ with

$$\tilde{v}_n(x,t) = Re\left(\sum_{i=1}^{n} \frac{\rho_i}{\sqrt{4\pi(t+t_0)}} e^{-\frac{(x-c_i)^2}{4(t+t_0)}}\right). \tag{2.3.60}$$

In view of (2.1.13) and (2.3.58), we can say that rate of convergence of N-wave approximation is higher than that of Gaussian approximation. Further, we can notice that N-wave approximation u_n is in a simpler form whereas Gaussian approximation $\tilde{u}_n$ involves an integral in the denominator (2.3.2) to be evaluated numerically. It is further to be noted that Chung, Kim and Ni (2009) considered the class of initial functions for which odd number of moments exist whereas we dealt with the class of zero mass initial functions for which even number of moments exist. In this regard, an example is given in Section 2.3.2.

Remark 2.3.2. *We considered the Burgers equation (2.1.8) with respect to the zero mass initial data u_0 on $\mathbb{R}$ by imposing a condition on u_0 that*

$$(1 + |x|^{2n+1})u_0 \in L^1(\mathbb{R}). \tag{2.3.61}$$

The necessity of the condition (2.3.61) is to assure the existence of $2n+1$ moments for u_0. One may even replace the condition (2.3.61) by assuming that u_0 is bounded almost everywhere on the open interval $(-1,1)$ and $x^{2n+1}u_0 \in L^1(\mathbb{R})$.

2.3.1 Examples with single heat kernel

We pick up two initial functions for the Burgers equation (2.1.8) and construct corresponding approximate solutions, in the form of spatial derivate of a single heat kernel, to (2.1.8) with such initial functions. Then we show that these constructed solutions approach the true solutions with an error of order $O(t^{-2+1/2p})$ as $t \to \infty$ in L^p-norm, where $1 \leq p \leq \infty$. It is known that $u_0(x)$ is of zero mass if and only if $v_0(x) = -\frac{1}{2}u_0(x)\exp\left(-\frac{1}{2}\int_{-\infty}^{x} u_0(s)ds\right)$ is of zero mass. Hence, as we deal with zero mass initial data $u_0(x)$ for (2.1.8), (2.1.11), one may find the moments of the initial data v_0 of the concerned heat equation (2.2.24)-(2.2.25) with Yanagisawa (2007).

However, for the sake of easier calculations, we first consider the initial profiles for the heat equation and then describe the approximate solutions for the concerned initial value problem posed for Burgers equation (2.1.8).

The first example we see is the initial data

$$v_0(x) := \frac{1}{3\sqrt{\pi}}\left[e^{-(x-1)^2} - 3e^{-9(x+2)^2}\right] \tag{2.3.62}$$

to the problem (2.2.24)-(2.2.25). Then

$$\alpha_0 = 0,\ \alpha_1 = 1,\ \alpha_2(0) = -\frac{23}{27}.$$

In consideration of (2.2.32), we have

$$\tilde{\alpha_0} = -1,\ \tilde{\alpha_1} = \frac{23}{54}.$$

Hence the truncated moment problem

$$b_1 = \tilde{\alpha_0},\ b_1c_1 = \tilde{\alpha_1} \tag{2.3.63}$$

has a unique solution as the concerned Hankel matrix is non singular. The solution of the truncated moment problem (2.3.63) is

$$b_1 = -1, \quad c_1 = -\frac{23}{54}.$$

Thus the approximate solution of the problem (2.2.24)-(2.2.25) with (2.3.62) is given by

$$v_1(x,t) := -\frac{1}{\sqrt{4\pi(t+1)}}\partial_x\left(e^{-\frac{(x+23/54)^2}{4(t+1)}}\right).$$

Define

$$u(x,0) := -\frac{2v_0(x)}{1+\int_{-\infty}^{x} v_0(s)ds}, \quad x \in \mathbb{R}, \tag{2.3.64}$$

where v_0 is given in (2.3.62). Then the approximate solution of (2.1.8) subject to (2.3.64) is given by

$$u_1(x,t) := -2\frac{v_1(x,t)}{1+\int_{-\infty}^{x} v_1(y,t)dy} = \frac{x+23/54}{(1+t)\left[1-2\sqrt{\pi(1+t)}e^{\frac{(x+23/54)^2}{4(t+1)}}\right]}$$

and the error estimate is given by

$$||u(.,t) - u_1(.,t)||_p = O(t^{-2+1/2p}), \quad t \to \infty. \tag{2.3.65}$$

Let us consider another initial data

$$v_0(x) := \frac{1}{\sqrt{\pi}}\left[e^{-(x-1)^2} + e^{-4(x+2)^2} - \frac{9}{2}e^{-9x^2}\right] \tag{2.3.66}$$

to the problem (2.2.24)-(2.2.25). Then

$$\alpha_0 = 0, \ \alpha_1 = 0, \ \alpha_2(0) = \frac{167}{48}$$

In consideration of (2.2.32), we have

$$\tilde{\alpha_0} = 0, \ \tilde{\alpha_1} = -\frac{167}{48}.$$

Hence the truncated moment problem

$$b_1 = \tilde{\alpha_0}, \ b_1c_1 = \tilde{\alpha_1}$$

can not be solved as the concerned Hankel matrix is singular. Then by introducing $q_0(x) = \frac{e^{-x^2/2}}{2\sqrt{\pi}}$ and $\beta_k = \int_{-\infty}^{\infty} x^k q_0(x)dx$, one has

$$\begin{aligned} m_0 &= \tilde{\alpha_0} + i\beta_0 = i \\ m_1 &= \tilde{\alpha_1} + i\beta_1 = -\frac{167}{96}. \end{aligned}$$

Then the generalized truncated moment problem

$$b_1 = m_0, \ b_1c_1 = m_1$$

has a solution

$$b_1 = i, \quad c_1 = \frac{167}{96} i.$$

Thus the approximate solution of the problem (2.2.24)-(2.2.25) with (2.3.66) is given by

$$v_1(x,t) := \mathrm{Re}\left(\frac{i}{\sqrt{4\pi(t+1)}} \partial_x \left(e^{-\frac{(x-\frac{167}{96}i)^2}{4(t+1)}} \right) \right).$$

Define

$$u(x,0) := -\frac{2v_0(x)}{1+\int_{-\infty}^{x} v_0(s)ds}, \quad x \in \mathbb{R}, \tag{2.3.67}$$

where v_0 is given in (2.3.66). Then the approximate solution $u_1(x,t)$ of the Burgers equation (2.1.8) with reference to (2.3.67) satisfies the error estimate (2.3.65) and is given by

$$u_1(x,t) := -2\frac{\mathrm{Re}\left(\frac{i}{\sqrt{4\pi(t+1)}} \partial_x \left(e^{-\frac{(x-\frac{167}{96}i)^2}{4(t+1)}} \right) \right)}{1+\mathrm{Re}\left(\frac{i}{\sqrt{4\pi(t+1)}} \left(e^{-\frac{(x-\frac{167}{96}i)^2}{4(t+1)}} \right) \right)}.$$

2.3.2 An example with three heat kernels

Consider the initial value problem (2.1.8), (2.1.11) with

$$u_0(x) = \begin{cases} \dfrac{16}{x-8x^9}, & \text{if } x \leq -1, \\ -\dfrac{16x}{3+4x^2}, & \text{if } -1 < x < 1, \\ \dfrac{16}{x-8x^9}, & \text{if } x \geq 1. \end{cases} \tag{2.3.68}$$

It is seen that moments of u_0 exist upto the 7-th order only. Due to Cole-Hopf transformation, we now study heat equation (2.2.24)-(2.2.25) with

$$v_0(x) = \begin{cases} x, & -1 < x < 1, \\ \dfrac{1}{x^9}, & \text{otherwise}. \end{cases} \tag{2.3.69}$$

We then see that v_0 has moments upto the order 7 only. These are;

$$\alpha_0(0) = 0, \alpha_1(0) = 20/21, \ \alpha_2(0) = 0, \ \alpha_3(0) = 4/5,$$

$$\alpha_4(0) = 0, \alpha_5(0) = 20/21, \alpha_6(0) = 0, \alpha_7(0) = 20/9.$$

Then the backward moments at $t_0 = 1$ are

$$\alpha_0(-1) = 0, \alpha_1(-1) = 20/21,\ \alpha_2(-1) = 0,\ \alpha_3(-1) = -172/35,$$
$$\alpha_4(-1) = 0, \alpha_5(-1) = 884/21, \alpha_6(-1) = 0, \alpha_7(-1) = -4516/9,$$

where the algebraic relations (2.2.26)-(2.2.27) were used. Then (2.2.32) gives

$$\tilde{\alpha}_0 = -20/21, \tilde{\alpha}_1 = 0,\ \tilde{\alpha}_2 = 172/105,$$
$$\tilde{\alpha}_3 = 0,\ \tilde{\alpha}_4 = -884/105, \tilde{\alpha}_5 = 0.$$

Pick up a nonnegative function $q_0(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ and introduce $m_k = \tilde{\alpha}_k + i\beta_k$ with $\beta_k = \int_{-\infty}^{\infty} x^k q_0(x) dx$ for $k = 0, 1, 2, 3, 4, 5$. Then, we have

$$m_0 = -\frac{20}{21} + i, m_1 = 0,\ m_2 = \frac{172}{105} + \frac{i}{2},$$
$$m_3 = 0,\ m_4 = -\frac{884}{105} + \frac{3i}{4}, m_5 = 0.$$

Solving the Hankel system,

$$\begin{pmatrix} m_0 & m_1 & m_2 \\ m_1 & m_2 & m_3 \\ m_2 & m_3 & m_4 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} m_3 \\ m_4 \\ m_5 \end{pmatrix},$$

we have

$$\psi_0 = 0,\ \psi_1 = -\frac{1183309}{258722} + \frac{239820}{129361} i,\ \psi_2 = 0. \qquad (2.3.70)$$

Then the auxiliary polynomial $g_3(z) = z^3 - \psi_1 z$ has three distinct complex zeros, namely,

$$z = 0,\ z = c,\ z = -c,\ \text{where}$$
$$c = \sqrt{-\frac{1183309}{258722} + \frac{239820}{129361} i}. \qquad (2.3.71)$$

Now, solving the Augumented system,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & c & -c \\ 0 & c^2 & c^2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} m_0 \\ m_1 \\ m_2 \end{pmatrix},$$

we have

$$b_1 = -\frac{903556004}{1323264705} + \frac{15357222}{12602521}i, \ b_2 = b_3 = -\frac{178348048}{1323264705} - \frac{2754701}{25205042}i. \quad (2.3.72)$$

Hence, the N-wave approximation for (2.1.8), (2.1.11) with (2.3.68) is given by

$$u_3(x,t) = -2\frac{\mathrm{Re}\left[\frac{1}{\sqrt{4\pi(t+1)}}\partial_x\left(b_1e^{-\frac{x^2}{4(t+1)}} + b_2e^{-\frac{(x-c)^2}{4(t+1)}} + b_3e^{-\frac{(x+c)^2}{4(t+1)}}\right)\right]}{1+\mathrm{Re}\left[\frac{1}{\sqrt{4\pi(t+1)}}\left(b_1e^{-\frac{x^2}{4(t+1)}} + b_2e^{-\frac{(x-c)^2}{4(t+1)}} + b_3e^{-\frac{(x+c)^2}{4(t+1)}}\right)\right]}, \quad (2.3.73)$$

where b_1, b_2, b_3 and c are given in (2.3.72) and (2.3.71). The rate of convergence is

$$||u(.,t) - u_3(.,t)||_p = O(t^{-4+1/2p}), \quad t \to \infty. \quad (2.3.74)$$

For the same initial value problem (2.1.8), (2.1.11), (2.3.68), one can construct a Gaussian approximation (Jaywan et al., 2010) $\tilde{u}_2(x,t)$ as follows:

$$||u(.,t) - \tilde{u}_3(.,t)||_p = O(t^{-\frac{7}{2}+\frac{1}{2p}}), \quad t \to \infty, \quad (2.3.75)$$

where $1 \leq p \leq \infty$ and

$$\tilde{u}_3(x,t) := -2\frac{\tilde{v}_3(x,t)}{1+\int_{-\infty}^{x}\tilde{v}_3(y,t)dy}$$

is well defined for $t \geq T$ with

$$\tilde{v}_3(x,t) = \mathrm{Re}\left(\sum_{i=1}^{3}\frac{\rho_i}{\sqrt{4\pi(t+1)}}e^{-\frac{(x-c_i)^2}{4(t+1)}}\right),$$

where $\rho_i, c_i \in \mathbb{C}$ and $T > 0$.

We are now in a position to generalize the Theorem 2.1.1 as follows.

Corollary 2.3.3. *Let $u(x,t)$ be a solution to the Burgers equation (2.1.8) subject to (2.1.11) with the measurable function $u_0(x)$ satisfying $\int_{-\infty}^{\infty}x^kv_0(x)dx = 0, \ 0 \leq k < m$ and $v_0, \ x^{2n+m}v_0 \in L^1(\mathbb{R})$, where v_0 is given by (2.2.25). Then, there exist $b_i, c_i \in \mathbb{C}$ and $T > 0$ such that*

$$||u(.,t) - u_n(.,t)||_p = O(t^{-(\frac{2n+1+m}{2})+\frac{1}{2p}}), \quad t \to \infty,$$

where $1 \leq p \leq \infty$,

$$u_n(x,t) := -2\frac{v_n(x,t)}{1+\int_{-\infty}^{x}v_n(y,t)dy} \quad (2.3.76)$$

is well defined for $t \geq T$ *with*

$$v_n(x,t) = Re\left(\sum_{i=1}^{n} \frac{b_i}{\sqrt{4\pi t}} \partial_x^m \left(e^{-\frac{(x-c_i)^2}{4t}}\right)\right).$$

The proof of Corollary 2.3.3 is also repetition of the procedure discussed and hence we omit the proof.

2.4 Asymptotics for solutions of Adhesion model

In this section, we present main results about L^p-contraction rate of solutions to Adhesion Model.

We now extend the results obtained in previous Sections to the de-coupled system (2.1.9)-(2.1.10) with initial functions

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{2.4.77}$$

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbb{R}, \tag{2.4.78}$$

by proving the Theorem 2.1.2.

Proof of Theorem 2.1.2. In view of the lemma 2.2.6, it is enough to consider the partial differential equation (2.1.10) with initial condition (2.4.78). Using the generalized Cole-Hopf transformation (Joseph, 2009),

$$C(x,t) = -\frac{1}{2}\int_{-\infty}^{x} \rho(s,t)ds \;\; \exp\left(-\frac{1}{2}\int_{-\infty}^{x} u(s,t)ds\right),$$

the equations (2.1.10) and (2.4.77)-(2.4.78) lead to

$$C_t = C_{xx} \tag{2.4.79}$$

$$C(x,0) = C_0(x), \tag{2.4.80}$$

where C_0 is given by (2.1.15). Let $C_n(x,t)$ and $\rho_n(x,t)$ be approximations to $C(x,t)$ and $\rho(x,t)$ respectively. Further, let $C(x,t) - C_n(x,t) = \gamma(x,t)$ and recall $v(x,t) - v_n(x,t) = E(x,t)$. By virtue of the lemma 2.3.3, we have

$$\| \gamma(.,t) \|_p = O(t^{-\frac{2n+1+m}{2}+\frac{1}{2p}}), \quad t \to \infty.$$

Using the inverse generalized Cole-Hopf transformation, we now obtain

$$\rho(x,t) = \frac{\left[1+\int_{-\infty}^{x} v(s,t)ds\right]\partial_x C(x,t) - v(x,t)C(x,t)}{\left[1+\int_{-\infty}^{x} v(s,t)ds\right]^2}, \tag{2.4.81}$$

where

$$C(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} C_0(y)\,\exp\left(-\frac{(x-y)^2}{4t}\right)dy, \tag{2.4.82}$$

$$v(x,t) = -\frac{1}{4\sqrt{\pi t}}\int_{-\infty}^{\infty} u_0(y)\,\exp\left(-\frac{1}{2}\int_{-\infty}^{y} u_0(s)ds - \frac{(x-y)^2}{4t}\right)dy. \tag{2.4.83}$$

Consider

$$\begin{aligned}\rho(x,t)-\rho_n(x,t) =& \frac{\left[1+\int_{-\infty}^{x} v(s,t)ds\right]\partial_x C(x,t) - v(x,t)C(x,t)}{\left[1+\int_{-\infty}^{x} v(s,t)ds\right]^2}\\ &- \frac{\left[1+\int_{-\infty}^{x} v_n(s,t)ds\right]\partial_x C_n(x,t) - v_n(x,t)C_n(x,t)}{\left[1+\int_{-\infty}^{x} v_n(s,t)ds\right]^2} \qquad (2.4.84)\\ =& \frac{A(x,t)}{\left[1+\int_{-\infty}^{x} v(s,t)ds\right]^2\left[1+\int_{-\infty}^{x} v_n(s,t)ds\right]^2}, \qquad (2.4.85)\end{aligned}$$

where the numerator $A(x,t)$ is given by

$$\begin{aligned}A(x,t) =& -\Bigg[\int_{-\infty}^{x} E(s,t)ds\ \partial_x C_n(x,t) + 2\int_{-\infty}^{x} E(s,t)ds\int_{-\infty}^{x} v_n(s,t)ds\times\\ &\partial_x C_n(x,t) - 2C_n(x,t)v_n(x,t)\int_{-\infty}^{x} E(s,t)ds + \int_{-\infty}^{x} E(s,t)ds\times\\ &\left(\int_{-\infty}^{x} v_n(s,t)ds\right)^2\partial_x C_n(x,t) - 2C_n(x,t)v_n(x,t)\int_{-\infty}^{x} E(s,t)ds\int_{-\infty}^{x} v_n(s,t)ds\Bigg]\\ &-\left(\int_{-\infty}^{x} E(s,t)ds\right)^2\left[\partial_x C_n(x,t) + \partial_x C_n(x,t)\int_{-\infty}^{x} v_n(s,t)ds - C_n(x,t)v_n(x,t)\right]\\ &+\partial_x\gamma(x,t)\int_{-\infty}^{x} E(s,t)ds\partial_x\gamma(x,t) + 3\partial_x\gamma(x,t)\int_{-\infty}^{x} v_n(s,t)ds - E(s,t)\Big[C_n(x,t)\\ &+\gamma(x,t)\Big] - v_n(x,t)\gamma(x,t) + 2\partial_x\gamma(x,t)\int_{-\infty}^{x} E(s,t)ds\int_{-\infty}^{x} v_n(s,t)ds\\ &+3\partial_x\gamma(x,t)\left(\int_{-\infty}^{x} v_n(s,t)ds\right)^2 - 2E(x,t)C_n(x,t)\int_{-\infty}^{x} v_n(s,t)ds\end{aligned}$$

$$
\begin{aligned}
&- 2\gamma(x,t)v_n(x,t)\int_{-\infty}^{x} v_n(s,t)ds - 2E(x,t)\gamma(x,t)\int_{-\infty}^{x} v_n(s,t)ds \\
&+ \left(\int_{-\infty}^{x} v_n(s,t)ds\right)^2 \left[\partial_x\gamma(x,t)\int_{-\infty}^{x} E(s,t)ds + \int_{-\infty}^{x} v_n(s,t)ds\partial_x\gamma(x,t)\right. \\
&\left. -E(x,t)(C_n(x,t)+\gamma(x,t)) - v_n(x,t)\gamma(x,t)\right].
\end{aligned}
$$

The existence of $T > 0$ is obtained by repeating the proof of *Step 1* of the Theorem 2.1.1. The existence of $\tilde{b}_i, \tilde{c}_i$ with (2.1.19) is obtained by repeating the proof of Lemma 2.2.5. Eventually, we have

$$
\begin{aligned}
\left\|\int_{-\infty}^{\cdot} E(s,t)ds\right\|_p =& O(t^{-\frac{2n+m}{2}+\frac{1}{2p}}), \quad t\to\infty, \\
\|E(.,t)\|_p =& O(t^{-\frac{2n+1+m}{2}+\frac{1}{2p}}), \quad t\to\infty, \\
\|\partial_x\gamma(.,t)\|_p =& O(t^{-\frac{2n+2+m}{2}+\frac{1}{2p}}), \quad t\to\infty, \\
\|\partial_x C_n(.,t)\|_p =& O(t^{-\frac{2+m}{2}+\frac{1}{2p}}), \quad t\to\infty, \\
\left\|\int_{-\infty}^{\cdot} v_n(s,t)ds\right\|_p =& O(t^{-\frac{m}{2}+\frac{1}{2p}}), \quad t\to\infty, \\
\|v_n(.,t)\|_p =& O(t^{-\frac{1+m}{2}+\frac{1}{2p}}), \quad t\to\infty, \\
\|C_n(.,t)\|_p =& O(t^{-\frac{1+m}{2}+\frac{1}{2p}}), \quad t\to\infty,
\end{aligned}
$$

for $1 \le p \le \infty$. Taking L^p-norm on equation (2.4.84) and then making use Minkowski inequality with above order estimates, we conclude the proof. □

We conclude this section by proving the Proposition 2.1.3, which shows a way to construct Schwartz function sharing the moments with the given function.

Proof of Proposition 2.1.3. In view of (2.1.20), construct a polynomial p_n by

$$
p_n(x) = \sum_{k=0}^{n} \frac{(2\pi i)^k c_k}{k!} x^k
$$

Denote the Fourier transform of $p_n\phi$ by g as follows:

$$
g(y) = \mathcal{F}(p_n(x)\phi(x)) = \int_{-\infty}^{\infty} p_n(x)\phi(x)e^{-2\pi ixy}dx,
$$

where ϕ is any test function having compact support in $\mathbb{R}$ which is 1 in the interval $[-1,1]$ and zero out side $[-2,2]$.

Then g is a Schwartz class function and denote the Fourier transform of g by h as follows:

$$h(x) = \int_{-\infty}^{\infty} g(y)e^{-2\pi ixy}dy \tag{2.4.86}$$

As $h(x) = p_n(-x)$ for $-1 \leq x \leq 1$, we get

$$h^{(j)}(0) = (-2\pi i)^j c_j, \quad j = 0, 1 \ldots n, \tag{2.4.87}$$

where $h^{(j)}(x)$ is the jth derivative of $h(x)$.

On the other hand, the equation (2.4.86) gives

$$h^{(j)}(0) = (-2\pi i)^j \int_{-\infty}^{\infty} y^j g(y)dy \tag{2.4.88}$$

Comparing (2.4.87) and (2.4.88), we have $\int_{-\infty}^{\infty} y^j g(y)dy = c_j, \quad j = 0, 1 \ldots n.$

The rest part of the Proposition follows from the Theorem 2.2.3. □

Chapter 3

Generalized solutions for a de-coupled system and a forced Burgers equation

3.1 Introduction

The forced Burgers equation

$$u_t + uu_x = \nu u_{xx} + f(x,t), \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.1}$$

has wide variety of applications in different fields of science (Xu et al., 2007). In this chapter, we consider the forced Burgers equation

$$u_t + uu_x = \nu u_{xx} + \frac{k}{(2\beta t + 1)^{3/2}}, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.2}$$

where $\nu > 0$, $\beta > 0$ and k is a non-zero constant, subject to the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1.3}$$

with the assumption that $u_0(x) = o(|x|)$ for large $|x|$.

Hopf (1950) studied the vanishing viscosity behavior of solutions to the viscous Burgers equation

$$\begin{aligned} u_t + uu_x &= \nu u_{xx}, \quad x \in \mathbb{R}, \quad t > 0, & (3.1.4)\\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, & (3.1.5) \end{aligned}$$

with the assumption that the initial data u_0 satisfies $\int_0^x u_0(y)dy = o(x^2)$ for large $|x|$ and he proved that the solution of (3.1.4)-(3.1.5) converges to the weak solution of the concerned inviscid Burgers equation as the viscosity $\nu \to 0$.

Joseph (1993) considered a system of conservation laws

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ v_t + (uv)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{3.1.6}$$

with the initial conditions

$$(u,v)^t(x,0) = \begin{cases} (u_L, v_L)^t & \text{if } x < 0, \\ (u_R, v_R)^t & \text{if } x > 0 \end{cases} \tag{3.1.7}$$

which is known as Riemann problem. He analyzed the solution of above problem by using vanishing viscosity method. For which he took an approximate solution $(u^\nu(x,t), v^\nu(x,t))$ of (3.1.6)-(3.1.7) which is defined by the Riemann problem

$$\begin{cases} u_t^\nu + \left(\frac{{u^\nu}^2}{2}\right)_x = \frac{1}{2}\nu u_{xx}^\nu, \quad x \in \mathbb{R}, \quad t > 0, \\ v_t^\nu + (uv)_x^\nu = \frac{1}{2}\nu v_{xx}^\nu, \quad x \in \mathbb{R}, \quad t > 0, \end{cases} \tag{3.1.8}$$

with the initial conditions

$$(u^\nu, v^\nu)^t(x,0) = \begin{cases} (u_L^\nu, v_L^\nu)^t & if\ x < 0, \\ (u_R^\nu, v_R^\nu)^t & if\ x > 0. \end{cases} \tag{3.1.9}$$

He proved that the solution so obtained for the above Riemann problem will give the solution of (3.1.6)-(3.1.7) in the sense of distribution as $\nu \to 0$. The explicit solution for (3.1.6)-(3.1.7) given by Joseph (1993) is

(i) $.u_L > u_R$

$$(u^0(x,t), v^0(x,t)) = \begin{cases} (u_L, v_L) & \text{if } \ x < st, \\ (\frac{1}{2}(u_L + u_R), \frac{1}{2}(u_L - u_R)(v_L + v_R)t\delta_{x=st}) & \text{if } \ x = st, \\ (u_R, v_R) & \text{if } \ x > st, \end{cases} \tag{3.1.10}$$

where $s = \frac{1}{2}(u_L + u_R)$ and $\delta_{x=st}$ is the usual δ- measure concentrated along the line $x = st$.

(ii) $u_L < u_R$

$$(u^0(x,t), v^0(x,t)) = \begin{cases} (u_L, v_L) & \text{if } x < u_L t, \\ (x/t, 0) & \text{if } u_L t < x < u_R t, \\ (u_R, v_R) & \text{if } x > u_R t, \end{cases} \tag{3.1.11}$$

(iii) $u_L = u_R = \overline{u}$

$$(u^0(x,t), v^0(x,t)) = \begin{cases} (\overline{u}, v_L) & \text{if } x < \overline{u}t, \\ (\overline{u}, v_R) & \text{if } x > \overline{u}t, \end{cases} \tag{3.1.12}$$

Ding and Ding (2003) showed that, for fixed (x,t), the solution of

$$u_t + uu_x = \nu u_{xx} + 4x, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.1.13}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{3.1.14}$$

converges to the weak solution of relevant inviscid forced Burgers equation as $\nu \to 0$, assuming the initial data satisfies

$$u_0(x) = o(x), \quad |x| \to \infty.$$

This chapter is organized as follows. In the Section 3.2, we consider a Riemann problem for de-coupled system and obtain the explicit solution. In Section 3.3, we find the solution to the initial value problem for forced Burgers equation (3.1.2)-(3.1.3). Section 3.4 shows that the solution obtained in the Section 3.3 converges to the relevant inviscid forced Burgers equation as $\nu \to 0$.

3.2 Riemann problem for de-coupled system

In this section we construct the solution of the following Riemann problem for decoupled system.

$$\begin{aligned} & u_t + uu_x = \tfrac{k}{(2\beta t+1)^{3/2}}, \quad x \in \mathbb{R}, \quad t > 0 \\ & \rho_t + (u\rho)_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \end{aligned} \tag{3.2.15}$$

with the initial value

$$(u,\rho)^t(x,0) = \begin{cases} (u_L,\rho_L)^t & if\ x<0, \\ (u_R,\rho_R)^t & if\ x>0. \end{cases} \tag{3.2.16}$$

Assuming that there exists a v such that

$$v_x = \rho, \qquad v_t = -u\rho.$$

Then the system (3.2.15) with (3.2.16) reduces to

$$\begin{cases} u_t + uu_x = \frac{k}{(2\beta t+1)^{3/2}}, \\ v_t + uv_x = 0, \\ u(x,0) = \begin{cases} u_L, & x<0, \\ u_R, & x>0, \end{cases} \\ v(x,0) = \int_0^x \rho(x,0)dx = \begin{cases} \rho_L x, & x<0, \\ \rho_R x, & x>0. \end{cases} \end{cases} \tag{3.2.17}$$

We now solve the Riemann problem for inviscid forced Burgers equation using the method of characteristics.

The characteristic equations are

$$\frac{dx}{dt} = u, \qquad \frac{du}{dt} = \frac{k}{(2\beta t+1)^{3/2}}. \tag{3.2.18}$$

Solving this system of ODEs, we obtain the characteristic curve originated at the point $(x_0,0)$ and is given by

$$x(t) = \frac{k}{\beta^2}\left(1-\sqrt{2\beta t+1}\right) + \left(u(x_0,0)+\frac{k}{\beta}\right)t + x_0, \tag{3.2.19}$$

and the solution along the curve is

$$u(x(t),t) = \frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right) + u(x_0,0). \tag{3.2.20}$$

Depending on the values of u_L and u_R, we have the following cases.

Case 1: $u_L > u_R$

In this case, we will have a shock wave originated from $(0,0)$ and satisfies

$$\begin{cases} \frac{dx}{dt} = \frac{1}{2}\left[\frac{2k}{\beta}(1-\frac{1}{\sqrt{2\beta t+1}}) + u_L + u_R\right], \\ x(0) = 0. \end{cases} \tag{3.2.21}$$

Then the equation of shock wave is

$$\begin{aligned} x(t) &= \frac{1}{2}\left[\frac{2k}{\beta^2}\Big(1-\sqrt{2\beta t+1}\Big)+\Big(\frac{2k}{\beta}+u_L+u_R\Big)t\right] \\ &=: g(t). \end{aligned} \tag{3.2.22}$$

Thus the solution for the Riemann problem for inviscid forced Burgers equation is

$$u(x,t)=\begin{cases} \frac{k}{\beta}\Big(1-\frac{1}{\sqrt{2\beta t+1}}\Big)+u_L & \text{if } x<g(t), \\ c & \text{if } x=g(t), \\ \frac{k}{\beta}\Big(1-\frac{1}{\sqrt{2\beta t+1}}\Big)+u_R & \text{if } x>g(t), \end{cases} \tag{3.2.23}$$

where c is any constant.

Using the solution $u(x,t)$ given in (3.2.23), we solve the Riemann problem for the second equation in de-coupled system (3.2.17).

If $x<g(t)$, then we have

$$x(t) = \frac{k}{\beta^2}(1-\sqrt{2\beta t+1})+\left(u_L+\frac{k}{\beta}\right)t+\xi_1, \tag{3.2.24}$$

$$u(x,t) = \frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_L, \tag{3.2.25}$$

where $x(0)=\xi_1<0$. Hence,

$$\begin{aligned} v(x,t) &= \rho_L\xi_1 \\ &= \rho_L\left(x-\frac{k}{\beta^2}(1-\sqrt{2\beta t+1})-\left(u_L+\frac{k}{\beta}\right)t\right) \end{aligned} \tag{3.2.26}$$

for the case $x<g(t)$.

If $x>g(t)$, then we have

$$x(t) = \frac{k}{\beta^2}(1-\sqrt{2\beta t+1})+\left(u_R+\frac{k}{\beta}\right)t+\xi_2, \tag{3.2.27}$$

$$u(x,t) = \frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_R, \tag{3.2.28}$$

where $x(0)=\xi_2>0$. Hence,

$$\begin{aligned} v(x,t) &= \rho_R\xi_2 \\ &= \rho_R\left(x-\frac{k}{\beta^2}(1-\sqrt{2\beta t+1})-\left(u_R+\frac{k}{\beta}\right)t\right) \end{aligned} \tag{3.2.29}$$

for the case $x > g(t)$.

Here we find the distribution derivative of $v(x,t)$.

Let

$$e_L(t) = \frac{k}{\beta^2}(\sqrt{2\beta t+1}-1) + \left(u_L + \frac{k}{\beta}\right)t,$$
$$e_R(t) = \frac{k}{\beta^2}(\sqrt{2\beta t+1}-1) + \left(u_R + \frac{k}{\beta}\right)t.$$

$$\begin{aligned}
\rho(x,t) &= T_{\frac{\partial v}{\partial x}}(\phi) \\
&= -T_v\left(\frac{\partial \phi}{\partial x}\right) \\
&= -\left[\int_0^\infty \int_{\mathbb{R}} v(x,t)\frac{\partial \phi}{\partial x} dxdt\right] \\
&= -\left[\int_0^\infty \int_{x<g(t)} \rho_L[x+e_L(t)]\frac{\partial \phi}{\partial x} dxdt + \int_0^\infty \int_{x>g(t)} \rho_R[x+e_R(t)]\frac{\partial \phi}{\partial x} dxdt\right] \\
&= \int_0^\infty \int_{-\infty}^{g(t)} \rho_L \phi(x,t)dx - \int_0^\infty \rho_L[g(t)+e_L(t)]\phi(g(t),t) \\
&\quad + \int_{g(t)}^\infty \rho_R\phi(x,t)dx + \int_0^\infty \rho_R[g(t)+e_R(t)]\phi(g(t),t)dt \\
&= \int_{t=0}^\infty \int_{-\infty}^{g(t)} \rho_L\phi(x,t)dxdt + \int_{t=0}^\infty \int_{g(t)}^\infty \rho_R\phi(x,t)dxdt \\
&\quad + \int_{t=0}^\infty \left[\rho_R[g(t)+e_R(t)] - \rho_L[g(t)+e_L(t)]\rho_R\right]\phi(g(t),t)dt,
\end{aligned}$$

for every $C_c^\infty(\mathbb{R}\times(0,\infty))$. Thus

$$\rho(x,t) = \begin{cases} \rho_L & \text{if } x < g(t), \\ \rho_R & \text{if } x > g(t), \\ \left(\rho_R[g(t)+e_R(t)] - \rho_L[g(t)+e_L(t)]\right)\delta_{x=g(t)} & \text{if } x = g(t). \end{cases} \tag{3.2.30}$$

Case 2: $u_L < u_R$

From (3.2.19), the characteristic curves originated from $(x_0, 0)$ for the Riemann problem for inviscid forced Burgers equation are

$$x(t) \;=\; \begin{cases} \frac{k}{\beta^2}\left(1-\sqrt{2\beta t+1}\right) + \left(u_L + \frac{k}{\beta}\right)t + x_0 & \text{if } x_0 < 0, \\ \frac{k}{\beta^2}\left(1-\sqrt{2\beta t+1}\right) + \left(u_R + \frac{k}{\beta}\right)t + x_0 & \text{if } x_0 > 0, \end{cases} \tag{3.2.31}$$

and the solution along these curves respectively is

$$u(x,t) = \begin{cases} \dfrac{k}{\beta}\Big(1 - \dfrac{1}{\sqrt{2\beta t+1}}\Big) + u_L, \\ \dfrac{k}{\beta}\Big(1 - \dfrac{1}{\sqrt{2\beta t+1}}\Big) + u_R. \end{cases} \tag{3.2.32}$$

Let

$$f_L(t) = \frac{k}{\beta^2}(1 - \sqrt{2\beta t+1}) + \Big(\frac{k}{\beta} + u_L\Big)t, \tag{3.2.33}$$

$$f_R(t) = \frac{k}{\beta^2}\left(1 - \sqrt{2\beta t+1}\right) + \Big(\frac{k}{\beta} + u_R\Big)t. \tag{3.2.34}$$

It is clear that the region $\{f_L(t) < x < f_R(t)\}$ is not covered by the characteristics. Therefore, the solution in this case is

$$u(x,t) = \begin{cases} \dfrac{k}{\beta}\Big(1 - \dfrac{1}{\sqrt{2\beta t+1}}\Big) + u_L & \text{if } x < f_L(t), \\ \dfrac{1}{t}\Big[x - \frac{k}{\beta^2}(1 - \sqrt{2\beta t+1})\Big] - \frac{k}{\beta\sqrt{2\beta t+1}} & \text{if } f_L(t) < x < f_R(t), \\ \dfrac{k}{\beta}\Big(1 - \dfrac{1}{\sqrt{2\beta t+1}}\Big) + u_R & \text{if } x > f_R(t) \end{cases} \tag{3.2.35}$$

and

$$\rho(x,t) = \begin{cases} \rho_L, & \text{if } x < f_L(t), \\ 0, & \text{if } f_L(t) < x < f_R(t), \\ \rho_R, & \text{if } x > f_R(t). \end{cases} \tag{3.2.36}$$

Now we prove that the solution so obtained is a weak solution of (3.2.15)-(3.2.16).

First we prove that u is the weak solution of the first equation in (3.2.15), i.e. to show that u satisfies the integral equation

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt &= -\int_{-\infty}^0 u_L\phi(x,0)dx - \int_0^\infty u_R\phi(x,0)dx \\ &\quad - \int_0^\infty \int_{-\infty}^\infty \frac{k}{(2\beta t+1)^{3/2}}\phi(x,t)dxdt. \end{aligned} \tag{3.2.37}$$

for every $\phi(x,t) \in C_c^\infty(\mathbb{R} \times (0,\infty)$.

Case 1: $u_L > u_R$

Consider

$$\begin{aligned}\int_0^\infty \int_{-\infty}^\infty (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt &= \int_0^\infty \int_{-\infty}^{g(t)} (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt + \int_0^\infty \int_{g(t)}^\infty (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt \\ &= I_1 + I_2.\end{aligned}$$

Then

$$\begin{aligned}I_1 &= \int_0^\infty \int_{-\infty}^{g(t)} \left\{ \left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_L\right]\phi_t + \frac{1}{2}\left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_L\right]^2 \phi_x \right\} dxdt \\
&= \int_0^\infty \left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_L\right]\left(\int_{-\infty}^{g(t)} \phi_t dx\right)dt \\
&\quad + \int_0^\infty \frac{1}{2}\left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_L\right]^2 \left(\int_{-\infty}^{g(t)} \phi_x dx\right) dt \\
&= \int_0^\infty \left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_L\right]\left(\frac{d}{dt}\int_{-\infty}^{g(t)} \phi(x,t)dx - \phi(g(t),t)g'(t)\right)dt \\
&\quad + \int_0^\infty \frac{1}{2}\left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_L\right]^2 \phi(g(t),t)dt \\
&= -\int_0^\infty \int_{-\infty}^{g(t)} \frac{k}{(2\beta t+1)^{3/2}}\phi(x,t)dxdt - u_L\int_{-\infty}^0 \phi(x,0)dx \\
&\quad - \int_0^\infty \left\{ \left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_L\right]g^{'}(t) - \frac{1}{2}\left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_L\right]^2 \right\}\phi(g(t),t)dt,\end{aligned} \tag{3.2.38}$$

and

$$\begin{aligned}I_2 &= \int_0^\infty \int_{g(t)}^\infty \left\{ \left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_R\right]\phi_t + \frac{1}{2}\left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_R\right]^2 \phi_x \right\} dxdt \\
&= \int_0^\infty \left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_R\right]\left(\int_{g(t)}^\infty \phi_t dx\right)dt \\
&\quad + \int_0^\infty \frac{1}{2}\left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_R\right]^2 \left(\int_{g(t)}^\infty \phi_x dx\right) dt \\
&= \int_0^\infty \left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_R\right]\left(\frac{d}{dt}\int_{g(t)}^\infty \phi(x,t)dx + \phi(g(t),t)g'(t)\right)dt \\
&\quad - \int_0^\infty \frac{1}{2}\left[\frac{k}{\beta}\left(1 - \frac{1}{\sqrt{2\beta t+1}}\right) + u_R\right]^2 \phi(g(t),t)dt\end{aligned}$$

$$= -\int_0^\infty \int_{g(t)}^\infty \frac{k}{(2\beta t+1)^{3/2}}\phi(x,t)dxdt - u_R\int_0^\infty \phi(x,0)dx$$
$$+\int_0^\infty \left\{\left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_R\right]g^{'}(t)-\frac{1}{2}\left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_R\right]^2\right\}\phi(g(t),t)dt. \tag{3.2.39}$$

Thus

$$\begin{aligned}\int_0^\infty\int_{-\infty}^\infty (u\phi_t+(\frac{u^2}{2})\phi_x)dxdt &= \int_0^\infty\int_{-\infty}^\infty \frac{k}{(2\beta t+1)^{3/2}}\phi(x,t)dxdt - u_L\int_{-\infty}^0 \phi(x,0)dx\\
&- u_R\int_0^\infty \phi(x,0)dx + \int_0^\infty \Bigg(\left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_R\right]\\
&- \left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_L\right]\Bigg)g^{'}(t)\phi(g(t),t)dt\\
&+\int_0^\infty \frac{1}{2}\Bigg[\left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_L\right]^2\\
&- \left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_R\right]^2\Bigg]\phi(g(t),t)dt\\
&= \int_0^\infty\int_{-\infty}^\infty \frac{k}{(2\beta t+1)^{3/2}}\phi(x,t)dxdt - u_L\int_{-\infty}^0 \phi(x,0)dx\\
&- u_R\int_0^\infty \phi(x,0)dx.\end{aligned} \tag{3.2.40}$$

Hence, $u(x,t)$ given in (3.2.23) is the weak solution of inviscid forced Burgers equation (3.2.15).

Case 2: $u_L < u_R$

We prove that $u(x,t)$ given in (3.2.35) is a weak solution of (3.2.15)-(3.2.16). For that we show that $u(x,t)$ satisfies (3.2.37).

Consider

$$\begin{aligned}\int_0^\infty\int_{-\infty}^\infty (u\phi_t+\left(\frac{u^2}{2}\right)\phi_x)dxdt &= \int_0^\infty\int_{-\infty}^{f_L(t)} (u\phi_t+\left(\frac{u^2}{2}\right)\phi_x)dxdt+\\
&\int_0^\infty\int_{f_L(t)}^{f_R(t)} (u\phi_t+\left(\frac{u^2}{2}\right)\phi_x)dxdt\\
&+\int_0^\infty\int_{f_R(t)}^{\infty} (u\phi_t+\left(\frac{u^2}{2}\right)\phi_x)dxdt\\
&=: I_1+I_2+I_3.\end{aligned}$$

Then

$$\begin{aligned}
I_1 &= \int_0^\infty \int_{-\infty}^{f_L(t)} \left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_L\right]\phi_t dxdt \\
&+ \int_0^\infty \int_{-\infty}^{f_L(t)} \left(\frac{1}{2}\left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_L\right]^2\right)\phi_x dxdt \\
&= -\int_{-\infty}^{0} u_L\phi(x,0)dx - \int_0^\infty \int_{-\infty}^{f_L(t)} \frac{k}{(2\beta t+1)^{3/2}}\phi(x,t)dxdt \\
&+ \frac{1}{2}\int_0^\infty \left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_L\right]^2 \phi(f_L(t),t)dt.
\end{aligned}$$

Since I_2 has singularity at $t=0$, we consider $I_2 = \lim_{\epsilon\to 0} I_{2,\epsilon}$.

$$\begin{aligned}
I_{2,\epsilon} &= \int_\epsilon^\infty \int_{f_L(t)}^{f_R(t)} (u\phi_t + \left(\frac{u^2}{2}\right)\phi_x)dxdt. \\
&= \int_\epsilon^\infty \int_{f_L(t)}^{f_R(t)} \left(\frac{1}{t}\left[x-\frac{k}{\beta^2}(1-\sqrt{2\beta t+1})\right]-\frac{k}{\beta\sqrt{2\beta t+1}}\right)\phi_t dxdt \\
&+ \frac{1}{2}\int_\epsilon^\infty \int_{f_L(t)}^{f_R(t)} \left(\frac{1}{t}\left[x-\frac{k}{\beta^2}(1-\sqrt{2\beta t+1})\right]-\frac{k}{\beta\sqrt{2\beta t+1}}\right)^2 \phi_x dxdt \\
&= -\frac{1}{2}\int_\epsilon^\infty \left(u_R+\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)\right)^2 \phi(f_R(t),t)dt \\
&+ \frac{1}{2}\int_\epsilon^\infty \left(u_L+\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)\right)^2 \phi(f_L(t),t)dt \\
&- \int_\epsilon^\infty \int_{f_L(t)}^{f_R(t)} \frac{k}{(2\beta t+1)^{3/2}\phi(x,t)}dxdt.
\end{aligned}$$

$$\begin{aligned}
I_2 &= \lim_{\epsilon\to 0} I_{2,\epsilon} \\
&= \frac{1}{2}\int_0^\infty \left(u_R+\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)\right)^2 \phi(f_R(t),t)dt \\
&- \frac{1}{2}\int_0^\infty \left(u_L+\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)\right)^2 \phi(f_L(t),t)dt \\
&- \int_0^\infty \int_{f_L(t)}^{f_R(t)} \frac{k}{(2\beta t+1)^{\frac{3}{2}}}dxdt.
\end{aligned}$$

Also,

$$
\begin{aligned}
I_3 &= \int_0^\infty \int_{f_R(t)}^\infty \left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_R\right]\phi_t dxdt \\
&\quad + \int_0^\infty \int_{f_R(t)}^\infty \left(\frac{1}{2}\left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_R\right]^2\right)\phi_x dxdt \\
&= -\int_{-\infty}^0 u_R\phi(x,0)dx - \int_0^\infty \int_{f_R(t)}^\infty \frac{k}{(2\beta t+1)^{3/2}}\phi(x,t)dxdt \\
&\quad - \frac{1}{2}\int_0^\infty \left[\frac{k}{\beta}\left(1-\frac{1}{\sqrt{2\beta t+1}}\right)+u_R\right]^2 \phi(f_R(t),t)dt.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\int_0^\infty \int_{-\infty}^\infty (u\phi_t + (\frac{u^2}{2})\phi_x)dxdt &= I_1 + I_2 + I_3 \\
&= -\int_{-\infty}^0 u_L\phi(x,0)dx - \int_0^\infty u_R\phi(x,0)dx \\
&\quad - \int_0^\infty \int_{-\infty}^\infty \frac{k}{(2\beta t+1)^{3/2}}dxdt.
\end{aligned}
$$

Similarly, one may prove that, the solution $\rho(x,t)$ given in (3.2.30) and (3.2.36) is a weak solution of (3.2.15), i.e. to prove the following,

$$
\begin{aligned}
&\int_0^\infty \int_{-\infty}^{f_L(t)} \Big[\rho\phi_t + (u\rho)\phi_x\Big]dxdt + \int_0^\infty \int_{f_L(t)}^{f_R(t)} \Big[\rho\phi_t + (u\rho)\phi_x\Big]dxdt \\
&+ \int_0^\infty \int_{f_R(t)}^\infty \Big[\rho\phi_t + (u\rho)\phi_x\Big]dxdt = -\int_{-\infty}^0 \rho_L\phi(x,0)dx - \int_0^\infty \rho_R\phi(x,0)dx.
\end{aligned}
\tag{3.2.41}
$$

for every $\phi(x,t) \in C_c^\infty(\mathbb{R} \times (0,\infty))$

3.3 Solution of the forced Burgers equation

In this section, we show that the solution of the forced Burgers equation (3.1.2) converges to the generalized solution of relevant inviscid forced Burgers equation

$$u_t + uu_x = \frac{k}{(2\beta t+1)^{3/2}}, \quad x \in \mathbb{R}, \quad t > 0 \tag{3.3.42}$$

as $\nu \to 0$.

From Satyanarayana et al. (2017), we can see that under the Cole-Hopf like transformation

$$u(x,t) = -\frac{2\nu}{\sqrt{2\beta t+1}}\frac{\psi_\eta(\eta,t)}{\psi(\eta,t)}, \tag{3.3.43}$$

where $\eta = \frac{x}{\sqrt{2\beta t+1}}$, the problem (3.1.2)-(3.1.3) reduces to

$$\begin{aligned} (2\beta t+1)\psi_t &= \beta\eta\psi_\eta + \nu\psi_{\eta\eta} - \frac{k}{2\nu}\eta\psi, && (3.3.44)\\ \psi(\eta,0) &= \exp\left\{-\frac{1}{2\nu}\int_0^\eta u_0(s)ds\right\} \equiv \psi_0(\eta). && (3.3.45)\end{aligned}$$

Scaling the variables

$$\eta = \sqrt{\nu}\eta', \quad \psi(\eta,t) = \psi(\sqrt{\nu}\eta',t) = \varphi(\eta',t), \tag{3.3.46}$$

in (3.3.44)-(3.3.45), we find that

$$\begin{aligned} (2\beta t+1)\varphi_t &= \beta\eta'\varphi_{\eta'} + \varphi_{\eta'\eta'} - \frac{k}{2\sqrt{\nu}}\eta'\varphi, && (3.3.47)\\ \varphi(\eta',0) &= \psi_0(\sqrt{\nu}\eta') = \varphi_0(\eta'). && (3.3.48)\end{aligned}$$

A solution for (3.3.47)-(3.3.48) was obtained by Satyanarayana et al. (2017) by assuming that the initial data $u_0 \in L^1(\mathbb{R})$ and u_0 is continuous over $\mathbb{R}$, which is

$$\varphi(\eta',t) = (2\beta t+1)^{\frac{k^2}{8\nu\beta^3}}\int_{-\infty}^{\infty} K(\eta',y',t)\varphi_0(y')dy', \tag{3.3.49}$$

where,

$$K(\eta',y',t) = \frac{1}{2\sqrt{\pi t}}\exp\left\{\frac{k}{2\sqrt{\nu}\beta}(\eta'-y') - \frac{1}{4t}\left[\sqrt{2\beta t+1}\left(\eta' + \frac{k}{\sqrt{\nu}\beta^2}\right) - \left(y' + \frac{k}{\sqrt{\nu}\beta^2}\right)\right]^2\right\}. \tag{3.3.50}$$

Scaling back to η and ψ, we get a solution of (3.3.44)-(3.3.45), which is

$$\begin{aligned}\psi(\eta,t) &= \frac{(2\beta t+1)^{\frac{k^2}{8\nu\beta^3}}}{2\sqrt{\pi t\nu}} \\ &\times \int_{-\infty}^{\infty} \exp\left\{\frac{k}{2\beta\nu}(\eta-y) - \frac{1}{4\nu t}\left[\sqrt{2\beta t+1}\left(\eta+\frac{k}{\beta^2}\right) - \left(y+\frac{k}{\beta^2}\right)\right]^2\right\}\psi_0(y)dy. \end{aligned} \tag{3.3.51}$$

Let $t' = \dfrac{1}{\sqrt{2\beta t+1}}$, then (3.3.51) becomes

$$\begin{aligned}\psi(\eta,t) &= \frac{\sqrt{\beta}}{\sqrt{2\pi\nu(1-t'^2)}} t'^{\left(1-\frac{k^2}{4\nu\beta^3}\right)} \\ &\times \int_{-\infty}^{\infty} \exp\left\{\frac{k}{2\beta\nu}(\eta-y) - \frac{\beta t'^2}{2\nu(1-t'^2)}\left[\frac{1}{t'}\left(\eta+\frac{k}{\beta^2}\right) - \left(y+\frac{k}{\beta^2}\right)\right]^2\right\}\psi_0(y)dy. \end{aligned} \tag{3.3.52}$$

Set

$$\begin{aligned}F(\eta,y,t') &= -\left\{\frac{k}{\beta}(\eta-y) - \frac{\beta t'^2}{(1-t'^2)}\left[\frac{1}{t'}\left(\eta+\frac{k}{\beta^2}\right) - \left(y+\frac{k}{\beta^2}\right)\right]^2\right\} + \int_0^y u_0(s)ds \\ &= \frac{\beta}{1-t'^2}\eta^2 + \frac{\beta t'^2}{1-t'^2}y^2 + \frac{k}{\beta}\left(\frac{1-t'}{1+t'}\right)\left[\eta - y + \frac{k}{\beta^2}\right] - \frac{2\beta t'}{1-t'^2}\eta y \\ &\quad + \int_0^y u_0(s)ds. \end{aligned} \tag{3.3.53}$$

Then, from (3.3.52) we see that

$$\psi(\eta,t) = \frac{\sqrt{\beta}}{\sqrt{2\pi\nu(1-t'^2)}} t'^{\left(1-\frac{k^2}{4\nu\beta^3}\right)} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\nu}F(\eta,y,t')\right\}dy. \tag{3.3.54}$$

One can find

$$F_\eta(\eta,y,t') = \frac{2\beta(\eta-t'y)}{1-t'^2} + \frac{k}{\beta}\left(\frac{1-t'^2}{1+t'^2}\right).$$

Therefore, the solution of the Cauchy problem (3.1.2)-(3.1.3) is

$$\begin{aligned}u(x,t) &= -\frac{2\nu}{\sqrt{2\beta t+1}}\frac{\psi_\eta(\eta,t)}{\psi(\eta,t)} = -2\nu t'\frac{\psi_\eta(\eta,t)}{\psi(\eta,t)} \\ &= \frac{t'\int_{-\infty}^{\infty} F_\eta(\eta,y,t')\exp\left\{-\frac{1}{2\nu}F(\eta,y,t')\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\nu}F(\eta,y,t')\right\}dy} \\ &= \frac{\int_{-\infty}^{\infty}\left[\frac{2\beta t'(\eta-t'y)}{1-t'^2} + \frac{kt'}{\beta}\left(\frac{1-t'}{1+t'}\right)\right]\exp\left\{-\frac{1}{2\nu}F(\eta,y,t')\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\nu}F(\eta,y,t')\right\}dy} \\ &=: \hat{u}(\eta,t'), \end{aligned} \tag{3.3.55}$$

where F is given in (3.3.53), $\eta = \dfrac{x}{\sqrt{2\beta t+1}}$ and $t' = \dfrac{1}{\sqrt{2\beta t+1}}$.

3.4 Vanishing viscosity behavior

In this section, denoting the solution in (3.3.55) by $u(x,t,\nu)$, we show that $u(x,t,\nu)$ converges to the generalized solution of inviscid forced Burgers equation (3.3.42) as $\nu \to 0$. In order to prove this, we use the original ideas from Hopf (1950).

Since

$$u_0(y) = o(y) \quad as \quad |y| \to \infty,$$

it can be seen that

$$\frac{F(\eta,y,t')}{y^2} \to \frac{\beta t'^2}{1-t'^2} \quad \text{as} \quad |y| \to \infty, \quad \text{for every fixed} \quad \eta, t'. \tag{3.4.56}$$

Then there exists a natural number N such that

$$\frac{1}{2}\left(\frac{\beta t'^2}{1-t'^2}\right) < F(\eta,y,t') < \frac{3}{2}\left(\frac{\beta t'^2}{1-t'^2}\right), \quad \forall \quad |y| > N. \tag{3.4.57}$$

Since F is continuous in y for all $|y| \leq N$ and by the virtue of (3.4.57), minimum of $F(\eta,y,t')$ exists for all y. Say,

$$\min_y F(\eta,y,t') =: m(\eta,t').$$

Let $y_*(\eta,t')$ and $y^*(\eta,t')$ be the smallest and largest value of y for which $F(\eta,y,t')$ attains its minimum.

Lemma 3.4.1. *We have the following*

(i). $y^*(\eta,t') \leqslant y_*(\eta',t') \quad$ *if* $\quad \eta \leqslant \eta'$;

(ii). $y_*(\eta-0,t') = y_*(\eta,t'),\ y^*(\eta+0,t') = y^*(\eta,t')$;

(iii). $y_*(+\infty,t') = +\infty,\ y^*(-\infty,t') = -\infty$.

Proof. Setting

$$G(\eta,y,t') := F(\eta,y,t') - \frac{\beta}{1-t'^2}\eta^2 - \frac{k}{\beta}\left(\frac{1-t'}{1+t'}\right)\left[\eta + \frac{k}{\beta^2}\right], \tag{3.4.58}$$

it can be seen that $G(\eta,y,t')$ also attains its minimum at $y_*(\eta,t')$ and $y^*(\eta,t')$. So

$$G(\eta,y,t') - G(\eta,y^*,t') \begin{cases} \geqslant 0 & \text{if} \quad y < y^*, \\ > 0 & \text{if} \quad y > y^*, \end{cases} \tag{3.4.59}$$

where $y^* = y^*(\eta, t')$. In view of (3.4.58) and (3.3.53), we have

$$G(\eta, y, t') = G(0, y, t') - \frac{2\beta t'}{1-t'^2}\eta y \tag{3.4.60}$$

and

$$G(\eta+a, y, t') - G(\eta+a, y^*, t') = G(\eta, y, t') - G(\eta, y^*, t') - \frac{2\beta t' a}{1-t'^2}(y-y^*). \tag{3.4.61}$$

Then from (3.4.59),(3.4.60), we find that

$$G(\eta+a, y, t') - G(\eta+a, y^*, t') > 0 \quad \text{for} \quad y < y^* \ \text{and} \ a > 0. \tag{3.4.62}$$

Therefore, $G(\eta+a, y, t')$ attains its minimum for $y \geqslant y^*$. Hence (i) is proved.

We will prove $y^*(\eta+0, t') = y^*(\eta, t')$, i.e, we will prove that for a given $\epsilon > 0$ there exists a real number a such that

$$|y^*(\eta+h, t') - y^*(\eta, t')| < \epsilon \quad \forall \quad h \in (0, a). \tag{3.4.63}$$

Let $\epsilon > 0$ be given. Set

$$g_{\eta,t'}(y) := \frac{G(\eta, y, t') - G(\eta, y^*, t')}{y - y^*}. \tag{3.4.64}$$

Then in view of (3.4.59), we have $g_{\eta,t'}(y) > 0$ for $y > y^*$.

From (3.4.56), we can say that there exists a natural number N such that

$$\frac{1}{2}\left(\frac{\beta t'^2}{1-t'^2}\right)y^2 < F(\eta, y, t') \qquad \forall \ |y| > N.$$

The above inequality leads to

$$\frac{1}{2}\left(\frac{\beta t'^2}{1-t'^2}\right)\frac{y^2}{y-y^*} - \frac{F(\eta, y*, t')}{y-y^*} < \frac{F(\eta, y, t') - F(\eta, y*, t')}{y-y^*} = g_{\eta,t'}(y).$$

Thus, we find that $g_{\eta,t'}(y) \to \infty$ as $y \to \infty$. Then, corresponding to $M > 0$, there exists a $N > 0$, such that

$$g_{\eta,t'}(y) > M \quad \forall \ y > N. \tag{3.4.65}$$

On the other hand, from (3.4.59), $g_{\eta,t'}(y) > 0$ and is continuous in $[y^* + \epsilon, N]$, so it attains a positive minimum say $d > 0$ in $[y^* + \epsilon, N]$ in the case that $y^* + \epsilon < N$, the other case is trivial.

Let $\alpha = \min\{d, M\}$, then

$$g_{\eta,t'}(y) \geq \alpha > 0, \quad \forall \; y > y^* + \epsilon. \tag{3.4.66}$$

In view of (3.4.61) and (3.4.66), we can choose sufficiently small a, such that

$$\begin{aligned} \frac{G(\eta+h, y, t') - G(\eta+h, y^*, t')}{y - y^*} &= g_{\eta,t'}(y) - \frac{2\beta t' h}{1 - t'^2} \\ &> \alpha - \frac{2\beta t' a}{1 - t'^2}, \\ &> 0, \end{aligned} \tag{3.4.67}$$

for all $h \in (0, a)$ and $y > y^* + \epsilon$. Hence, from (3.4.62) and (3.4.67) it is clear that

$$G(\eta+h, y, t') - G(\eta+h, y^*, t') > 0, \tag{3.4.68}$$

holds when $y < y^*$ or $y > y^* + \epsilon$. This means,

$$y^* \leqslant y^*(\eta + h, t') \leqslant y^* + \epsilon \quad \forall \; h \in (0, a).$$

Therefore, (3.4.63) follows. Similarly, we can prove $y_*(\eta - 0, t') = y_*(\eta, t')$. Hence, (ii) is proved.

We will prove $y_*(+\infty, t') = +\infty$, i.e., we will prove that for every real number $I > 0$, there exists $\eta_0 > 0$ such that

$$y_*(\eta, t') > I \quad \forall \; \eta > \eta_0. \tag{3.4.69}$$

Let the minimum of $G(0, y, t')$ be m for fixed t'. Take a real number A such that

$$A > y^*(0, t'). \tag{3.4.70}$$

Then,

$$\begin{aligned} 0 &< \frac{1 - t'^2}{2\beta t'}[G(0, A+1, t') - n] \\ &=: \eta_0 \end{aligned}$$

And in view of (3.4.60), we have

$$G(\eta_0, y, t') - n + \frac{2\beta\eta_0 t'}{1 - t'^2}A = G(0, y, t') - n - \frac{2\beta\eta_0 t'}{1 - t'^2}(y - A). \tag{3.4.71}$$

Hence, $G(\eta_0, y, t')$ and left hand side of (3.4.71) attain their minimum at same points. Clearly,

$$G(\eta_0, y, t') - n + \frac{2\beta\eta_0 t'}{1-t'^2} A \begin{cases} > 0 & \text{if} \quad y < A, \\ = 0 & \text{if} \quad y = A + 1. \end{cases} \tag{3.4.72}$$

Thus, the function $G(\eta_0, y, t')$ attains its minimum only for $y \geqslant A$. That is $y_*(\eta_0, t') \geqslant A$. Hence, in view of (i), we get

$$y_*(\eta, t') \geqslant A \quad \forall \quad \eta > \eta_0. \tag{3.4.73}$$

Case(a): If $I \leqslant A$, then (3.4.69) follows from (3.4.73).
Case(b): If $I > A$, then (3.4.69) follows by replacing A with I in (3.4.70) and proceeding the same upto (3.4.73).
Similarly we can prove $y^*(-\infty, t') = -\infty$. Hence, (iii) is proved. □

Lemma 3.4.2. *The function*

$$m(\eta, t') = F(\eta, y_*(\eta, t'), t') = F(\eta, y^*(\eta, t'), t')$$

is continuous in the region $\{(\eta, t') : \eta \in \mathbb{R},\ 0 < t' < 1\}$.

Proof. Take an arbitrary point (η_0, t'_0) in the region $\{(\eta, t') : \eta \in \mathbb{R},\ 0 < t' < 1\}$. We will show that $m(\eta, t')$ is continuous at (η_0, t'_0).
Denote

$$y_* = y_*(\eta_0, t'_0),\ y^* = y^*(\eta_0, t'_0) \text{ and } m' = F(\eta_0, y_*, t'_0) = F(\eta_0, y^*, t'_0).$$

Since m' is the minimum of $F(\eta_0, y, t'_0)$, there exists a constant $p > 0$ such that

$$F(\eta_0, y, t'_0) > m' + p \ \text{ whenever } \ y < y_* - 1 \quad \text{and} \quad y > y^* + 1. \tag{3.4.74}$$

In view of (3.3.53), $F(\eta, y, t') \to \infty$ uniformly as $|y| \to \infty$ in a neighborhood of (η_0, t'_0). Hence, in view of (3.4.74), there exists $0 < q < t'_0$, such that

$$F(\eta, y, t') > m' + \frac{p}{2}, \ \text{ whenever } \ y < y_* - 1,\ \ y > y^* + 1 \ \text{ and } \ |\eta - \eta_0| + |t' - t'_0| \leqslant q. \tag{3.4.75}$$

Besides, from the fact that $F(\eta, y, t')$ is continuous at the point (η_0, y_*, t'_0), we see that there exists $r > 0,\ r < q$, such that

$$F(\eta, y, t') < m' + \frac{p}{2}, \quad \text{whenever} \quad |\eta - \eta_0| + |t' - t'_0| \leqslant r. \tag{3.4.76}$$

In view of (3.4.75) and (3.4.76), we conclude that, for all (η, t') satisfying $|\eta - \eta_0| + |t' - t'_0| \leqslant r$, $F(\eta, y, t')$ attains its minimum in

$$y_* - 1 \leqslant y \leqslant y^* + 1.$$

Thus, there exists y' in $[y_* - 1, y^* + 1]$ such that $F(\eta, y, t')$ attains its minimum for all $(\eta,\ t')$ satisfying $|\eta - \eta_0| + |t' - t'_0| \leqslant r$. Then uniform continuity of $F(\eta, y, t')$ over the region

$$\{(\eta, y, t') : y_* - 1 \leqslant y \leqslant y^* + 1 \quad \text{and} \quad |\eta - \eta_0| + |t' - t'_0| \leqslant r\}$$

will imply the existence of $0 < \delta_1 \leqslant r$ such that

$$F(\eta_0, y', t'_0) - F(\eta, y', t') < \epsilon \quad \text{whenever} \quad |\eta - \eta_0| + |t' - t'_0| < \delta_1.$$

Therefore,

$$\begin{aligned} m(\eta_0, t'_0) - m(\eta, t') &= \min_y F(\eta_0, y, t'_0) - \min_y F(\eta, y, t') \\ &\leqslant F(\eta_0, y', t'_0) - F(\eta, y', t') < \epsilon, \end{aligned} \tag{3.4.77}$$

whenever $|\eta - \eta_0| + |t' - t'_0| < \delta_1$.

Consider,

$$\begin{aligned} m(\eta, t') - m(\eta_0, t'_0) &= \min_y F(\eta, y, t') - F(\eta_0, y_*, t'_0) \\ &\leqslant F(\eta, y_*, t') - F(\eta_0, y_*, t'_0). \end{aligned}$$

Since $F(\eta, y, t')$ is continuous at (η_0, y_*, t'_0), we get

$$m(\eta, t') - m(\eta_0, t'_0) < \epsilon \quad \text{whenever} \quad |\eta - \eta_0| + |t' - t'_0| < \delta_2. \tag{3.4.78}$$

Thus, in view of (3.4.77) and (3.4.78), choosing δ to be the minimum of δ_1 and δ_2 we infer that $m(\eta, t')$ is continuous at (η_0, t'_0). $\square$

Lemma 3.4.3. *The functions $y_*(\eta, t')$ and $y^*(\eta, t')$ are lower and upper semi continuous respectively. And at the point (η, t') where $y_*(\eta, t') = y^*(\eta, t')$, they are continuous.*

Proof. Let (η_0, t'_0) be a fixed point in $t > 0$, $t' = \frac{1}{\sqrt{2\beta t+1}}$ and (η_n, t'_n) be a sequence of points approaching to (η_0, t'_0). And also let U be an open and bounded set that contains (η_0, t'_0) and (η_n, t'_n) $\forall\ n = 1, 2, 3, \cdots$. Firstly, we prove that the sequence $\{y_*(\eta_n, t'_n)\}$ is bounded, for which let us assume contrarily that $\{y_*(\eta_n, t'_n)\}$ is unbounded. The fact that $m(\eta, t')$ is continuous proved in Lemma 3.4.2 will imply that

$$\sup_{\overline{U}} m(\eta, t') =: M$$

exists. Due to (3.4.56), corresponding to this $M > 0$ there exists a $N > 0$ such that

$$F(\eta, y, t') > M, \quad \text{whenever} \quad |y| > N, \tag{3.4.79}$$

for all $(\eta, t') \in U$. Since $\{y_*(\eta_n, t'_n)\}$ is unbounded, so corresponding to this $N > 0$ there exists a point (η_i, t'_i) such that $|y_*(\eta_i, t'_i)| > N$. From (3.4.79), we get $F(\eta_i, y_*(\eta_i, t'_i), t'_i) > M$, which is a contradiction, since M is supremum of $m(\eta, t')$. Thus $\{y_*(\eta_n, t'_n)\}$ is bounded.
Let

$$\underline{\lim}_{(\eta_n, t'_n)\to(\eta_0, t'_0)} y_*(\eta_n, t'_n) =: \bar{y}. \tag{3.4.80}$$

Then the sequence $\{y_*(\eta_n, t'_n)\}$ has a subsequence $\{y_*(\eta_{n'}, t'_{n'})\}$, such that

$$\lim_{(\eta_{n'}, t'_{n'})\to(\eta_0, t'_0)} y_*(\eta_{n'}, t'_{n'}) = \bar{y}. \tag{3.4.81}$$

Though we use the general notation of limit in (3.4.81), our meaning here is to say that (η_n, t'_n) converges to (η_0, t'_0) along the specific curve.
If y is arbitrary, then from

$$F(\eta_{n'}, y, t'_{n'}) \geqslant F(\eta_{n'}, y_*(\eta_{n'}, t'_{n'}), t'_{n'}),$$

we get

$$F(\eta_0, y, t'_0) \geqslant F(\eta_0, \bar{y}, t'_0),$$

in view of (3.4.81), by letting $(\eta_{n'}, t'_{n'}) \to (\eta_0, t'_0)$. Therefore, it is clear that $\bar{y}$ is also a point at which $F(\eta_0, y, t'_0)$ attain its minimum. So

$$y_*(\eta_0, t'_0) \leqslant \bar{y} = \underline{\lim}_{(\eta_n, t'_n)\to(\eta_0, t'_0)} y_*(\eta_n, t'_n). \tag{3.4.82}$$

Thus, $y_*(\eta, t')$ is lower semi continuous at (η_0, t'_0). Similarly, we can show that $y^*(\eta, t')$ is upper semi continuous. It is obvious that, if $y_*(\eta_0, t'_0) = y^*(\eta_0, t'_0)$, then

$$\begin{aligned}\lim_{(\eta,t')\to(\eta_0,t'_0)} y_*(\eta, t') &= y_*(\eta_0, t'_0),\\ \lim_{(\eta,t')\to(\eta_0,t'_0)} y^*(\eta, t') &= y^*(\eta_0, t'_0).\end{aligned}$$

So $y_*(\eta, t')$ and $y^*(\eta, t')$ are continuous at (η_0, t'_0). $\square$

Theorem 3.4.4. *Suppose $u(x, t, \nu)$ be a solution of (3.1.2) subject to (3.1.3) with $u_0(x) = o(x)$ as $|x| \to \infty$. Then for $x, t > 0$ we have,*

$$\overline{\lim_{(\alpha,\tau,\nu)\to(\eta,t',0)}} \hat{u}(\alpha, \tau, \nu) \leqslant \frac{2\beta t'[\eta - t'y_*(\eta, t')]}{1 - t'^2} + \frac{kt'}{\beta}\left(\frac{1 - t'}{1 + t'}\right), \tag{3.4.83}$$

$$\frac{2\beta t'[\eta - t'y^*(\eta, t')]}{1 - t'^2} + \frac{kt'}{\beta}\left(\frac{1 - t'}{1 + t'}\right) \leqslant \underline{\lim_{(\alpha,\tau,\nu)\to(\eta,t',0)}} \hat{u}(\alpha, \tau, \nu). \tag{3.4.84}$$

Proof. We will prove (3.4.83). We know from (3.3.55) that

$$\hat{u}(\eta, t') = \frac{\int_{-\infty}^{\infty}\left[\frac{2\beta t'(\eta - t'y)}{1-t'^2} + \frac{kt'}{\beta}\left(\frac{1-t'}{1+t'}\right)\right]\exp\left\{-\frac{1}{2\nu}F(\eta, y, t')\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2\nu}F(\eta, y, t')\right\}dy},$$

where $t' = \frac{1}{\sqrt{2\beta t+1}}$, $\eta = \frac{x}{\sqrt{2\beta t+1}}$ and F is as given in (3.3.53). Now we have

$$\hat{u}(\alpha, \tau, \nu) = \frac{\int_{-\infty}^{\infty}\left[\frac{2\beta\tau'(\xi - \tau'y)}{1-\tau'^2} + \frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right)\right]\exp\left\{-\frac{1}{\nu}P(\xi, y, \tau')\right\}dy}{\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{\nu}P(\xi, y, \tau')\right\}dy}, \tag{3.4.85}$$

where $\tau' = \frac{1}{\sqrt{2\beta\tau+1}}$, $\xi = \frac{\alpha}{\sqrt{2\beta\tau+1}}$ and

$$P(\xi, y, \tau') = \frac{1}{2}[F(\xi, y, \tau') - m(\xi, \tau')] \tag{3.4.86}$$

with

$$m(\xi, \tau') = F(\xi, y_*(\xi, \tau'), \tau') = F(\xi, y^*(\xi, \tau'), \tau').$$

Now, we notice that

$$P(\xi, y, \tau') = \begin{cases} > 0 & \text{if } \ y < y_*(\xi, \tau') \ \text{ or } \ y > y^*(\xi, \tau'),\\ = 0 & \text{if } \ y = y_*(\xi, \tau') \ \text{ or } \ y = y^*(\xi, \tau')\end{cases} \tag{3.4.87}$$

and from Lemma 3.4.2, $P(\xi, y, \tau')$ is continuous in ξ, y, τ'. In view of (3.3.53), we see that

$$\lim_{|y|\to\infty} \frac{P(\xi, y, \tau')}{y^2} = \frac{\beta\tau'^2}{2(1 - \tau'^2)}, \tag{3.4.88}$$

holds uniformly with respect to ξ, τ' on every closed set.

We choose an arbitrary point (η, t') and then let

$$Y_* = y_*(\eta, t'), \quad Y^* = y^*(\eta, t').$$

For any small real number $\epsilon > 0$, if a and b are chosen sufficiently small, then we find that

$$\begin{aligned} l &:= \frac{2\beta t'(\eta - t'Y^*)}{1 - t'^2} + \frac{kt'}{\beta}\left(\frac{1-t'}{1+t'}\right) - \epsilon \\ &< \frac{2\beta\tau'(\xi - \tau' y)}{1-\tau'^2} + \frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right) \\ &< \frac{2\beta t'(\eta - t'Y_*)}{1-t'^2} + \frac{kt'}{\beta}\left(\frac{1-t'}{1+t'}\right) + \epsilon =: L. \end{aligned} \tag{3.4.89}$$

holds when

$$|\xi - \eta| + |\tau' - t'| < a, \text{ and } Y_* - 2b < y < Y^* + 2b.$$

From Lemma 3.4.3, since $y_*(\xi, \tau')$ is lower semi continuous at (η, t'), corresponding to this $b > 0$ there exists $a_1 > 0$ such that

$$Y_* - b < y_*(\xi, \tau'),$$

whenever $|\xi - \eta| + |\tau' - t'| < a_1$. At the same time, since $y^*(\xi, \tau')$ is upper semi continuous at (η, t'), corresponding to the same $b > 0$ there exists $a_2 > 0$ such that

$$y_*(\xi, \tau') < Y_* + b,$$

when $|\xi - \eta| + |\tau' - t'| < a_2$.

Therefore, if $a = \min\{a_1,\ a_2\}$, then

$$Y_* - b < y_*(\xi, \tau') \leqslant y^*(\xi, \tau') < Y^* + b, \tag{3.4.90}$$

when

$$|\xi - \eta| + |\tau' - t'| < a. \tag{3.4.91}$$

Now, in view of (3.4.89) and (3.4.90), the numerator in right hand side of (3.4.85) is less than

$$\begin{aligned} L\int_{-\infty}^{\infty} \exp\left\{-\frac{P}{\nu}\right\} dy + \int_{-\infty}^{Y_*-2b} \left[\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2} + \frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right) - L\right] \exp\left\{-\frac{P}{\nu}\right\} dy \\ + \int_{Y^*+2b}^{\infty} \left[\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2} + \frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right) - L\right] \exp\left\{-\frac{P}{\nu}\right\} dy, \end{aligned} \tag{3.4.92}$$

whenever $\xi,\ \tau'$ satisfy (3.4.91), where P is as given in (3.4.86).

Claim: There exist sufficiently small positive numbers a and b such that

$$\left|\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2}+\frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right)-L\right|\leq K(y-Y^*),$$

for all $y>Y^*+2b$, $|\xi-\eta|+|\tau'-t'|<a$ and for some constant K.

We have

$$\lim_{y\to\infty}\left|\frac{\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2}+\frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right)-L}{y-Y^*}\right|=\frac{2\beta\tau'^2}{1-\tau'^2}. \tag{3.4.93}$$

Then for $d>0$ there exists M such that

$$\begin{aligned}\left|\frac{\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2}+\frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right)-L}{y-Y^*}\right| &< \frac{2\beta\tau'^2}{1-\tau'^2}+d.\\ &\leq \frac{2\beta(t'+a)^2}{1-(t'+a)^2}+d=:K_2,\end{aligned} \tag{3.4.94}$$

for all $y>M$ and $|\xi-\eta|+|\tau'-t'|<a$. In fact a is chosen sufficiently small so that $t'+a<1$, remember that $0<t'<1$. Therefore,

$$\begin{aligned}\left|\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2}+\frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right)-L\right| &\leq K_2(y-Y^*),\quad \text{for all}\quad y>M\\ \text{and}\quad |\xi-\eta|+|\tau'-t'|<a.\end{aligned} \tag{3.4.95}$$

If $Y^*+2b\geq M$, then the claim holds. If $Y^*+2b<M$, then

$$\left|\frac{\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2}+\frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right)-L}{y-Y^*}\right|$$

is continuous on $[Y^*+2b,M]$. Hence it is bounded by K_1, for some real K_1.

Thus,

$$\left|\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2}+\frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right)-L\right|\leq K(y-Y^*), \tag{3.4.96}$$

for all $y>Y^*+2b$ and $|\xi-\eta|+|\tau'-t'|<a$, where $K=sup\{K_1,K_2\}$. Claim holds.

Similarly one can obtain that

$$\left|\frac{2\beta\tau'(\xi-\tau'y)}{1-\tau'^2}+\frac{k\tau'}{\beta}\left(\frac{1-\tau'}{1+\tau'}\right)-L\right|\leq K(Y^*-y), \tag{3.4.97}$$

for all $y < Y^* - 2b$,$|\xi - \eta| + |\tau' - t'| < a$ and for some constant $\hat{K}$.

Therefore, from (3.4.92), (3.4.96) and (3.4.97), we have

$$\hat{u}(\alpha,\tau,\nu) < L + K\frac{\int_{-\infty}^{Y_*-2b}(Y_* - y)\exp\left\{-\frac{P}{\nu}\right\}dy}{\int_{-\infty}^{y_*(\xi,\tau')}\exp\left\{-\frac{P}{\nu}\right\}dy} + \hat{K}\frac{\int_{Y^*+2b}^{\infty}(y - Y^*)\exp\left\{-\frac{P}{\nu}\right\}dy}{\int_{y^*(\xi,\tau')}^{\infty}\exp\left\{-\frac{P}{\nu}\right\}dy} \tag{3.4.98}$$

(3.4.99)

when $|\xi - \eta| + |\tau' - t'| < a$ is satisfied.

Claim : $\lim\limits_{\substack{\nu\to 0\\ \xi\to\eta\\ \tau'\to t'}} \quad \hat{K}\frac{\int_{Y^*+2b}^{\infty}(y - Y^*)\exp\left\{-\frac{P}{\nu}\right\}dy}{\int_{y^*(\xi,\tau')}^{\infty}\exp\left\{-\frac{P}{\nu}\right\}dy} = 0.$

To prove this, we first show that $\frac{P(\xi,y,\tau')}{(y-Y^*)^2}$ is bounded below by a positive constant $\xi,\ y,$ and τ' satisfy $y > Y^*+2b$ and $|\xi-\eta|+|\tau'-t'| < a$. Since $\frac{P(\xi,y,\tau')}{(y-Y^*)^2} \to \frac{\beta\tau'^2}{2(1-\tau'^2)}$ as $|y| \to \infty$ uniformly on any closed set of (ξ,τ'), there exist a natural number N such that $\frac{P(\xi,y,\tau')}{(y-Y^*)^2}$ is bounded below by$\frac{\beta\tau'^2}{4(1-\tau'^2)}$ $=: A_1$ for all $y > N$ and $|\xi-\eta|+|\tau'-t'| < a$ is satisfied.

On the other hand $\frac{P(\xi,y,\tau')}{(y-Y^*)^2}$ attains its positive minimum on $B = \{(\xi,y,\tau')/Y^*+2b \le y \le N,\ |\xi-\eta|+|\tau'-t'| < a\}$ say A_2.

Take $A = min\{A_1, A_2\}$. Then

$\frac{A}{2} < \frac{P(\xi,y,\tau')}{(y-Y^*)^2}$, when $\xi,\ y,\ \tau'$ satisfy $y > Y^* + 2b$ and $|\xi - \eta| + |\tau' - t'| < a\}$.

Further, a calculation shows that

$$\begin{aligned}\int_{Y^*+2b}^{\infty}(y - Y^*)\exp\left\{-\frac{P}{\nu}\right\}dy &< \int_{Y^*+2b}^{\infty}(y - Y^*)\exp\left\{-\frac{A}{2\nu}(y - Y^*)^2\right\}dy\\ &= \int_{2b}^{\infty} r\exp\left\{-\frac{A}{2\nu}r^2\right\}dr\\ &= \frac{\nu}{A}\exp\left\{-\frac{2Ab^2}{\nu}\right\},\end{aligned} \tag{3.4.100}$$

whenever $\xi,\ \tau'$ satisfy $|\xi - \eta| + |\tau' - t'| < a$. Meanwhile, the uniform continuity of P on the set

$$\{(\xi,y,\tau') : |\xi - \eta| + |\tau' - t'| \leqslant a \text{ and } Y_* - 2b \leqslant y \leqslant Y^* + 2b\}$$

and the fact (3.4.87) imply that there exists a positive δ such that $P < 2Ab^2$ holds whenever $\xi,\ y,$ and τ' satisfy $y^*(\xi,\tau')-\delta < y < y^*(\xi,\tau')+\delta$ and $|\xi-\eta|+|\tau'-t'| < a$.

Thus

$$\int_{y^*(\xi,\tau')}^{\infty} \exp\left\{-\frac{P}{\nu}\right\} dy = \int_{y^*(\xi,\tau')}^{y^*(\xi,\tau')+\delta} \exp\left\{-\frac{2Ab^2}{\nu}\right\} dy + \int_{y^*(\xi,\tau')+\delta}^{\infty} \exp\left\{-\frac{2Ab^2}{\nu}\right\} dy$$
$$> \int_{y^*(\xi,\tau')}^{y^*(\xi,\tau')+\delta} \exp\left\{-\frac{2Ab^2}{\nu}\right\} dy$$
$$= \delta \exp\left\{-\frac{2Ab^2}{\nu}\right\} \quad (3.4.101)$$

whenever $|\xi-\eta|+|\tau'-t'|<a$ is true.

Therefore, the third term in right hand side of the inequality (3.4.99) is less than $\nu/A\delta$ and hence it tends to 0 as $\nu \to 0$, whenever $\xi,\ \tau'$ satisfy $|\xi-\eta|+|\tau'-t'|<a$. Claim holds. Similarly, second term in (3.4.99) also vanishes as $\nu \to 0$.

Thus we have proved (3.4.83). Similarly we can prove (3.4.84). □

In view of Theorem 3.4.4, we observe that, at the point (η, t') where $y_*(\eta,t') = y^*(\eta,t')$,

$$\lim_{(\alpha,\tau,\nu)\to(\eta,t',0)} \hat{u}(\alpha,\tau,\nu)$$

exists and

$$\lim_{(\alpha,\tau,\nu)\to(\eta,t',0)} \hat{u}(\alpha,\tau,\nu) = \frac{2\beta t'[\eta - t'y_*(\eta,t')]}{1-t'^2} + \frac{kt'}{\beta}\left(\frac{1-t'}{1+t'}\right)$$
$$= \frac{2\beta t'[\eta - t'y^*(\eta,t')]}{1-t'^2} + \frac{kt'}{\beta}\left(\frac{1-t'}{1+t'}\right) \quad (3.4.102)$$
$$=: \hat{u}(\eta,t')$$
$$=: u(x,t).$$

By virtue of Lemma 3.4.1, $y_*(\eta,t'),\ y^*(\eta,t')$ are monotonic in $\mathbb{R}$ for fixed t'. Then the set of points η at which $y_*(\eta,t')$and $y^*(\eta,t')$ are discontinuous is at most countable.

Further, suppose $y_*(\eta,t')$ is continuous at $\tilde{\eta}$. Take a sequence η_k such that $\eta_k \leq \tilde{\eta}$ and η_k converges to η.

Then

$$y^*(\eta_k,t') \leq y_*(\widetilde{\eta},t').$$

Therefore,

$$\lim_{k\to\infty} y^*(\eta_k,t') \leq y_*(\widetilde{\eta},t').$$

Hence,

$$y^*(\widetilde{\eta}, t') = y_*(\widetilde{\eta}, t').$$

Thus, we obtain,

$$y_*(\eta, t') = y^*(\eta, t') \quad a.e.$$

Finally, for every $t > 0$,

$$\lim_{\nu \to 0} u(x, t, \nu) = u(x, t) \quad a.e., \tag{3.4.103}$$

But we know that the weak formulation of (3.1.2) is

$$\begin{aligned} -\int_0^\infty \int_{-\infty}^\infty \left[u(x,t,\nu)\Psi_t + \frac{(u(x,t,\nu))^2}{2}\Psi_x \right] dxdt + \int_{-\infty}^\infty \Psi(x,0)u(x,0,\nu)dx \\ = \nu \int_0^\infty \int_{-\infty}^\infty u(x,t,\nu)\Psi_{xx} dxdt + k \int_0^\infty \int_{-\infty}^\infty \frac{\Psi}{(2\beta t+1)^{3/2}} dxdt, \end{aligned} \tag{3.4.104}$$

for any $\Psi \in C_0^\infty(\mathbb{R}_+^2)$, $\mathbb{R}_+^2 = \mathbb{R} \times (0, \infty)$. Let $\text{supp}(\Psi) =: K_1 \times K_2 \subseteq \mathbb{R}_+^2$. Therefore, passing to $\nu \to 0$ for the limit function $u(x, t)$, we have

$$\int_{K_2} \int_{K_1} \left[u(x,t)\Psi_t + \frac{(u(x,t))^2}{2}\Psi_x + \frac{k}{(2\beta t+1)^{3/2}}\Psi \right] dxdt - \int_{K_1} \Psi(x,0)u(x,0)dx = 0. \tag{3.4.105}$$

Hence, $u(x, t)$ is the generalized solution of

$$u_t + uu_x = \frac{k}{(2\beta t+1)^{3/2}}, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.4.106}$$

where $\beta > 0$ and k is a non-zero constant, with the initial condition (3.1.3).

Chapter 4

Conclusions and future work

- We assumed that $u_0 \in L^1(\mathbb{R}),\ x^{2n+1}u_0 \in L^1(\mathbb{R})$ with $\int_{-\infty}^{\infty} u_0(x)dx = 0$, then we showed the existence of approximate solution $u_n(x,t)$ to the true solution $u(x,t)$ of

$$\begin{cases} u_t + uu_x = \mu u_{xx}, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{4.0.1}$$

  such that $||u(.,t) - u_n(.,t)||_p = O(t^{-(n+1)+1/2p})$, as $t \to \infty$.
  For which, an asymptotic N-wave approximate solution was constructed for heat equation in such a way that, the moments of exact solution of the heat equation agree with those of approximate solution and achieved higher order estimates. In the case of Burgers solutions also, we obtained the same higher order convergence, even though the moments of exact and approximate solution of Burgers equation are not equal. The proposed approximate solution is much simpler and the rate of convergence of the N-wave approximate solution is higher when it is compared with the Gaussian approximation proposed by Kim (2011) in certain cases. Finally, we constructed an approximate solution for a de-coupled system and obtained higher order convergence.

- Using generalization of truncated moment problem, we just showed the existence of approximate solutions for (4.0.1). However, one can further study which one among all those approximate solutions would be convenient to applications.

- It will be more challenging to consider the problem (4.0.1) with $(1+|x|^{2n+1})u_0 \in L^1(\mathbb{R})$ and then looking for an approximate solution $u(x,t)$ such that $||u(.,t) - v(.,t)||_p = O(t^{-m+1/2p})$, as $t \to \infty$, where $m > n+1$. Perhaps one has to introduce time shifts in approximate solution in addition to considering space shift parameters. It is worthy to consider $n = 2$ case first and then one can look for generalizing it.

- It will be exciting to consider the non planar Burgers equation

$$\begin{cases} u_t + uu_x + \frac{ju}{2t} = \gamma u_{xx}, \\ u(x,0) = u_0(x), \end{cases} \tag{4.0.2}$$

where $\int_{-\infty}^{\infty} u_0(x)dx = 0$ and $u_0 \in C(\mathbb{R})$. By mimicking the solution, constructed in Chapter 2, one may investigate the behavior of solution of (4.0.2). It will be more worthy to obtain approximate N-wave solutions with precise error estimations.

- We constructed approximate solutions for

$$u_t + uu_x = u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0, \tag{4.0.3}$$

$$\rho_t + (u\rho)_x = \rho_{xx}, \qquad x \in \mathbb{R}, \quad t > 0. \tag{4.0.4}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{4.0.5}$$

$$\rho(x,0) = \rho_0(x), \qquad x \in \mathbb{R}, \tag{4.0.6}$$

under the assumption that

$$V_0,\ x^{2n+m}V_0,\ C_0,\ x^{2n+m}C_0 \in L^1(\mathbb{R}),$$
$$\int_{-\infty}^{\infty} x^k V_0(x)dx = \int_{-\infty}^{\infty} x^k C_0(x)dx = 0,\ 0 \le k < m.$$

One can look for the solution of (4.0.3)-(4.0.5) under the conditions that $u_0 \in C^\infty(\mathbb{R})$ and $\rho_0 \in C^\infty(\mathbb{R})$. One can also investigate the behavior of solutions for the generalization of (4.0.3)-(4.0.5) in higher dimensions.

- We constructed generalized solutions for the following problem

$$
\begin{aligned}
&u_t + uu_x = \tfrac{k}{(2\beta t+1)^{3/2}}, \quad x \in \mathbb{R}, \quad t > 0 \\
&\rho_t + (u\rho)_x = 0, \quad x \in \mathbb{R}, \quad t > 0,
\end{aligned} \tag{4.0.7}
$$

subject to

$$
(u, \rho)^t(x, 0) = \begin{cases} (u_L, \rho_L)^t & \text{if } x < 0, \\ (u_R, \rho_R)^t & \text{if } x > 0. \end{cases} \tag{4.0.8}
$$

Also, we obtained the generalized solutions for

$$
u_t + uu_x = \frac{k}{(2\beta t + 1)^{3/2}}, \quad x \in \mathbb{R}, \quad t > 0 \tag{4.0.9}
$$

subject to $u(x, 0) = u_0(x)$, where $u_0(x) = o(|x|)$ as $|x| \to \infty$. For which, we made use of viscosity method.

If one can try to study (4.0.7) subject to the general initial data

$$
(u(x, 0), \rho(x, 0)) = (u_0(x), \rho_0(x)), \tag{4.0.10}
$$

where

$$
\begin{cases} u_0(x) = o(|x|), & \text{as } |x| \to \infty, \\ \rho_0(x) = o(|x|), & \text{as } |x| \to \infty, \end{cases} \tag{4.0.11}
$$

it gives good insight into the properties of solutions and hence it will be helpful in dealing with the problem (4.0.9) with $f(x, t)$ in place of specific expression $\frac{k}{(2\beta t+1)^{3/2}}$.

- One can even though study the initial value problem (4.0.9)-(4.0.11) regarding uniqueness of solutions by following the well known results from Evans (1998).

- While studying (4.0.7)-(4.0.10), we completed the cases $u_L < u_R$ and $u_L > u_R$. One can also investigate the case $u_L = u_R$.

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