RADIO *k*-COLORING AND *k*-DISTANCE COLORING OF GRAPHS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

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December 2020

Dedicated to

My Parents and Sisters

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **RADIO *k*-COLORING AND *k*-DISTANCE COLORING OF GRAPHS** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfilment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me*. The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

This is to *certify* that the Research Thesis entitled **RADIO *k*-COLORING AND *k*-DISTANCE COLORING OF GRAPHS** submitted by **Mr. NIRANJAN P K**, (Register Number: 165069 MA16F03) as the record of the research work carried out by him, is *accepted as the Research Thesis submission* in partial fulfilment of the requirements for the award of degree of **Doctor of Philosophy**.

Dr. SRINIVASA RAO KOLA
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Chairman - DRPC
(Signature with Date and Seal)

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ABSTRACT

The frequency assignment problem is the problem of assigning frequencies to transmitters in an optimal way and with no interference. Interference can occur if transmitters located sufficiently close to each other receive close frequencies. The frequency assignment problem motivates many graph coloring problems. Motivated by this, we study radio k-coloring and k-distance coloring of graphs. In this thesis, we study radio k-coloring of paths, trees, Cartesian product of graphs and corona of graphs; k-distance coloring of trees, cycles and cactus graphs. A radio k-coloring of a simple connected graph G is an assignment f of positive integers (colors) to the vertices of G such that for every pair of distinct vertices u and v in G, $|f(u)-f(v)|$ is at least $1+k-d(u,v)$. The span of f, $rc_k(f)$, is the maximum color assigned by f. The radio k-chromatic number $rc_k(G)$ is $\min\{rc_k(f) : f$ is a radio k-coloring of $G\}$. If d is the diameter of G, then a radio d-coloring and the radio d-chromatic number are referred as a radio coloring and the radio number $rn(G)$ of G. Since finding the radio k-chromatic number is highly non-trivial, it is known for very few graphs and that too for some particular values of k only. For $k \geq 6$, we determine the radio k-chromatic number of path P_n for $\frac{2n+1}{7} \leq k \leq n-4$ if k is odd and for $\frac{2n-4}{5} \leq k \leq n-5$ if k is even. For some classes of trees, we obtain an upper bound for the radio k-chromatic number when k is at least the diameter of the tree. Also, for the same, we give a lower bound which matches with the upper bound when k and the diameter of the tree are of the same parity. Further, we determine the radio d-chromatic number of larger trees constructed from the trees of diameter d in some subclasses of the above classes. We determine the radio number for some classes of the Cartesian product of complete graphs and cycles. We obtain a best possible upper bound for the radio k-chromatic number of corona $G \odot H$ of arbitrary graphs G and H. Also, we obtain a lower bound and an improved upper bound for the radio number of $Q_n \odot H$ and $P_{2p+1} \odot H$. A k-distance coloring of a simple connected graph G is an assignment f of positive integers to the vertices of G such that no two vertices at distance less than or equal to k receive the same color. If α is the maximum color assigned by f, then f is referred as a k-distance α-coloring. The k-distance chromatic number $\chi_k(G)$ is the minimum α such that G has a k-distance α-coloring. We determine the k-distance chromatic number for trees and cycles. Also, we determine the 2-distance chromatic number of cactus graphs.

Keywords: radio k-coloring; span; radio k-chromatic number; radio coloring; radio number; k-distance coloring; distance coloring; k-distance chromatic number; 2-distance coloring; 2-distance chromatic number

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Nomenclature and Abbreviations

Abbreviations

FAP	:	Frequency Assignment Problem

Basic Graph Theory Symbols

G,H	:	Graphs
$V(G)$	:	Vertex set of G
$E(G)$	:	Edge set of G
$n(G),n$	:	Order of G
$deg_G(v)$, $deg(v)$	:	Degree of v in G
$d_G(u,v)$, $d(u,v)$	:	Distance between u and v in G
$e_G(v)$, $e(v)$	:	Eccentricity of v in G
$diam(G)$	:	Diameter of G
$rad(G)$	:	Radius of G
$\Delta(G)$, Δ	:	Maximum degree of G
$\delta(G)$, δ	:	Minimum degree of G
$N(v)$	:	Set of all neighbors of a vertex v in G
$N(S)$	:	Set of all neighbors of a vertices in $S \subseteq V(G)$
$g(G)$	:	Girth of G
$\alpha(G)$	:	Independence number of G
$\omega(G)$	:	Clique number of G
$w(G)$	:	Weight of G
$ad(G)$	:	Average degree of G
$mad(G)$	:	Maximum average degree of G
$\chi(G)$	:	Chromatic number of G

Operations and Relations of Graphs

$G \cong H$	:	G and H are isomorphic
$\overline{G}$	:	Complement of G
$G - v$	:	Graph obtained by deleting vertex v from G
$G[S]$	:	Subgraph of G induced by S
$G \Box H$	:	Cartesian product of G and H
$G \times H$	:	Direct product of G and H
$G \odot H$	:	Corona of G and H
G^r	:	r^{th} power of G
$M(G)$	:	Middle graph of G

Graph Classes

P_n	:	Path on n vertices
C_n	:	Cycle on n vertices
K_n	:	Complete graph on n vertices
$K_{n,m}$	:	Complete bipartite graph with partite sets of size m and n
$K_{1,n}$	:	Star graph
Q_n	:	n-dimensional hypercube, hypercube
$Ci_n(l)$	:	Circulant graph on n vertices
$GP(n,r)$	:	Generalized Petersen graph
$P_n \Box P_m$	:	Grid graph
$C_n \Box C_m$	:	Toroidal grid graph
$P_n \Box C_m$	:	Generalized prism graph
$T_{r,m}$	:	Complete m-ary tree of height r
B_n	:	Binomial tree
BFT_n	:	Binary Fibonacci tree
FT_n	:	Fibonacci tree

$B(n,r)$	:	Banana tree
$F(n,r)$	:	Firecracker tree
W_n	:	Wheel graph
$J_{t,n}$	:	Generalized gear graph
H_n	:	Helm graph

Notations in Radio *k*-coloring

$rc_k(f)$	:	Span of a radio k-coloring f
$rn(f)$	:	Span of a radio coloring f (span of f)
$rc_k(G)$	:	Radio k-chromatic number of G
$rn(G)$	:	Radio number of G
$ac(G)$	:	Antipodal number of G
$ac'(G)$	:	Nearly antipodal number of G

Notations in *k*-distance Coloring

$\chi_k(G)$	:	k-distance chromatic number
$\chi_2(G)$	:	2-distance chromatic number

CHAPTER 1

INTRODUCTION

"Graph coloring is arguably the most popular subject in graph theory."
- Noga Alon (1993)

Graphs can be used to model many types of relations and processes in physical, biological, social, and information systems. Many of the real-world practical problems can be represented by graphs. Beginning with the origin of the Four Color Problem in 1852, the field of graph colorings has developed into one of the most popular areas of Graph Theory. Many graph colorings are motivated by the frequency assignment problem. Due to the rapid growth of wireless networks and the relatively scarce radio spectrum, the importance of the frequency assignment problem is growing significantly. Motivated by this, we discuss two graph coloring problems, namely, radio k-coloring of graphs and k-distance coloring of graphs. Before going deep into these areas, we give some basic definitions of Graph Theory in the following section. We have referred the textbook "Introduction to Graph Theory" by West (1996) for all the terminologies and definitions.

1.1 BASIC DEFINITIONS OF GRAPH THEORY

A *simple graph* G is an ordered pair $(V(G), E(G))$, where $V(G)$ is a non-empty finite set and $E(G)$ is a subset of the set of all two-element subsets of $V(G)$. The elements of $V(G)$ are called as *vertices* of G and the elements of $E(G)$ as *edges* of G. Unless there is no ambivalence in the graph under discussion, $V(G)$ and $E(G)$ are represented by V and E, respectively. The *order* and the *size* of a graph are the number of vertices and the number of edges in G, respectively. An edge $e = \{u, v\} \in E$ is simply represented by uv, and u and v are called as the *end vertices* of e. An edge e is said to be *incident* on the vertices u and v if u and v are the end vertices of e. Two vertices u and v are said to be *adjacent* in G if uv is an edge of G. If a vertex u is adjacent to a vertex v, then v is called a *neighbor* of u. The *degree* of a vertex v in a graph is the number of edges incident on v, and is denoted by $deg_G(v)$ or simply $deg(v)$. A vertex of degree one is called a *pendant vertex* or a *leaf*. A Graph H is said to be a *subgraph* of a graph G if the vertex set and the edge set of H are subsets of $V(G)$ and $E(G)$, respectively. If H is a subgraph of G, then we say that G contains H. For a vertex v in G, $G - v$ denotes the graph with vertex set $V(G)\backslash\{v\}$ and edges in $G - v$ are all those edges of G which are not incident on v. An *induced subgraph* $G[S]$ of G induced by S is a subgraph of G obtained by deleting the vertices in $V(G)\backslash S$ from G. The complement of a graph G is the graph $\overline{G}$ with vertex set $V(\overline{G}) = V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G.

A *walk* in a graph G is an alternating sequence of vertices and edges, starting and ending with vertices, and every edge is incident on vertices preceding and succeeding to it. The number of edges in a walk is called the *length* of the walk. If a walk starts and ends at the same vertex, then it is called a *closed walk*. If all the edges of a walk are distinct, then the walk is said to be a *trail*. If all the vertices of a walk are distinct, then it is called a *path*. If a path has u and v as end vertices, then it is called a u, v-path. If all the vertices of a closed walk are distinct (except starting and end vertices), then it is called a *cycle*. A graph G is said to be *connected* if there exists a path between every pair of vertices. A *disconnected* graph is a graph which is not connected. The *distance* between two vertices u and v in a graph G is the length of a shortest u, v-path if it exist, otherwise the distance is ∞, and it is denoted by $d_G(u, v)$ or simply $d(u, v)$. In a connected graph G, the eccentricity $e_G(v)$ (or simply $e(v)$) of a vertex v is the $\max\{d(v, u) : u \in V(G)\}$. The *diameter*, $diam(G)$, and the *radius*, $rad(G)$, of a connected graph G are the maximum and the minimum of the set $\{e(v) : v \in V(G)\}$, respectively. The center of a graph is the subgraph induced y the vertices of minimum eccentricity. The *girth* $g(G)$ of G, containing a cycle, is the length of the shortest cycle in G. A subset of the vertex set of a graph is said to be an *independent set* if no two vertices in it are adjacent. The *independence number* $\alpha(G)$ of G is the maximum size of an independent set in G. A subset of the vertex set of a graph is said to be a *clique* if every pair of vertices in it are adjacent. The *clique number* $\omega(G)$ of G is the maximum size of a clique in G.

A graph is said to be a *path* if its vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the order. Path of order n is denoted by P_n. A graph is said to be a *cycle* if its vertices can be placed on a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. Cycle of order n is denoted by C_n. A *complete graph* of order n, denoted by K_n, is the graph in which every pair of vertices are adjacent. A *tree* is a connected graph without cycles. A *rooted tree* is a tree with one vertex u chosen as root. For each vertex v, let $P(v)$ be the unique v,u-path. The *parent* of v is its neighbor on $P(v)$; its *children* are its other neighbors. The *level of a vertex* $L(v)$ is the distance of it from the root. The *height of the rooted tree* is $\max\{L(v) : v \in V(T)\}$. A tree is said to be a *caterpillar* if deleting all the pendant vertices of the tree results a path graph. An *m-distant tree* is a tree T in which there is a path P of maximum length (this path is referred as the central path) such that every vertex in $V(T)\backslash V(P)$ is at distance at most m from P. A *unicyclic graph* is a connected graph containing exactly one cycle. A *cactus graph* is a connected graph in which no two cycles have a common edge. A graph G is said to be *bipartite* if its vertex set can be partitioned into two independent sets X and Y. The partition $\{X,Y\}$ is called a *bipartition* of G. A *complete bipartite graph* is a bipartite graph in which every vertex in one partite set is adjacent to every vertex in the other partite set, and is denoted by $K_{m,n}$, where m and n are the cardinalities of the partite sets. The complete bipartite graph $K_{1,n}$ is known as a *star graph*. For an integer $r \geq 2$, an *r-partite graph* or simply *multipartite graph* is a graph whose vertex set can be partitioned into r independent sets. A *complete r-partite graph* or simply *complete multipartite graph* is an r-partite graph such that every vertex in every partite set is adjacent to all the vertices in all the other partite sets. An *r-regular graph* is a graph in which the degree of every vertex is r. An *n-dimensional hypercube* Q_n is the graph whose vertices are the n-tuples with entries in $\{0,1\}$ and two vertices are adjacent if they differ in exactly one position.

Two graphs G and H are said to be *isomorphic* if there exists a bijection f from $V(G)$ to $V(H)$ such that two vertices u and v of G are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in H. If G and H are isomorphic, then we denote $G \cong H$. An *automorphism* of G is an isomorphism from G to G. A graph G is said to be a *vertex-transitive graph* if for every pair $u, v \in V(G)$, there is an automorphism that maps u to v. The hypercube Q_n is a vertex-transitive graph. The *Cartesian product* $G \Box H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and two vertices (u,x) and (v,y) in $G \Box H$ are adjacent if $u = v$ and $xy \in E(H)$, or $x = y$ and $uv \in E(G)$. It is easy to see that $diam(G \Box H)$ is $diam(G) + diam(H)$. The *direct product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and two vertices (u,x) and (v,y) are adjacent if $uv \in E(G)$ and $xy \in E(H)$. Let G and H be two graphs with vertex sets $\{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_1, u_2, u_3, \ldots, u_m\}$, respectively. Then the *corona* $G \odot H$ of G and H is the graph with vertex set $V(G) \bigcup \Big(\bigcup\limits_{i=1}^{n} \{v_i^j : 1 \leq j \leq m\} \Big)$ and edge set $E(G) \bigcup \Big(\bigcup\limits_{i=1}^{n} \{v_i v_i^j : 1 \leq j \leq m\} \Big) \bigcup \Big(\bigcup\limits_{i=1}^{n} \{v_i^l v_i^j : u_l u_j \in E(H)\} \Big)$. Equivalently, $G \odot H$ is the graph obtained by taking one copy of G and for each vertex v_i of G, one copy of H, say H_i, and joining v_i to each and every vertex of H_i by an edge. It is easy to see that $G \odot H \ncong H \odot G$ if and only if $G \ncong H$. Also, $diam(G \odot H) = diam(G) + 2$. For a positive integer r, the r^{th} *power of a graph* G, denoted by G^r, is the graph with vertex set $V(G)$ and vertices u and v in G^r are adjacent if and only if $d_G(u,v) \leq r$.

A *coloring* of a graph is an assignment of positive integers to the vertices of it. A coloring f is said to be a *proper coloring* if no two adjacent vertices receive the same color. If k is the maximum color used in a proper coloring f of G, then f is called a *k-coloring* of G. The *chromatic number* $\chi(G)$ of G is the minimum k such that G has a k-coloring. A graph G is said to be a *planar graph* if it has a drawing without crossing of edges.

1.2 THE FREQUENCY ASSIGNMENT PROBLEM

The Frequency Assignment Problem (FAP) emerges in a wide variety of real-world situations. Several such problems, perhaps modeled as an optimization problem in the following manner. Given a collection of transmitters to be assigned operating frequencies, obtain an assignment that meets multiple constraints, and that minimizes the value of a given objective function. FAP has applications in wireless networks. Due to rapid growth of wireless networks and to the relatively scarce radio spectrum, the importance of FAP is growing significantly. The first FAP emerged from the discovery that transmitters receiving the identical or closely related frequencies had the potential to interfere with each other. Consequently, the primary approach to FAP is to minimize or eradicate this potential interference. In this strategy, the significant constraints are the operating bandwidth of the transmitters, the band of the electromagnetic spectrum which the transmitters are capable of using, and the total number of frequencies available for assignment to the transmitters. An easy approach to minimize the interference is to assign different transmitters distinct non-interfering frequencies. Such an approach to frequency assignment is tied up a lot of the spectrum but persisted viable so long as the growth of the available spectrum kept pace with the growth in demand of it. The growth of the usable spectrum slowed while the demand of it grown exponentially. This turn of the event forced to consider different approaches. A type of constraint specifies that if the distance between a pair of transmitters is less than a designated minimum number of miles, then some combinations of assignments to this pair of transmitters are excluded. Such constraints employ both frequency and distance separation to decrease interference and are called frequency-distance constraints. An FAP in which the interference limiting constraints are all frequency-distance constraints is called a frequency-distance constrained FAP. One more type of interference restricting constraint stipulates that some specific combinations of assignments are forbidden for a given couple of transmitters. Such constraints employ only frequency separation to alleviate interference and are called frequency constraints. An FAP in which the interference restricting constraints are all frequency constraints is called a frequency constrained FAP.

In 1980, Hale has modeled FAPs as optimization problems. Most of them are graph coloring problems. The modeling is as follows. Transmitters are represented by vertices of a graph and those vertices corresponding to transmitters which are very close are joined by edges. Frequency assignment to transmitters is nothing but assignment of positive integers to the vertices of the corresponding graph.

1.2.1 Radio *k*-coloring of Graphs

In 2001, Gary Chartrand, David Erwin, Frank Harary, and Ping Zhang have considered a variation of FAP, in which maximum interference occurs among transmitters corresponding to adjacent vertices. Interference decreases as distance between transmitters increases. Assigning frequencies to transmitters is same as assigning positive integers (colors) to the vertices. To get efficient assignment, we should assign colors to the vertices so that the adjacent vertices' colors differ a lot and other vertices' colors difference decreases as distance increases. Also, we have to minimize the maximum color assigned.

Definition 1.2.1. *(Chartrand et al., 2001) For a connected graph G and an integer k,* $1 \leq k \leq diam(G)$*, a radio k-coloring of G is an assignment f of positive integers (referred as colors) to the vertices of G such that for every pair u and v of distinct vertices in G,* $|f(u) - f(v)| \geq 1 + k - d(u,v)$*. The maximum color assigned by f is called the span* $rc_k(f)$ *of f.*

Definition 1.2.2. *(Chartrand et al., 2001) The radio k-chromatic number $rc_k(G)$ of a connected graph G is the minimum of spans over all radio k-colorings of G. A radio k-coloring with span $rc_k(G)$ is referred as a minimal radio k-coloring of G.*

Example 1.2.3. A radio 3-coloring of a graph with span 16 is given in Figure 1.1. A minimal radio 3-coloring of the same graph is given in Figure 1.2.

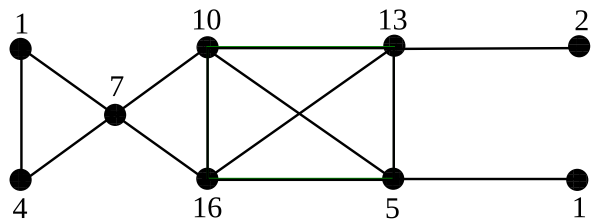

Figure 1.1 A radio 3-coloring of a graph

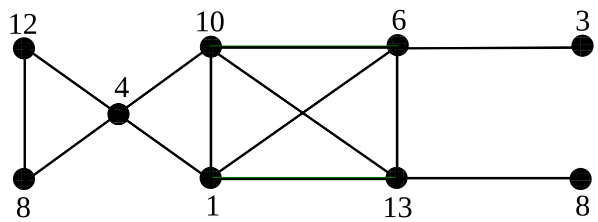

Figure 1.2 A minimal radio 3-coloring of the graph Figure 1.1

For some special values of k, there are special names of radio k-colorings and as well as the radio k-chromatic numbers in the literature which are given in Table 1.1. Radio k-coloring of graphs is a generalization of proper coloring and $L(2,1)$-coloring (introduced by Griggs and Yeh (1992)) of graphs. In a graph G, two vertices u and v are said to be antipodal vertices if $d(u,v) = diam(G)$. In a radio $(d-1)$-coloring of a graph, antipodal vertices can receive the same colors.

Few researchers have studied radio k-coloring of graphs as Multi-level distance labeling of graphs. Also, few authors have considered the radio k-coloring is a mapping of non-negative integers satisfying the same condition. The radio k-chromatic number obtained by considering radio k-coloring as a mapping of non-negative integers is one

k	**Name of the coloring**	**Radio k-chromatic number**	$rc_k(G)$
1	Proper coloring	Chromatic number	$\chi(G)$
2	$L(2,1)$-coloring	λ-number or $L(2,1)$-number	$\lambda(G)$
$diam(G)$	Radio coloring	Radio number	$rn(G)$
$diam(G)-1$	Antipodal coloring	Antipodal number	$ac(G)$
$diam(G)-2$	Nearly antipodal coloring	Nearly antipodal number	$ac'(G)$

Table 1.1 Radio k-colorings and the radio k-chromatic numbers for some special values of k

less than that obtained by considering radio k-coloring as a mapping of positive integers. Also, if the minimum color r assigned by a radio k-coloring f is greater than 1, then a coloring g defined by $g = f - r + 1$ is a radio k-coloring whose span is $r - 1$ less than that of f. So, we assume that every radio k-coloring of a graph assigns the color 1 to at least one vertex of the graph.

1.2.2 k-distance Coloring of Graphs

In 1969, Florica Kramer and Horst Kramer have introduced k-distance coloring of graphs as a generalization of proper coloring of graphs. In recent times, some authors have studied k-distance coloring as an FAP.

Definition 1.2.4. *(Kramer and Kramer, 1969b) Given a connected graph G and a positive integer k, a k-distance coloring of G is an assignment f of positive integers (referred as colors) to the vertices of G such that no two vertices at distance less than or equal to k receive the same color. If α is the maximum integer assigned by f, then f is referred as a k-distance α-coloring.*

Definition 1.2.5. *(Kramer and Kramer, 1969b) The k-distance chromatic number $\chi_k(G)$ is the smallest α for which G has a k-distance α-coloring.*

Example 1.2.6. A 3-distance 7-coloring of a graph is given in Figure 1.3. It is easy to see that 7 is the 3-distance chromatic number of the graph.

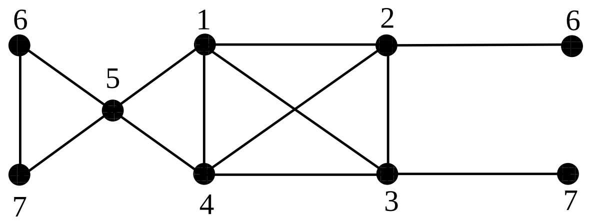

Figure 1.3 A 3-distance 7-coloring of a graph

1.3 LITERATURE SURVEY

In this section, we give a detailed literature survey for radio k-coloring of graphs and k-distance coloring of graphs. There are two survey papers on radio k-coloring of graphs, one is by Chartrand and Zhang (2007) and the other by Panigrahi (2009). Also, a literature survey of radio k-coloring can be found in the dynamic survey of graph labelings by Gallian (2019). A survey paper of k-distance coloring of graph is published by Kramer and Kramer (2008). Throughout this section and in the subsequent chapters, unless we mention, graph means a connected graph.

1.3.1 Radio k-coloring

In the introductory paper, Chartrand et al. (2001) have studied the radio numbers of some well known graphs, namely, cycles, complete multipartite graphs and graphs with diameter 2. They have computed the radio numbers of C_n for $n \leq 8$, and have given bounds for other values of n. Also, they have found the radio number of a complete t-partite graph $K_{n_1,n_2,n_3,\ldots,n_t}$ as $(t-1)+\sum_{i=1}^{t} n_i$. Further, they have proved that $n \leq rn(G) \leq 2n-2$ for any graph G of order n and diameter 2. For $1 \leq k \leq n-1$, Chartrand et al. (2004) have proved that $rc_k(P_n)$ is at the most $\frac{k^2+2k+1}{2}$ if k is odd and at the most $\frac{k^2+2k+2}{2}$ if k is even. Although, Chartrand et al. (2001) have defined radio k-coloring of a graph G for $k \leq diam(G)$, one can also see this problem for $k > diam(G)$, as it is useful to find the radio k-chromatic number of larger graphs containing G. Kchikech et al. (2007) have determined the radio k-chromatic number of path P_n for $k \geq n$ as $(n-1)k - \frac{1}{2}n(n-2)+1$ if n is even and $(n-1)k - \frac{1}{2}(n-1)^2+2$ if n is odd, and conjectured as below.

Conjecture 1.3.1. *(Kchikech et al., 2007) For any integer $k \geq 5$,*

$$\lim_{n\to\infty} rc_k(P_n) = \begin{cases} \frac{k^2+2k+1}{2} & if\ \ k\ \ is\ odd, \\ \frac{k^2+2k+2}{2} & if\ \ k\ \ is\ even. \end{cases}$$

Liu and Zhu (2005) have determined the radio number of P_n for $n \geq 4$ as $2p^2-2p+2$ if $n=2p$ and $2p^2+3$ if $n=2p+1$. Khennoufa and Togni (2005) have found the antipodal number of P_n as $2p^2-4p+5$ if $n=2p$ and $2p^2-2p+3$ if $n=2p+1$. Kola and Panigrahi (2009a) have determined the nearly antipodal number of P_n as $2p^2-6p+8$ if $n=2p$ and $2p^2-4p+6$ if $n=2p+1$. Kola and Panigrahi (2009b) have determined the radio $(n-4)$-chromatic number of P_n as $\frac{n^2-8n+25}{2}$ when n is odd and given an upper bound for the same as $\frac{n^2-8n+26}{2}$ when n is even. Das et al. (2017) have improved the lower bound for the radio k-chromatic number of infinite path, towards Conjecture 1.3.1, as $\frac{k^2+k+2}{2}$. In an attempt to prove Conjecture 1.3.1, Kola and Panigrahi (2013) have improved the upper bound of $rc_k(P_n)$ for different intervals of $n \leq \left\lfloor \frac{k^2+2k}{2} \right\rfloor$ as below.

Theorem 1.3.2. *(Kola and Panigrahi, 2013) For $k \geq 7$ and $4 \leq s \leq \lfloor \frac{k+1}{2} \rfloor$,*

$$rc_k(P_{k+s}) \leq \begin{cases} \frac{k^2+2s+1}{2} & if\ k\ is\ odd, \\ \frac{k^2+2s+2}{2} & if\ k\ is\ even. \end{cases}$$

Theorem 1.3.3. *(Kola and Panigrahi, 2013) For any odd $k \geq 5$,*

$$rc_k(P_n) \leq \begin{cases} \frac{k^2+k+2}{2} & if\ \frac{3k+1}{2} < n \leq \frac{5k-1}{2}, \\ \frac{k^2+k+2s+4}{2} & if\ \frac{(5+2s)k+1}{2} \leq n \leq \frac{(7+2s)k-1}{2},\ s=0,1,2,\ldots,\frac{k-5}{2}. \end{cases}$$

Theorem 1.3.4. *(Kola and Panigrahi, 2013) For any even* $k \geq 6$*,*

$$rc_k(P_n) \leq \begin{cases} \frac{k^2+k+2}{2} & if\ n = \frac{3k+2}{2}, \\ \frac{k^2+k+2s+4}{2} & if\ \frac{(3+2s)k+2s+4}{2} \leq n \leq \frac{(5+2s)k+2s+4}{2},\ s = 0,1,2,\ldots,\frac{k-4}{2}. \end{cases}$$

Further, Kola and Panigrahi (2013) have re-conjectured Conjecture 1.3.1 as below.

Conjecture 1.3.5. *(Kola and Panigrahi, 2013) For any integer* $k \geq 5$ *and* $n \geq n_0$*,* $rc_k(P_n) = n_0$*, where* $n_0 = \frac{k^2+2k+1}{2}$ *if k is odd and* $n_0 = \frac{k^2+2k+2}{2}$ *if k is even.*

For a tree T of order n, Liu (2008) has showed that $rn(T) \geq (n-1)(diam(T)+1)+1-2w(T)$, where $w(T)$ is the weight of T, defined by $w(T) = \min_{u \in V(T)} \left\{ \sum_{v \in V(T)} d(u,v) \right\}$. Also, she has given a lower bound for the radio number of spider graph (tree with at most one vertex of degree more than two) and characterized the spider graphs achieving this bound. Given integers $m \geq 2$ and $r \geq 1$, the complete m-ary tree of height r, denoted by $T_{r,m}$, is a rooted tree such that each non-pendant vertex has m children and all the pendant vertices are at distance r from the root. Li et al. (2010) have found the radio number of complete m-ary trees. Marinescu Ghemeci (2010) has determined the radio number of caterpillars in which all non-pendant vertices have degree 3. Also, she has determined the radio number of the tree obtained by attaching r pendant vertices to each pendant vertex of star $K_{1,n}$, by an edge. A binomial tree B_n consists of two copies of B_{n-1} such that the root of one is the leftmost child of the root of the other, where B_0 is the one vertex tree. Binary Fibonacci trees BFT_0 and BFT_1 are paths P_1 and P_2 respectively. For $n \geq 2$, a binary Fibonacci tree BFT_n is a rooted tree in which the left subtree and the right subtree are BFT_{n-1} and BFT_{n-2}. Fibonacci trees FT_0 and FT_1 are path P_1. For $n \geq 2$, a Fibonacci tree FT_n consists FT_{n-1} and FT_{n-2} such that the root of FT_{n-2} is the leftmost child of the root of FT_{n-1}. An uniform caterpillar is a caterpillar in which all the non-pendant vertices are of the same degree. Reddy and Iyer (2011) have given upper bounds for the radio number of binomial trees, binary

Fibonacci trees, Fibonacci trees and uniform caterpillars. Benson et al. (2013) have determined the radio number of all graphs of order n and diameter $n-2$ (these graphs are caterpillars having exactly 3 pendant vertices). Kola and Panigrahi (2015b) have determined the radio number of some classes of caterpillars. Also, Kola and Panigrahi (2014) have found the radio number of some m-distant trees. A banana tree $B(n,r)$ is a tree obtained by making adjacent one pendant vertex from each of n copies of a $(r-1)$-star to a new vertex. A firecracker tree, denoted by $F(n,r)$, is the tree obtained by taking a path P_n and n copies of $(r-1)$-star, and making each vertex of P_n adjacent to a pendant vertex in the corresponding $(r-1)$-star. Bantva et al. (2015) have found the radio number of symmetric trees (trees in which all the non-pendant vertices have the same degree and all the pendant vertices have the same eccentricity). Also, Bantva et al. (2017) have determined the radio number of banana trees, Fire cracker trees and some classes of caterpillars.

Chartrand et al. (2001) have computed the radio numbers of C_n for $n \leq 8$ and proved that the radio number of C_n, $n \geq 6$, is at least $3\left\lceil \frac{n}{2} \right\rceil - 1$. Also, they have obtained an upper bound for the radio number of the same as $\frac{n^2-2n+1}{4}$ if n is odd and $\frac{n^2-2n+4}{4}$ if n is even. Liu and Zhu (2005) have improved the bounds for $rn(C_n)$ and proved that the radio number of cycle C_n, $n \geq 3$, is $\frac{n-2}{2}\phi(n)+1$ if n is even and $\frac{n-1}{2}\phi(n)$ if n is odd, where $\phi(n)$ is equal to $s+1$ if $n = 4s+1$ and $s+2$ if $n = 4s+r$, $r = 0,2,3$. Chartrand et al. (2000) have proved that $ac(C_n) = 2s(s+1)+1$, where $n = 4s+2$ and in the other cases they have given lower bounds. Juan and Liu (2006) have showed that the lower bounds for $ac(C_n)$ given by Chartrand et al. (2000) are exact for $n \equiv 1,3$ $(mod\ 4)$ and for $n \equiv 0$ $(mod\ 4)$ conjectured as below.

Conjecture 1.3.6. *(Juan and Liu, 2006) For any $s \geq 1$, $ac(C_{4s}) = 2s^2$.*

Kola and Panigrahi (2013) have shown that if Conjecture 1.3.1 is true, then Conjecture 1.3.6 is also true. Karst et al. (2017) have given a lower bound for $rc_k(C_n)$, $k > diam(C_n)$, as $\frac{\Phi(k,n)(n-2)}{2} - \frac{n}{2} + k + 1$ if n is even and $\frac{\Phi(k,n)(n-1)}{2}$ if n is odd, where

$\Phi(k,n) = \lceil \frac{3k-n+3}{2} \rceil$. Also, they have proved that the lower bound is exact when $k = diam(C_n) + 1$.

The generalized Petersen graph $GP(n,r)$, $n \geq 3$ and $1 \leq r \leq \lfloor \frac{n-1}{2} \rfloor$, is a graph with vertex set $\{u_1, u_2, u_3, \ldots, u_n, v_1, v_2, v_3, \ldots, v_n\}$ and edge set $\{u_iu_{i+1}, u_iv_i, v_i, v_{i+r} : i = 1,2,3,\ldots,n\}$, where the subscripts are taken modulo n. Kola and Panigrahi (2011) have determined the radio number of generalized Petersen graphs $GP(n,1)$ when $n \equiv 0\ (mod\ 4)$; $n \equiv 1\ (mod\ 4)$; $n = 4m+2$ and m odd; $n = 4m+3$ and m even; and $n = 4m+3$ and $m \equiv 1,2\ (mod\ 3)$. Zhang et al. (2019) have determined the radio number of $GP(n,2)$ when $n = 4m+2$ and they have obtained a lower bound for the radio number of $GP(4m,2)$. Kousar et al. (2015) have determined the radio number of $GP(n,3)$ for $n \equiv 4\ (mod\ 6)$.

For the hypercube Q_n, $n \geq 2$, and $k \geq 2$, Kchikech et al. (2008) have proved $(2^n - 1)k - 2^{n-1}(2n-3) + n - 1 \leq rc_k(Q_n) \leq (2^n-1)k - 2^{n-1} + 2$. Kola and Panigrahi (2010) have improved the lower bound given by Kchikech et al. (2008) and proved that the improved lower bound is sharp for the radio number. The radio number of hypercube Q_n is $\left(\frac{n+4}{2}\right) 2^{n-1} + \frac{n}{2}$ if n is even and $\left(\frac{n+3}{2}\right) 2^{n-1} + \frac{n-1}{2}$ if n is odd. Khennoufa and Togni (2011) have determined the antipodal number ($(2^{n-1}-1)\lceil \frac{n}{2} \rceil + \varepsilon(n)$, where $\varepsilon(n)$ is 1 if $n \equiv 0\ (mod\ 4)$, else 0) of hypercube Q_n. Kchikech et al. (2008) have given an upper bound for the radio k-chromatic number of Cartesian product $G\square H$ of arbitrary graphs as $rc_k(G\square H) \leq \chi(H^k)(rc_k(G) + k - 1) - k + 1$. Also, they have given bounds for $rc_k(P_n\square P_n)$ when $k \geq 2n-3$. Jiang (2014) has completely found the radio number of grid graph $P_n\square P_m$ by improving both the upper and lower bounds given by Kchikech et al. (2008). The radio number of $P_n\square P_m$ is $\frac{n^2m+m^2n}{2} - mn - m - n + 6$ if both m and n are even; $\frac{n^2m+m^2n-m-n}{2} - mn + 2$ if both m and n are odd; $\frac{n^2m+m^2n-n}{2} - mn - n + 2$ if m is odd and n is even. Kim et al. (2015a) have determined the radio number of $P_n\square K_m$ as $\frac{mn^2-2n+4}{2}$ if n is even and $\frac{mn^2-2n+m+4}{2}$ if n is odd. Martinez et al. (2011) have determined the radio number of generalized prism graph $P_n\square C_m$, for $n = 1,2,3$. Morris-Rivera et al. (2015) have determined $rn(C_n\square C_n)$ as $2p^3 + 4p^2 - p$ if $n = 2p$ and is $2p^3+4p^2+2p+1$ if $n=2p+1$. Saha and Panigrahi (2013) have found the radio number of toroidal grid $C_m\square C_n$ when at least one of m and n is even. Nazeer and Kousar (2014) have proved that $rn(P_2\odot P_n)=2n+4, n\geq 5$ and $rn(P_2\odot K_{1,m})=2m+7, m\geq 2$.

For a graph G of order n and diameter d, Saha and Panigrahi (2015) have proved that $\left\lceil \frac{rn(G)}{2}\right\rceil \leq rn(G^2)\leq \left\lfloor \frac{rn(G)+n-1}{2}\right\rfloor$ if d is even and $\left\lceil \frac{rc_{d+1}(G)}{2}\right\rceil \leq rn(G^2)\leq \left\lfloor \frac{rc_{d+1}(G)+n-1}{2}\right\rfloor$ if d is odd. Also, they have determined the radio number of G^2 when G is an even order graph of diameter d except for $\frac{d}{2}\equiv 0\ (mod\ 4)$ and hence they have obtained the radio number of Q_n^2, square of the hypercube, when $n\not\equiv 0\ (mod\ 4)$, and $(C_m\square C_n)^2$, square of the toroidal grid, when $n+m\not\equiv 0,5,7\ (mod\ 8)$. For the square of paths, Liu and Xie (2009) have determined the radio number as follows. If $n\equiv 1\ (mod\ 4)$ and $n>8$, then $rn(P_n^2)=\left\lfloor\frac{n}{2}\right\rfloor^2+3$, else $rn(P_n^2)=\left\lfloor\frac{n}{2}\right\rfloor^2+2$. Rao et al. (2018) have completely determined the radio number of r^{th} power of P_n. For $2\leq r\leq n-2$, $p=\left\lfloor\frac{n}{2r}\right\rfloor$ and $m=n-2pr$, the radio number of P_n^r is $2rp^2+2$ if $m=0$ or $m=1$ and $n<4r+1$; $2rp^2+3$ if $m=0$ or $m=1$ and $n\geq 4r+1$; $2rp^2+2rp+m+1$ if $2\leq m\leq r+1$; $2rp^2+2rp+m$ if $m=r+1$; and $2rp^2+2rp+2r+2$ if $r+2\leq m\leq 2r-1$. Liu and Xie (2004) have proved that the radio number of square of cycle C_n^2 is $\frac{2p^2+5p+1}{2}$ if $n=4p$ and p is odd; $\frac{2p^2+3p+2}{2}$ if $n=4p$ and p is even; p^2+5p+2 if $n=4p+2$ and p is odd; p^2+4p+2 if $n=4p+2$ and p is even; p^2+2p+2 if $n=4p+1$ and p is even; p^2+p+1 if $n=4p+1$ and $p\equiv 3\ (mod\ 4)$; $\frac{2p^2+9p+6}{2}$ if $n=4p+3$ and $p\equiv 0\ (mod\ 4)$; $\frac{2p^2+9p+6}{2}$ if $n=4p+3$ and $p=4m+2$, $m\not\equiv 5\ (mod\ 7)$; $\frac{2p^2+7p+5}{2}$ if $n=4p+3$ and $p=4m+1$, $m\equiv 0,1\ (mod\ 3)$; $\frac{2p^2+7p+7}{2}$ if $n=4p+3$ and $p=4m+1$, $m\equiv 2\ (mod\ 3)$. For the remaining cases, they have given upper and lower bounds. Nazeer et al. (2015) have determined the antipodal number of C_{4p+2}^2 as p^2+p if p is odd and p^2+2p if p is even. Sooryanarayana and Raghunath (2007) have determined the radio number of C_n^3 for some classes of n.

For any list l chosen from $\{1,2,3,\ldots,\lfloor\frac{n}{2}\rfloor\}$, a circulant graph $Ci_n(l)$ is a graph on the vertices $v_1,v_2,v_3,\ldots,v_n$ such that each v_i, $1\leq i\leq n$, is adjacent to v_{i+j} and v_{i-j} (subscripts are taken modulo n) for every j in the list l. It is easy to see that

$C_n^r = Ci_n(1,2,3,\ldots,r)$. Kenneth et al. (2013) have determined the antipodal number of $Ci_n(1,2,3,\ldots,\lfloor \frac{n}{2} \rfloor - 1)$ as $\lceil \frac{n}{2} \rceil$. Also, they have obtained upper bounds for the antipodal number of circulant graphs $Ci_{4n}(1,2n)$, $n \geq 4$; $Ci_{3n}(1,n)$, $n \geq 2$; and $Ci_{10n}(1,2n)$, $n \geq 1$. Kang et al. (2016) have determined the radio number and the antipodal number of $Ci_{4mp+2m}(1,2m)$ when m is even and they have given a lower bound for the radio number of $Ci_{4mp+2m}(1,2m)$ when m is odd.

The wheel graph W_n ($n \geq 3$) consists of an n-cycle together with a center vertex that is adjacent to all n vertices of the cycle. The generalized gear graph $J_{t,n}$ is obtained from the wheel W_n by replacing each edge on n-cycle by a path of length $t+1$, that is, by introducing t-vertices between every pair of adjacent vertices on the n-cycle of the wheel. Fernandez et al. (2008) have found the radio number for the wheel graph W_n and gear graph $J_{1,n}$. Rahim et al. (2012) have given an upper bound for the radio number of generalized gear graph $J_{t,n}$ when $n \geq 7$ and $t < n-1$. Later, Ali et al. (2012) have given a lower bound for the radio number of generalized gear graph which matches with the upper bound given by Rahim et al. (2012). A helm graph H_n is obtained from the wheel W_n by attaching a vertex to each of the n vertices of the cycle of the wheel by an edge. Rahim and Tomescu (2012) have determined the radio number of the helm graph as $rn(H_3) = 13$, $rn(H_4) = 21$ and $rn(H_n) = 4n+2$ for every $n \geq 5$. The middle graph $M(G)$ of a graph G is the graph such that $V(M(G)) = V(G) \cup E(G)$ and two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident on it. Bantva (2017) has determined the radio number of $M(P_n)$ for all n. Vaidya and Vihol (2012) have determined the radio number of $M(C_n)$ for all n.

For the radio number of a graph, the order of the graph is a trivial lower bound. Sooryanarayana and Raghunath (2007) have characterized the graphs C_n^3 for which the radio number is n. Niedzialomski (2016) has proved that the radio number for the Cartesian product of t copies of K_n is n^t for $n \geq 3$ and $2 \leq t \leq n$. Also, she has proved that the radio number of Hamming graph $K_{n_1}\square K_{n_2}\square K_{n_3}\square\ldots\square K_{n_r}$ is $\prod_{i=1}^{r} n_i$ if $n_1,n_2,n_3,\ldots,n_r$ are relatively prime.

Kola and Panigrahi (2015a) have given a lower bound for $rc_k(G)$ of an arbitrary graph G (see Theorem 1.4.4, page 22) and proved that the lower bound is sharp for the radio number of cycle C_n. Using this lower bound, they have given a lower bound for $rc_k(C_n\square P_m)$. Further, they have proved that the lower bound is exact for $C_n\square P_2$, when $n \equiv 1\ (mod\ 4)$ and $n \equiv 2\ (mod\ 8)$. Das et al. (2017) have given a lower bound technique for the radio k-chromatic number of a graph (see Theorem 1.4.3, page 21)). Bantva (2019) has improved the lower bound technique of Das et al. (2017) for the radio number. Saha and Panigrahi (2019) have given two algorithms to produce radio k-colorings for general graphs. Using the algorithms, they have obtained minimal radio k-colorings for several graphs and many values of k as given in Table 1.2.

Graphs	**Values of *k***	**Values of *n***
C_n, $6 \le n \le 400$	$diam(C_n)$	All n
	$diam(C_n)-1$	$n \equiv 1,2\ (mod\ 4)$
	$diam(C_n)-2$	$n \equiv 2\ (mod\ 4)$
$C_n\square P_2$ $6 \le n \le 200$	$diam(C_n\square P_2)$	n odd
	$diam(C_n\square P_2)-1$	$n \not\equiv 1\ (mod\ 4)$
	$diam(C_n\square P_2)-2$	$n \equiv 2\ (mod\ 4)$
$C_n\square C_4$ $6 \le n \le 100$	$diam(C_n\square C_4)$	$n \equiv 0\ (mod\ 4)$
	$diam(C_n\square C_4)-1$	$n \equiv 2,3\ (mod\ 4)$
	$diam(C_n\square C_4)-2$	$n \equiv 1\ (mod\ 4)$

Table 1.2 The graphs and values of k for which minimal radio k-colorings are given by Saha and Panigrahi (2019) using the algorithms

1.3.2 *k*-distance Coloring

As k-distance coloring is trivial for $k \ge diam(G)$, it is studied for $k < diam(G)$. Since $k+1$ is trivial lower bound for $\chi_k(G)$, Kramer and Kramer (1969a,b) characterized the graphs with $\chi_k(G) = k+1$ as below.

Theorem 1.3.7. *(Kramer and Kramer, 1969a,b) For any graph G,* $\chi_k(G) = k+1$ *if and only if G satisfies one of the following.*

(i) $|V(G)| = k+1$.

(ii) G is a path of length greater than k.

(iii) G is a cycle of length multiple of $k+1$.

Fertin et al. (2003) have determined the k-distance chromatic number of two dimensional grid $P_m \square P_n$ as $\left\lceil \frac{(k+1)^2}{2} \right\rceil$. Also, they have determined the 2-distance chromatic number of m-dimensional grid graph $P_{n_1} \square P_{n_2} \square P_{n_3} \square \ldots \square P_{n_m}$ as $2m+1$. Sevcikova (2001) have found the exact value of $\chi_k(G)$ for triangular lattice G as $\left\lceil \frac{3(k+1)^2}{2} \right\rceil$. Jacko and Jendrol (2005) have determined the k-distance chromatic number of hexagonal lattice as $\left\lceil \frac{3(k+1)^2}{8} \right\rceil$ if k is odd and $\left\lceil \frac{3}{8}\left(k+\frac{4}{3}\right)^2 \right\rceil$ if k is even. Jendrol and Skupien (2001) have given an upper bound for the k-distance chromatic number of an arbitrary planar graph as below.

Theorem 1.3.8. *(Jendrol and Skupien, 2001) If G is a planar graph with maximum degree* Δ *and* $N = \max\{\Delta, 8\}$*, then*

$$\chi_k(G) \leq \frac{3N+3}{N-2}((N-1)^{k-1} - 1) + 6.$$

Definition 1.3.9. *For a non-negative integer r and a vertex v of a graph G, the graph* G_v^r *denotes the subgraph of G induced by the vertices of G which are at distance less than or equal to r from v.*

The following result of Sharp (2007) gives a lower bound for the k-distance chromatic number of an arbitrary graph.

Theorem 1.3.10. *(Sharp, 2007) For any graph G and a positive integer k,*

$$\chi_k(G) \geq \begin{cases} \max\limits_{v \in V(G)} \left| V\left(G_v^{\frac{k}{2}}\right) \right| & if\ k\ is\ even, \\ \max\limits_{v \in V(G)} \left| V\left(G_v^{\frac{k-1}{2}}\right) \right| + 1 & if\ k\ is\ odd. \end{cases}$$

Kramer and Kramer (1986) have given an upper bound for $\chi_3(G)$ of a bipartite graph G with maximum degree Δ as $2(1+\Delta(\Delta-1))$. Also, they have proved that $\chi_3(G) \leq 8$ for a bipartite planar graph G with maximum degree $\Delta \leq 3$.

Although, k-distance coloring is defined for all positive integers k, it is mostly studied for $k=2$ and $k=3$. For any planar graph G with maximum degree Δ, Wegner (1977) has proved that $\chi_2(G) \leq 8$ if $\Delta \leq 3$, and conjectured as below.

Conjecture 1.3.11. *(Wegner, 1977) For any planar graph G with maximum degree* Δ*,*

$$\chi_2(G) \leq \begin{cases} 7 & if\ \Delta = 3, \\ \Delta + 5 & if\ 4 \leq \Delta \leq 7, \\ \left\lfloor \frac{3\Delta}{2} \right\rfloor & if\ \Delta \geq 8. \end{cases}$$

The average degree of a graph G, denoted $ad(G)$, is $\frac{1}{|V(G)|} \sum\limits_{v \in V(G)} deg(v)$. The maximum average degree of a graph G, denoted $mad(G)$, is the maximum of $ad(H)$ on every subgraph H of G. Bonamy et al. (2011) have proved that $\chi_2(G)$ is $\Delta+1$ for any planar graph G with $\Delta \geq 5$ and girth $g(G) \geq 12$; $\Delta \geq 6$ and $g(G) \geq 10$; $\Delta \geq 8$ and $g(G) \geq 9$. Also, Bonamy et al. (2014) have found the 2-distance chromatic number of a graph with maximum degree $\Delta \geq 4$ and maximum average degree less than $\frac{7}{3}$ as $\Delta+1$. Wong (1996) have given an upper bound for $\chi_2(G)$ of planar graph G with maximum degree Δ as $3\Delta+5$. van den Heuvel and McGuinness (2003) have improved the upper bound given by Wong (1996) for planar graph with maximum degree $\Delta > 20$ as

$2\Delta+25$. For planar graphs with $\Delta \geq 241$, Molloy and Salavatipour (2005) have proved that $\chi_2(G) \leq \lceil \frac{5\Delta}{3} \rceil + 25$. For a planar graph G with girth at least 5, Dong and Lin (2016) have improved the existing lower bound as $\chi_2(G) \leq \Delta(G)+8$. Bu and Lv (2016) have further improved the upper bound for $\chi_2(G)$ of a planar graph G without cycles of length 3, 4 and 7 and $\Delta \geq 15$ as $\Delta+4$. For a planar graph G with maximum degree at least 5 and girth 339, Dong and Xu (2017) have proved that $\chi_2(G) \leq \Delta+3$. For a planar graph G, Zhu and Bu (2018) have proved that $\chi_2(G) \leq 5\Delta(G)-7$ if $\Delta(G) \geq 6$ and $\chi_2(G) \leq 20$ if $\Delta(G) \leq 5$. Dong and Xu (2019) have proved that if G is a planar graph without 4-cycle and 5-cycle, and $\Delta(G) \geq 185760$, then G is 2-distance $(\Delta(G)+2)$-colorable. Also, they have proved that, the upper bound $\Delta(G)+2$ is best possible. Kim et al. (2015b) have determined the 2-distance chromatic number of direct product of two cycles and direct product of path and cycle.

1.4 LOWER BOUNDS FOR THE RADIO k-CHROMATIC NUMBER OF AN ARBITRARY GRAPH

In this section, we provide some basic results related to radio k-coloring of graphs. The definition and the lemma below are first used by Khennoufa and Togni (2005). The definition below, gives how much extra is the difference between any two consecutive colors used in a radio k-coloring.

Definition 1.4.1. *For a graph G of order n and a radio k-coloring f of G, let $x_1, x_2, x_3, \ldots, x_n$ be an ordering of vertices of G such that $f(x_i) \leq f(x_{i+1})$, $1 \leq i \leq n-1$. Define $\varepsilon_i = f(x_i) - f(x_{i-1}) - (1+k-d(x_i,x_{i-1}))$, $2 \leq i \leq n$.*

We refer the sums $\sum_{i=2}^{n} d(x_i,x_{i-1})$ and $\sum_{i=2}^{n} \varepsilon_i$ as distance sum and epsilon sum, respectively. Lemma below gives the span of a radio k-coloring in terms of k, order of the graph, distance sum and epsilon sum.

Lemma 1.4.2. *Let f be a radio k-coloring of G and let $x_1, x_2, x_3, \ldots, x_n$ be an ordering of vertices of G such that $f(x_1) \le f(x_2) \le f(x_3) \le \cdots \le f(x_n)$ and $\varepsilon_i = f(x_i) - f(x_{i-1}) - (1 + k - d(x_i, x_{i-1}))$, $2 \le i \le n$. Then*

$$rc_k(f) = (n-1)(1+k) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \varepsilon_i + 1.$$

Proof:

$$\begin{aligned} f(x_n) - f(x_1) &= \sum_{i=2}^{n} [f(x_i) - f(x_{i-1})] \\ &= \sum_{i=2}^{n} [1 + k - d(x_i, x_{i-1}) + \varepsilon_i] \\ &= (n-1)(1+k) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \varepsilon_i. \end{aligned}$$

Since $f(x_1) = 1$, $rc_k(f) = f(x_n) = (n-1)(1+k) - \sum_{i=2}^{n} d(x_i, x_{i-1}) + \sum_{i=2}^{n} \varepsilon_i + 1$. □

For a given k and a graph G, the term $(n-1)(k+1)+1$ is constant. To get a lower bound, we need to maximize $\sum_{i=2}^{n} d(x_i, x_{i-1}) - \sum_{i=2}^{n} \varepsilon_i$, precisely, we have to maximize the distance sum simultaneously minimizing the epsilon sum. For a subset S of the vertex set of a graph G, let $N(S)$ denote the set of all vertices of G adjacent to at least one vertex of S. Das et al. (2017) have given a lower bound technique for the radio k-chromatic number of a graph G as in Theorem 1.4.3. Since we use Theorem 1.4.3 and its proof frequently in the subsequent chapters, we give a proof of it.

Theorem 1.4.3. *(Das et al., 2017) If f is a radio k-coloring of a graph G, then*

$$rc_k(f) \ge |D_k| - 2p + 2\sum_{i=0}^{p} |L_i|(p-i) + \alpha + \beta, \tag{1.4.1}$$

where D_k and L_i's are defined as follows. If $k = 2p+1$, then $L_0 = V(C)$, where C is a maximal clique in G. If $k = 2p$, then $L_0 = \{v\}$, where v is a vertex of G. Recursively define $L_{i+1} = N(L_i) \setminus (L_0 \cup L_1 \cup L_2 \cup \cdots \cup L_i)$ for $i = 0, 1, 2, \ldots, p-1$. Let $D_k = L_0 \cup L_1 \cup$

$L_2 \cup \cdots \cup L_p$. The minimum and the maximum colored vertices among the vertices of D_k are in L_α and L_β respectively.

Proof: Let $x_1, x_2, x_3, \ldots, x_{|D_k|}$ be an ordering of vertices of D_k such that $f(x_s) < f(x_{s+1})$. For $1 \leq s < |D_k|$, if $x_s \in L_i$ and $x_{s+1} \in L_j$, then $d(x_s, x_{s+1}) \leq i + j$ if k is even and $d(x_s, x_{s+1}) \leq i + j + 1$ if k is odd. Now, by the radio k-coloring condition, $f(x_{s+1}) - f(x_s) \geq 1 + k - d(x_s, x_{s+1}) \geq 2p + 1 - (i + j)$. Let g be a mapping from $\{1, 2, 3, \ldots, |D_k|\}$ to $\{0, 1, 2, \ldots, p\}$ defined by $g(s) = i$ if $x_s \in L_i$. Now,

$$\begin{aligned}
f(x_{|D_k|}) - f(x_1) &= \sum_{s=2}^{|D_k|} [f(x_s) - f(x_{s-1})] \\
&\geq \sum_{s=2}^{|D_k|} [2p + 1 - (g(s) + g(s-1))] \\
&= |D_k| - 1 + \sum_{s=2}^{|D_k|} [(p - g(s)) + (p - g(s-1))] \\
&= |D_k| - 1 - (p - g(1)) - (p - g(|D_k|)) + 2 \sum_{s=1}^{|D_k|} [p - g(s)] \\
&= |D_k| - 1 - 2p + g(1) + g(|D_k|) + 2 \sum_{s=1}^{|D_k|} [p - g(s)] \\
&= |D_k| - 1 - 2p + 2 \sum_{i=0}^{p} |L_i|(p - i) + \alpha + \beta,
\end{aligned}$$

where α and β are such that $x_1 \in L_\alpha$ and $x_{|D_k|} \in L_\beta$. Since $f(x_1) \geq 1$, $rc_k(f) \geq f(x_{|D_k|}) \geq |D_k| - 2p + 2 \sum_{i=0}^{p} [p - i] + \alpha + \beta$. □

For a given graph and a radio k-coloring of it, α and β are at most p. To get a better lower bound, we have to choose L_0 such that $|D_k|$ and $\sum_{i=0}^{p} |L_i|(p - i)$ are maximum. Kola and Panigrahi (2015a) have given a lower bound for $rc_k(G)$ of an arbitrary graph G described as below.

Theorem 1.4.4. *(Kola and Panigrahi, 2015a) Let G be a graph of order n. If $d(x, y) +$*

$d(x,z)+d(y,z) \leq M$ *for every triple x, y and z of G, then*

$$rc_k(G) \geq \begin{cases} \frac{(n-1)(3(k+1)-M)}{4}+1 & if\ n\ is\ odd\ and\ M \not\equiv k\ (mod\ 2), \\ \frac{(n-1)(3(k+1)-M+1)}{4}+1 & if\ n\ is\ odd\ and\ M \equiv k\ (mod\ 2), \\ \frac{(n-2)(3(k+1)-M)}{4}+k-diam(G)+2 & if\ n\ is\ even\ and\ M \not\equiv k\ (mod\ 2), \\ \frac{(n-2)(3(k+1)-M+1)}{4}+k-diam(G)+2 & if\ n\ is\ even\ and\ M \equiv k\ (mod\ 2). \end{cases}$$

The minimum positive real number M such that $d(x,y)+d(x,z)+d(y,z) \leq M$ for any three vertices of G is called as the triameter of G. If M is the triameter of G, then Theorem 1.4.4 gives a better lower bound for $rc_k(G)$. The following proposition gives an upper bound for the radio k-chromatic number of a graph.

Proposition 1.4.5. (Khennoufa and Togni, 2005) *For a graph G of order n having diameter d and for integers k and l with* $1 \leq k \leq l \leq d$, $rc_l(G) \leq rc_k(G)+(n-1)(l-k)$.

The proposition below gives bounds for the radio number of a graph.

Proposition 1.4.6. (Khennoufa and Togni, 2005) *For a graph G of order n and having diameter d,* $n \leq rn(G) \leq (n-1)d+1$.

1.5 SOME IMPORTANT ASSUMPTIONS IN THE THESIS

1. In all chapters, except Chapter 5, a graph means a connected graph.
2. In Chapter 5, whenever we consider $G \odot H$, the first graph G is always connected and the second graph H need not be connected.
3. The symbols k, p, r, n and m are positive integers.
4. For a radio k-coloring f, we refer the condition $|f(u)-f(v)| \geq 1+k-d(u,v)$ as the radio k-coloring condition.
5. The minimum color used by any radio k-coloring of a graph is 1.

6. By connecting two graphs G and H at u and v, $u \in V(G)$, $v \in V(H)$, we mean adding an edge between u and v.

7. In Chapter 4, moving on a cycle, unless we mention, we mean clockwise.

1.6 CHAPTERIZATION

The thesis consists of seven chapters of which Chapter 1 contains introduction and detailed literature survey of radio k-coloring and k-distance coloring of graphs. The next five chapters are the contributed chapters, and the last chapter is dedicated for conclusion and future scope.

In Chapter 2, we study radio k-coloring of path P_n. We show that the upper bounds given by Kola and Panigrahi (2013) are exact for $k+4 \leq n \leq \frac{7k-1}{2}$ if $k \geq 7$ is odd and for $k+4 \leq n \leq \frac{5k+4}{2}$ if $k > 7$ is even. In Chapter 3, we give an upper bound for the radio k-chromatic number of some classes of trees when k is at least the diameter of the tree. Also, we show that the upper bound is exact when the diameter of the tree and k are of the same parity. Further, we determine the radio d-chromatic number of infinitely many trees and graphs of large diameter constructed from the trees of diameter d in some subclasses of the above classes. In Chapter 4, we determine the radio number for the Cartesian product of complete graph K_n and cycle C_m when n even and m odd; any n and $m \equiv 6 \ (mod\ 8)$; and n odd and $m \equiv 5 \ (mod\ 8)$. In Chapter 5, we first obtain a best possible upper bound for the radio k-chromatic number of corona $G \odot H$ of arbitrary graphs. Later, we improve the upper bound for the radio number of $Q_n \odot H$ and $P_{2p+1} \odot H$, and also obtain a lower bound for the same. In Chapter 6, we determine the k-distance chromatic number of trees and cycles. Also, we determine the 2-distance chromatic number of cactus graphs.

CHAPTER 2

THE RADIO k-CHROMATIC NUMBER OF PATH P_n FOR SOME INTERVALS OF n

"*It would not be hard to present the history of graph theory as an account of the struggle to prove the four color conjecture, or at least to find out why the problem is difficult.*"

- William Thomas Tutte (1967)

Paths are the simplest class of graphs. For a path P_n, the radio k-chromatic number is known for $k \in \{n-1, n-2, n-3\}$. In this chapter, in an attempt towards Conjecture 1.3.5, we determine the radio k-chromatic number of P_n for $\frac{2n-4}{5} \leq k \leq n-5$ if k is even and $\frac{2n+1}{7} \leq k \leq n-4$ if k is odd.

2.1 PRELIMINARIES

To obtain lower bounds for the radio k-chromatic number of the paths, we use the lower bound technique for radio k-coloring given by Das et al. (2017). For convenience, we state Theorem 1.4.3 again.

Theorem 1.4.3. *(Das et al., 2017) If f is a radio k-coloring of a graph G, then*

$$rc_k(f) \geq |D_k| - 2p + 2\sum_{i=0}^{p} |L_i|(p-i) + \alpha + \beta, \tag{1.4.1}$$

where D_k and L_i's are defined as follows. If $k = 2p+1$, then $L_0 = V(C)$, where C is a maximal clique in G. If $k = 2p$, then $L_0 = \{v\}$, where v is a vertex of G. Recursively *define* $L_{i+1} = N(L_i) \backslash (L_0 \cup L_1 \cup L_2 \cup \cdots \cup L_i)$ *for* $i = 0, 1, 2, \ldots, p-1$. *Let* $D_k = L_0 \cup L_1 \cup L_2 \cup \cdots \cup L_p$. *The minimum and the maximum colored vertices among the vertices of* D_k *are in* L_α *and* L_β *respectively.*

From the proof of Theorem 1.4.3 (see page 21), the right hand side of Equation (1.4.1) is a lower bound for the induced subgraph $G[D_k]$. In a minimal radio k-coloring f of G, it is not necessary that the colors 1 and $span(f)$ are used to color a vertex in D_k. So, we have the theorem below.

Theorem 2.1.1. *Let G be a graph, and* L_i *and* D_k *be as in Theorem 1.4.3. If f is a radio k-coloring of G, and* $\lambda_{min} \in L_\alpha$ *and* $\lambda_{max} \in L_\beta$ *respectively are the minimum and the maximum colors among the vertices of* D_k*, then*

$$\lambda_{max} - \lambda_{min} + 1 \geq |D_k| - 2p + 2\sum_{i=0}^{p-1} |L_i|(p-i) + \alpha + \beta.$$

For path P_n, if k is odd, we choose L_0 as two adjacent vertices which are at distance at least $\frac{k-1}{2}$ from the pendant vertices of P_n, and if k is even, we choose L_0 as one vertex which is at distance at least $\frac{k}{2}$ from the pendant vertices of P_n. For $k = 2p+1$, we get $|L_i| = 2$ for all $i = 0, 1, 2, \ldots, p$, and for $k = 2p$, we get $|L_0| = 1$ and $|L_i| = 2$ for all $i = 1, 2, 3, \ldots, p$. In any case, D_k induces P_{k+1} for which L_0 is the center. Then by Theorem 2.1.1, we get the result below.

Theorem 2.1.2. *If f is a radio k-coloring of* P_n*, then*

$$rc_k(f) \geq \lambda_{max} \geq \begin{cases} \frac{k^2+3}{2} + \alpha + \beta + \lambda_{min} - 1 & if\ k\ is\ odd, \\ \frac{k^2+2}{2} + \alpha + \beta + \lambda_{min} - 1 & if\ k\ is\ even. \end{cases}$$

We use the following lemmas in the sequel.

Lemma 2.1.3. *If f is a radio k-coloring of a graph G with span λ, then there exists a radio k-coloring g of G with span λ such that the vertices of G receiving 1 and λ by f receive λ and 1, respectively, by g.*

Proof: The radio k-coloring g of G defined as $g(v) = \lambda + 1 - f(v)$ for every vertex v of G is one of such colorings. □

Lemma 2.1.4. *If n_1 and n_2 are positive integers such that $n_1 < n_2$, then $rc_k(P_{n_1}) \leq rc_k(P_{n_2})$.*

Proof: Since restriction of any radio k-coloring of the path P_{n_2} to the path P_{n_1} is a radio k-coloring of P_{n_1}, $rc_k(P_{n_1}) \leq rc_k(P_{n_2})$. □

For convenience in analyzing the results, in Table 2.1, the existing radio k-chromatic numbers of paths are given in terms of k.

2.2 THE RADIO *k*-CHROMATIC NUMBER OF PATH FOR *k* ODD

In this section, for k odd, we determine the radio k-chromatic number of path P_n, $k+5 \leq n \leq \frac{7k-1}{2}$. We use Theorem 2.1.1 and Theorem 2.1.2 to get the lower bounds those match with the upper bounds given in Theorems 1.3.2 and Theorems 1.3.3.

Theorem 2.2.1. *If $k \geq 7$ is odd and $4 \leq s \leq \frac{k+1}{2}$, then $rc_k(P_{k+s}) = \frac{k^2+2s+1}{2}$.*

Proof: Let f be a minimal radio k-coloring of path $P_{k+s} : v_1v_2v_3 \ldots v_{k+s}$ with span λ. Let i and j be the least positive integers such that $f(v_i) = 1$ and $f(v_j) = \lambda$. Without loss of generality, we assume that $i < j$. Let $k = 2p+1$. To prove the result, depending on the positions of the maximum and the minimum colored vertices, we choose a P_{k+1} subpath

Value of n	$rc_k(P_n)$	**References**
$n < k$ and n is odd	$(n-1)k - \frac{1}{2}(n-1)^2 + 2$	Kchikech et al. (2007)
$n < k$ and n is even	$(n-1)k - \frac{1}{2}n(n-2) + 1$	
$n = k$ and k is odd	$\frac{k^2+3}{2}$	
$n = k$ and k is even	$\frac{k^2+2}{2}$	
$n = k+1$ and k is odd	$\frac{k^2+3}{2}$	Liu and Zhu (2005)
$n = k+1$ and k is even	$\frac{k^2+6}{2}$	
$n = k+2$ and k is odd	$\frac{k^2+5}{2}$	Khennoufa and Togni (2005)
$n = k+2$ and k is even	$\frac{k^2+6}{2}$	
$n = k+3$ and k is odd	$\frac{k^2+7}{2}$	Kola and Panigrahi (2009a)
$n = k+3$ and k is even	$\frac{k^2+8}{2}$	
$n = k+4$ and k is odd	$\frac{k^2+9}{2}$	Kola and Panigrahi (2009b)

Table 2.1 The radio k-chromatic numbers of paths in terms of k

(L_0 is the center of it) of P_n such that $\alpha+\beta \geq s-1$. If $\alpha+\beta \geq s-1$, we get the required lower bound and if $\alpha+\beta > s-1$, we get a contradiction to Theorem 1.3.2 (using Theorem 2.1.2). If $i \leq s$, then by considering the path $v_iv_{i+1}v_{i+2}\ldots v_{i+p}v_{i+p+1}\ldots v_{i+k}$, we get $\alpha = \frac{k-1}{2}$. Now, by using Theorem 2.1.2, we get $rc_k(f) \geq \frac{k^2+k+2}{2}$ which is a contradiction to Theorem 1.3.2 if $s \neq \frac{k+1}{2}$. If $s < i < p+1$, then by considering the path $v_sv_{s+1}v_{s+2}\ldots v_{s+p}v_{s+p+1}\ldots v_{s+k}$, we get $\alpha \geq s$. If $j \geq k+1$, then by considering the path $v_{j-k}v_{j-k+1}v_{j-k+2}\ldots v_{j-p-1}v_{j-p}\ldots v_j$, we get $\beta \geq \frac{k-1}{2}$ which is strictly greater than $s-1$ if $s \neq \frac{k+1}{2}$. If $p+s < j < k+1$, then by considering the path $v_1v_2v_3\ldots v_{p+1}v_{p+2}\ldots v_{k+1}$, we get $\beta \geq s-1$. Suppose $p+1 \leq i < j \leq p+s$.

Case I: $s = 2l$

If $i \geq p+l+1$, then by choosing the path $v_1v_2v_3\ldots v_{p+1}v_{p+2}\ldots v_{k+1}$, we get $\alpha \geq l-1$ and $\beta \geq l$. By Theorem 2.1.2, we get $rc_k(f) \geq \frac{k^2+3}{2} + l - 1 + l = \frac{k^2+2s+1}{2}$. If $j \leq p+l+1$, then by choosing $v_sv_{s+1}v_{s+2}\ldots v_{s+p}v_{s+p+1}\ldots v_{k+s}$ subpath, we get $\beta \geq l-1$ and

$\alpha \geq l$. So, $\alpha+\beta \geq s-1$. Suppose $p+1 \leq i < p+l+1 < j \leq p+s$. Let $i = p+l+1-l_1$ and $j = p+l+1+l_2$, where $1 \leq l_1 \leq l$ and $1 \leq l_2 \leq l-1$. Suppose that $l_1 < l_2$. Then by considering the path $v_1v_2v_3 \ldots v_{p+1}v_{p+2} \ldots v_{k+1}$, we get $\alpha = (p+l+1-l_1)-(p+2) = l-l_1-1$ and $\beta = (p+l+1+l_2)-(p+2) = l+l_2-1$. Now, by Theorem 2.1.2, $rc_k(f) \geq \frac{k^2+3}{2}+l-l_1-1+l+l_2-1 = \frac{k^2+3}{2}+2l+(l_2-l_1)-2 \geq \frac{k^2+2s+1}{2}$. Suppose that $l_1 > l_2$. Then by considering the path $v_sv_{s+1}v_{s+2} \ldots v_{s+p}v_{s+p+1} \ldots v_{k+s}$, we get $\alpha = (p+2l)-(p+l+1-l_1) = l+l_1-1$ and $\beta = (p+2l)-(p+l+1+l_2) = l-l_2-1$. So, $\alpha+\beta \geq s-1$. If $l_1 = l_2$, then we choose $L_0 = \{v_p, v_{p+1}\}$ (we get the path $v_1v_2v_3 \ldots v_k$). So, we get $|L_p| = 1$ and $|L_t| = 2$, $t = 0,1,\ldots,p-1$. Also, $\alpha+\beta = p+l+1-l_1-p+1+p+l+1+l_2-(p+1) = 2l = s$. Now, by Theorem 2.1.1, $rc_k(f) \geq 2p+1-2p+2\sum_{t=0}^{p-1} 2(p-t)+1 = \frac{k^2+2s+1}{2}$.

Case II: $s = 2l+1$

If $i \geq p+l+1$ or $j \leq p+l+2$, then as in Case I, we get $rc_k(f) \geq \frac{k^2+2s+1}{2}$. So, we assume $p+1 \leq i < p+l+1 < p+l+2 < j \leq p+s$. Let $i = p+l+1-l_1$ and $j = p+l+2+l_2$, where $1 \leq l_1 \leq l$ and $1 \leq l_2 \leq l-1$. Rest of the proof is similar to that of Case I. □

Theorem 2.2.2. *If $k \geq 7$ is odd and $\frac{3k+1}{2} < n \leq \frac{5k-1}{2}$, then $rc_k(P_n) = \frac{k^2+k+2}{2}$.*

Proof: From Theorem 2.2.1, we have $rc_k(P_{\frac{3k+1}{2}}) = \frac{k^2+k+2}{2}$. By Lemma 2.1.4 and Theorem 1.3.3, we get the result. □

Lemma 2.2.3. *Let $k \geq 7$ be odd and f be a minimal radio k-coloring of $P_n : v_1v_2 \ldots v_n$, where $n = \frac{5k-1}{2}$. If $f(v_i) = 1$ and $f(v_j) = \frac{k^2+k+2}{2}$, then $\{i, j\} = \{k, n-k+1\}$.*

Proof: Let $f(v_i) = 1$ and $f(v_j) = \lambda$, where $\lambda = \frac{k^2+k+2}{2}$. Without loss of generality, we assume that $i < j$. Let $k = 2p+1$. To prove $i = k$ and $j = n-k+1$, we first show that $j-i = p$ or $j-i = p+1$. If $j-i < p$ or $p+1 < j-i \leq k$, then we choose the path $v_{j-k}v_{j-k+1}v_{j-k+2}\ldots v_{j-p-1}v_{j-p}\ldots v_j$ if $j > k$, else we choose the path $v_iv_{i+1}v_{i+2}\ldots v_{i+p}v_{i+p+1}\ldots v_{i+k}$. In any case, we get one of α and β is $\frac{k-1}{2}$ and the other is at least 1. Now, by Theorem 2.1.2, $rc_k(f) \geq \frac{k^2+k+4}{2}$, which is a contradiction. Suppose that $j-i > k$. If the color λ is not used in the path $v_iv_{i+1}v_{i+2}\ldots v_{i+p}v_{i+p+1}\ldots v_{i+k}$, using Theorem 2.1.2, we get a contradiction. Suppose the color λ is used in the path $v_iv_{i+1}v_{i+2}\ldots v_{i+p}v_{i+p+1}\ldots v_{i+k}$, say $f(v_t) = \lambda$. Since $t-i \leq k$, $t-i = p$ or $t-i = p+1$. Since $f(v_t) = f(v_j) = \lambda$, $t+k < j \leq n$. If the color 1 is not used in the path $v_tv_{t+1}v_{t+2}\ldots v_{t+p}v_{t+p+1}\ldots v_{t+k}$, using Theorem 2.1.2, we get a contradiction. Suppose the color 1 is used in the path $v_tv_{t+1}v_{t+2}\ldots v_{t+p}v_{t+p+1}\ldots v_{t+k}$, say $f(v_l) = 1$. Since $l-t \leq k$, $l-t$ is p or $p+1$. Since $f(v_i) = f(v_l) = 1$, $l-i \geq k+1$. Therefore $l = i+k+1$. Now, the minimum color used in the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ (path on k vertices) is not less than $p+2$. So, the colors available to color the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ is from $p+2 = \frac{k+3}{2}$ to $\frac{k^2+k+2}{2}$. Since $rc_k(P_k) = \frac{k^2+3}{2}$ and $\frac{k^2+k+2}{2} - \frac{k+3}{3} + 1 = \frac{k^2+1}{2}$, the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ cannot be colored. Therefore in any case, $j-i \not> k$ and hence $j-i = p$ or $p+1$.

Next, we show that $k \leq i < j \leq n-k+1$ and $j-i \neq p$. For that, we first prove that the color 1 and λ are used only once by f. Suppose $f(v_l) = 1$ for some $l \neq i$. Since $f(v_i) = 1$, $l \geq i+k+1$ and hence $l > j$. So, $l-j$ is p or $p+1$. Therefore $l-i = l-j+j-i \leq k+1$ and hence $l-i = k+1$. Now, the minimum color used in the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ (path on k vertices) is not less than $p+2$. So, the colors available to color the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ is from $p+2 = \frac{k+3}{2}$ to $\frac{k^2+k+2}{2}$. Since $rc_k(P_k) = \frac{k^2+3}{2}$ and $\frac{k^2+k+2}{2} - \frac{k+3}{3} + 1 = \frac{k^2+1}{2}$, the path $v_{i+1}v_{i+2}v_{i+3}\ldots v_{l-1}$ cannot be colored. Hence, the color 1 is assigned to only v_i and by Lemma 2.1.3, the color λ is assigned only to v_j. If $i < k$, then $v_{i+1}, v_{i+2}v_{i+3}\ldots v_n$ is a path of at least $\frac{3k+1}{2}$ vertices. Since $rc_k(P_{\frac{3k+1}{2}}) = \frac{k^2+k+2}{2} = \lambda$ and the color 1 is not used in the path $v_{i+1}, v_{i+2}v_{i+3}\ldots v_n$, we get a contradiction. Hence $i \geq k$. Suppose that $j > n-k+1$. Then $v_1v_2v_3\ldots v_{j-1}$ is a path of at least $\frac{3k+1}{2}$ vertices and $rc_k(P_{\frac{3k+1}{2}}) = \frac{k^2+k+2}{2} = \lambda$. But the maximum color used for a vertex of $v_1v_2v_3\dots v_{j-1}$ is at most $\lambda-1$, which is a contradiction. Therefore $k\le i<j\le n-k+1$. If $j-i=p$, then $i=k$, $j=k+p$ or $i=k+1$, $j=k+p+1$. If $i=k$ and $j=k+p$, then by considering the path $v_{k+p}v_{k+p+1}v_{k+p+2}\dots v_{k+2p}v_{k+2p+1}\dots v_n$, we get $\beta=\frac{k-1}{2}$ and the color 1 is not used for $v_{k+p}v_{k+p+1}v_{k+p+2}\dots v_n$. Now, by using Theorem 2.1.2, we get $rc_k(f)\ge\frac{k^2+k+4}{2}$, which is a contradiction. If $i=k+1$ and $j=k+p+1$, then for the path $v_1v_2v_3\dots v_{p+1}v_{p+2}\dots v_{k+1}$, the color $\frac{k^2+k+2}{2}$ is not used and $\alpha=\frac{k-1}{2}$. Now, by Theorem 2.1.2, we get $rc_k(f)\ge\frac{k^2+k+4}{2}$, which is a contradiction. Therefore, $j-i=p+1$, that is, $i=k$ and $j=n-k+1$. □

Example 2.2.4. For $k=7$ and $n=\frac{5k-1}{2}=17$, only one minimal radio k-coloring (radio 7-coloring with span $\frac{k^2+k+2}{2}=29$) is possible for the path P_{17}, which is given in Figure 2.1. Here, the color 1 is used to the vertex $v_7=v_k$ and the span 29 is used to the vertex $v_{11}=v_{n-k+1}$.

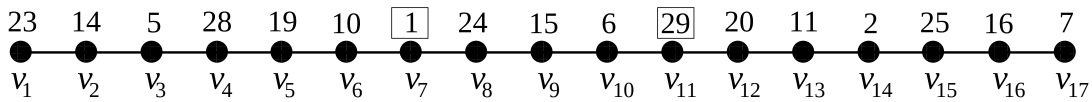

Figure 2.1 The minimal radio 7-coloring of P_{17}

Theorem 2.2.5. *If $k\ge7$ is odd and $\frac{5k+1}{2}\le n\le\frac{7k-1}{2}$, then $rc_k(P_n)=\frac{k^2+k+4}{2}$.*

Proof: Let $n=\frac{5k+1}{2}$, $P_n: v_1v_2V_3\dots v_n$ and $\lambda=\frac{k^2+k+2}{2}$. Suppose $rc_k(P_n)=\lambda$. Let f be a minimal radio k-coloring of P_n. Now, f restricted to $v_1v_2v_3\dots v_{n-1}$ is a minimal radio k-coloring of P_{n-1}. By Lemma 2.2.3, we get $\{f(v_k),f(v_{n-k})\}=\{1,\lambda\}$. By restricting f to the path $v_2v_3v_4\dots v_n$ and using Lemma 2.2.3, we get $\{f(v_{k+1}),f(v_{n-k+1})\}=\{1,\lambda\}$. Therefore, $rc_k(P_n)\ge\frac{k^2+k+4}{2}$ and hence by Theorem 1.3.3, $rc_k(P_n)=\frac{k^2+k+4}{2}$. □

2.3 THE RADIO k-CHROMATIC NUMBER OF PATH FOR k EVEN

In this section, for k even, we determine the radio k-chromatic number of path P_n, $k+4 \leq n \leq \frac{5k+4}{2}$. We use Theorem 2.1.1 and Theorem 2.1.2 to get the lower bounds those match with the upper bounds in Theorems 1.3.2 and Theorems 1.3.4.

Theorem 2.3.1. *If $k > 7$ is even and $4 \leq s \leq \frac{k}{2}$, then $rc_k(P_{k+s}) = \frac{k^2+2s+2}{2}$*

Proof: Let f be a minimal radio k-coloring of path $P_{k+s} : v_1v_2v_3 \dots v_{k+s}$ with span λ. Let i and j be the least positive integers such that $f(v_i) = 1$ and $f(v_j) = \lambda$. Without loss of generality, we assume that $i < j$. Let $k = 2p$. Analogous to the proof of Theorem 2.2.1, depending on the positions of the maximum and the minimum colored vertices, here also we choose a P_{k+1} subpath such that $\alpha + \beta \geq s$. If $i \leq s$, then we choose the path $v_iv_{i+1}v_{i+2} \dots v_{i+p} \dots v_{i+k}$. So, we get $\alpha = \frac{k}{2}$ and by Theorem 2.1.2, $rc_k(f) \geq \frac{k^2+k+2}{2}$, which is a contradiction to Theorem 1.3.2 if $s \neq \frac{k}{2}$. If $s < i \leq p$, then by choosing $v_sv_{s+1}v_{s+2} \dots v_{s+p} \dots v_{s+k}$ subpath, we get $\alpha \geq s$. If $j \geq k+1$, then as in Case I of the proof of Theorem 2.2.1, we get contradiction if $s \neq \frac{k}{2}$. Also, if $j > p+s$, then similar to the proof of Theorem 2.2.1, we get $\beta \geq s$. Suppose that $p+1 \leq i < j \leq p+s$.

Case I: $s = 2l$

If $i > p+l$, then by choosing the path $v_1v_2v_3 \dots v_{p+1} \dots v_{k+1}$, we get $\alpha \geq l$ and $\beta \geq l+1$. If $j \leq p+l$, then by considering the subpath $v_sv_{s+1}v_{s+2} \dots v_{s+p} \dots v_{s+k}$, we get $\beta \geq l$ and $\alpha \geq l+1$. Suppose $p+1 \leq i \leq p+l < j \leq p+s$. Let $i = p+l+1-l_1$ and $j = p+l+l_2$, where $1 \leq l_1 \leq l$ and $1 \leq l_2 \leq l$. The cases $l_1 < l_2$ and $l_1 > l_2$ are similar to Case I in proof of Theorem 2.2.1. If $l_1 = l_2$, we choose $L_0 = \{v_p\}$. So, we get $|L_0| = |L_p| = 1$ and $|L_t| = 2, t = 1,2,3,\dots,p-1$. Also, $\alpha + \beta = p+l+1-l_1-p+p+l+l_2-p = 2l+1 = s+1$. Now by Theorem 2.1.1, $rc_k(f) \geq 2p-2p+2p+2\sum_{t=1}^{p-1} 2(p-t)+s+1 = \frac{k^2+2s+2}{2}$.

Case II: $s = 2l + 1$

If $i > p+l+1$ or $j \leq p+l$, then as in Case I, we get $rc_k(f) \geq \frac{k^2+2s+1}{2}$. So, we assume that $p+1 \leq i < p+l+1 < p+l+2 < j \leq p+s$. Let $i = p+l+1-l_1$ and $j = p+l+1+l_2$, where $1 \leq l_1 \leq l$ and $0 \leq l_2 \leq l-1$. Rest of the proof is similar to that of Case I. □

Theorem 2.3.2. *If $k > 7$ is even and $n = \frac{3k+2}{2}$, then $rc_k(P_n) = \frac{k^2+k+2}{2}$.*

Proof: From Theorem 2.3.1, we have $rc_k(P_{\frac{3k}{2}}) = \frac{k^2+k+2}{2}$. By Lemma 2.1.4 and Theorem 1.3.4, we get the result. □

Lemma 2.3.3. *Let $k = 2p > 7$ and f be a minimal radio k-coloring of $P_n : v_1v_2\ldots v_n$, where $n = \frac{3k+2}{2}$. If $f(v_i) = 1$ and $f(v_j) = \frac{k^2+k+2}{2}$, then $\{i, j\} = \{p+1, n-p\}$.*

Proof: Let $f(v_i) = 1$ and $f(v_j) = \lambda$, where $\lambda = \frac{k^2+k+2}{2}$. Without loss of generality, we assume that $i < j$. To prove $i = p+1$ and $j = n-p$, we first show that $j - i = p$. Suppose that $j - i < p$. If $j > k$, then by choosing $v_{j-k}v_{j-k+1}v_{j-k+2}\ldots v_{j-p}\ldots v_j$ path and if $i \leq p+1$, then by choosing $v_iv_{i+1}v_{i+2}\ldots v_{i+p}\ldots v_{i+k}$ path, we get $\alpha + \beta \geq \frac{k}{2}+1$, a contradiction, by Theorem 2.1.2, to the fact that $rc_k(f) = \frac{k^2+k+2}{2}$. If $i \geq \lceil \frac{3p+1}{2} \rceil$, then by considering $L_0 = \{v_p\}$ and using Theorem 2.1.1, we get a contradiction as $\alpha + \beta \geq \frac{k}{2}+2$. If $j \leq \lceil \frac{3p+1}{2} \rceil$, then by considering the path $v_{p+1}v_{p+2}v_{p+3}\ldots v_{2p+1}\ldots v_n$ we get a contradiction. So, $p+1 < i < \lceil \frac{3p+1}{2} \rceil < j \leq k$. Let $i = \lceil \frac{3p+1}{2} \rceil - l_1$ and $j = \lceil \frac{3p+1}{2} \rceil + l_2$. By applying Theorem 2.1.1 with $L_0 = \{v_{2p+2}\}$ if $l_1 \geq l_2$ and with $L_0 = \{v_p\}$ if $l_1 < l_2$, we get a contradiction to the fact that $rc_k(f) = \frac{k^2+k+2}{2}$. Therefore $j - i \not< p$. If $j - i > p$, then by considering an appropriate subpath of $k+1$ vertices (starting with v_i or ending with v_j), again we get a contradiction. Therefore $j - i = p$.

Next, we show that $i = p+1$ and $j = n-p$. For that, we first show that the colors 1 and λ are not repeated. Suppose $f(v_l) = 1$ for some $l \neq i$. Then $l \geq i+k+1$ and

$l-j=p$. Therefore $l=j+p=i+2p=i+k$, which is a contradiction. Hence, the color 1 is assigned to only v_i and by Lemma 2.1.3, the color λ is assigned only to v_j. Suppose that $i \leq p$. Then $v_{i+1}v_{i+2}v_{i+3}\ldots v_{i+p+1}\ldots v_{i+k+1}$ does not contain the color 1. Let λ_{min} be the minimum color used in $v_{i+1}v_{i+2}v_{i+3}\ldots v_{i+p+1}\ldots v_{i+k+1}$, say $f(v_t)=\lambda_{min}$. Since $rc_k(P_{k+1}) = \frac{k^2+6}{2}$ and the maximum color used is $\frac{k^2+k+2}{2}$, $\lambda_{min} \leq p-1$. Now, $p-2 \geq \lambda_{min}-1 \geq 2p+1-d(v_i,v_t) = 2p+1-(t-i)$, that is $t \geq i+p+3$. So, $\alpha = t-(i+p+1) = (t-i)-(p+1) \geq 2p+1-\lambda_{min}+1-(p+1) = p+1-\lambda_{min}$ and $\beta = 1$. Now, by Theorem 2.1.2, we get $rc_k(f) \geq \frac{k^2+2}{2} + p+1-\lambda_{min}+1+\lambda_{min}-1 = \frac{k^2+k+4}{2}$ which is a contradiction. Similarly, by considering the path $v_{j-k-1}v_{j-k}v_{j-k+1}\ldots v_{j-p-1}\ldots v_{j-1}$, we get a contradiction if $j > n-p$. Therefore $j=n-p$ and $i=p+1$. □

Example 2.3.4. For $k=2p=10$ and $n=\frac{3k+2}{2}=16$, only one minimal radio k-coloring (radio 10-coloring with span $\frac{k^2+k+2}{2}=56$) is possible for the path P_{16}, which is given in Figure 2.2. Here, the color 1 is used to the vertex $v_6 = v_{p+1}$ and the span 56 is used to the vertex $v_{11} = v_{n-p}$.

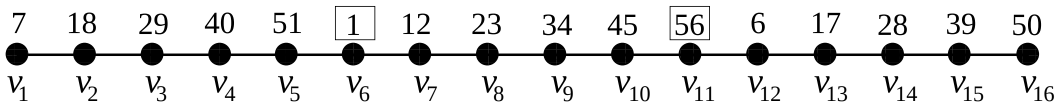

Figure 2.2 The minimal radio 10-coloring of P_{16}

Theorem 2.3.5. *If $k>7$ is even and $\frac{3k+4}{2} \leq n \leq \frac{5k+4}{2}$, then $rc_k(P_n) = \frac{k^2+k+4}{2}$.*

Proof: Let $k=2p$, $n=\frac{3k+4}{2}$, $P_n : v_1v_2\ldots v_n$ and $\lambda = \frac{k^2+k+4}{2}$. Suppose $rc_k(P_n)=\lambda$. Let f be a minimal radio k-coloring of P_n. Now, f restricted to $v_1v_2v_3\ldots v_{n-1}$ is a minimal radio k-coloring of P_{n-1}. By Lemma 2.3.3, we get $\{f(v_{p+1}), f(v_{n-1-p})\} = \{1,\lambda\}$. By restricting f to the path $v_2v_3\ldots v_n$ and using Lemma 2.2.3, we get $\{f(v_{p+2}), f(v_{n-p+1})\} = \{1,\lambda\}$. Therefore, $rc_k(P_n) \geq \frac{k^2+k+4}{2}$ and hence by Theorem 1.3.4, $rc_k(P_n) = \frac{k^2+k+4}{2}$.

□

2.4 SUMMARY

The radio k-chromatic number for path P_n is known for $k \geq n-3$ if k is odd and $k \geq n-4$ if k is even. In this chapter, we have determined $rc_k(P_n)$ for $\frac{2n+1}{7} \leq k \leq n-4$ if k is odd and for $\frac{2n-4}{5} \leq k \leq n-5$ if k is even. From Theorem 2.2.5 and Theorem 2.3.5, for the infinite path P_∞, $rc_k(P_\infty) \geq \frac{k^2+k+4}{2}$ which improves the lower bound given by Das et al. (2017) by one, a step towards Conjecture 1.3.5.

CHAPTER 3

THE RADIO k-CHROMATIC NUMBER OF TREES

> "*A diagram is worth of thousand proofs.*"
> - Carl E. Linderholm (1971)

Trees are used to analyze networks or structures and naturally arise in many areas of computer science, especially in data storage, searching and communication. We dedicate this chapter completely for the radio k-chromatic number of trees. Let G and H be two graphs and let u and v be two vertices of G and H respectively. By connecting G and H at u and v, we mean adding an edge between u and v. The graph obtained by merging u and v into a single vertex is called the concatenation of G and H at the vertices u and v. Let T_1 and T_2 be rooted trees such that the number of vertices in the i^{th} level of T_1 is equal to the the number of vertices in the $(p+1-i)^{\text{th}}$ level of T_2, $i = 1, 2, 3, \ldots, p$, where $p \geq 1$ is the number of levels in T_1 and T_2. Let T be the tree obtained by connecting the trees T_1 and T_2 at the roots and let $\mathscr{G}$ be the set of all such trees over all trees T_1 and T_2. Let T_1 and T_2 be rooted trees as described in the above with an extra condition that T_1 has at least two vertices in the first level. Let T' be the tree obtained by concatenation of T_1 and T_2 at the roots and let $\mathscr{G}'$ be the set of all such trees over all trees T_1 and T_2. Recall that finding the radio k-chromatic number of a graph G for $k > diam(G)$, is useful to determine the radio k-chromatic number of graphs containing G.

In this chapter, we first give an upper bound for the radio k-chromatic number of any tree T in $\mathscr{G}$ or $\mathscr{G}'$, when $k \geq diam(T)$, and prove that if k and $diam(T)$ are of the same parity, then the upper bound matches with the lower bound obtained from the lower bound technique given by Das et al. (2017). Later, we determine the radio d-chromatic number of trees and graphs constructed from the trees of diameter d in some subclasses of $\mathscr{G}$ or $\mathscr{G}'$.

3.1 ON THE RADIO k-CHROMATIC NUMBER OF TREES IN $\mathscr{G}$ AND $\mathscr{G}'$

In this section, for $k \geq diam(T)$, we define a radio k-coloring of any tree T in $\mathscr{G}$ or $\mathscr{G}'$, whose span, when k and $diam(T)$ are of the same parity, matches with the lower bound that we obtain using Theorem 3.1.1. Also, when k and $diam(T)$ are of different parity, we obtain a lower bound for $rc_k(T)$, which is $n+1$ less than the upper bound, where n is the order of T. As a direct consequence of Theorem 1.4.3, we have the theorem below which we use to get the lower bound for the radio k-chromatic number of trees under discourse.

Theorem 3.1.1. *For any graph G and any positive integer k, we have*

$$rc_k(G) \geq \begin{cases} |D_k| - 2p + 2\sum_{i=0}^{p} |L_i|(p-i) & if\ k = 2p+1, \\ |D_k| - 2p + 2\sum_{i=0}^{p} |L_i|(p-i) + 1 & if\ k = 2p. \end{cases}$$

Let $T \in \mathscr{G}$. Then $diam(T)$ is $2p+1$ and so the center of T is an edge, say uv. Let $L_0^u = \{u\}$, $L_0^v = \{v\}$, $L_1^u = N(L_0^u) \backslash L_0^v$, $L_1^v = N(L_0^v) \backslash L_0^u$, $L_{i+1}^u = N(L_i^u) \backslash L_{i-1}^u$ and $L_{i+1}^v = N(L_i^v) \backslash L_{i-1}^v$, $i = 1, 2, 3, \ldots, p-1$. Let $T' \in \mathscr{G}'$. Then $diam(T')$ is $2p$ and so the center of T' is a vertex, say u. If T' is the concatenation of trees T_1 and T_2, then u is the merged vertex. Let $L_0 = \{u\}$, $L_1^l = N(L_0) \cap V(T_1)$, $L_1^r = N(L_0) \cap V(T_2)$. Now, we define $L_2^l = N(L_1^l)\backslash L_0$, $L_2^r = N(L_1^r)\backslash L_0$ and $L_{i+1}^l = N(L_i^l)\backslash L_{i-1}^l$ and $L_{i+1}^r = N(L_i^r)\backslash L_{i-1}^r$, $i = 2,3,4,\ldots,p-1$. For $T \in \mathscr{G}$, if L_0 in Theorem 1.4.3 is $\{u,v\}$, then $L_i = L_i^u \cup L_i^v$, $i = 0,1,2,\ldots,p$. For $T \in \mathscr{G}'$, if L_0 in Theorem 1.4.3 is $\{u\}$, then $L_i = L_i^l \cup L_i^r$, $i = 1,2,3,\ldots p$. It is easy to see that $|L_i^u| = |L_{p+1-i}^v|$ and $|L_i^l| = |L_{p+1-i}^r|$ for $i = 1,2,\ldots,p$.

In the following two theorems, we give upper bounds for the radio k-chromatic number of trees in $\mathscr{G}$ and trees in $\mathscr{G}'$ when k is at least diameter.

Theorem 3.1.2. *If $T \in \mathscr{G}$ is a tree of order n, diameter $d = 2q+1$ and $k \geq d$, then*

$$rc_k(T) \leq \begin{cases} (2p-q)n + 2(q-p) + 2 & if\ k = 2p+1, \\ (2p-q-1)n + 2(q-p) + 3 & if\ k = 2p. \end{cases}$$

Proof: First, we order the vertices of T as follows. Let uv be the center of T. Let $x_1 = u$ and $x_n = v$. We label the vertices of L_{q+1-i}^v, $i = 1,2,3,\ldots,q$, as x_{2j}, $j = 1,2,3,\ldots,\frac{n}{2}-1$, starting from the vertices of L_q^v, and once all the vertices of L_q^v are labeled, we label the vertices of L_{q-1}^v and so on. Now, we label the vertices of L_i^u, $i = 1,2,3,\ldots,q$, as x_{2j+1}, $j = 1,2,3,\ldots,\frac{n}{2}-1$, starting from the vertices of L_1^u, and once all the vertices of L_1^u are labeled, we label the vertices of L_2^u and so on. In this labeling, if $x_s \in L_{q+1-i}^v$, then $x_{s-1} \in L_i^u$ or L_{i-1}^u. If $x_s \in L_{q+1-i}^v$ and $x_{s-1} \in L_i^u$, then $d(x_s,x_{s-1}) = p+2$. If $x_s \in L_{q+1-i}^v$ and $x_{s-1} \in L_{i-1}^u$, then $d(x_s,x_{s-1}) = q+1$. For each $i = 1,2,3,\ldots,q+1$, $x_s \in L_{q-i}^v$ and $x_{s-1} \in L_i^u$ happens exactly once. If $x_s \in L_i^u$, then $x_{s-1} \in L_{q+1-i}^v$ and $d(x_s,x_{s-1}) = q+2$. Therefore,

$$\begin{aligned} \sum_{s=2}^{n} d(x_s,x_{s-1}) &= (q+1)(q+1) + ((n-1)-(q+1))(q+2) \\ &= (q+2)n - 2q - 3. \end{aligned}$$

Now, we define a coloring f by $f(x_1) = 1$ and $f(x_j) = f(x_{j-1}) + (1 + k - d(x_j,x_{j-1}))$,

$2 \le j \le n$. Now, we show that f is a radio k-coloring of T. We have, $f(x_{s+1}) - f(x_s) \ge k - q - 1$ as $d(x_{s+1,x_s}) \le q+2$. So, $|f(x_t) - f(x_s)| \ge 2k - 2q - 2 > k$ if $1 \le t < s-2$ or $s+2 < t \le n$. Therefore, it is enough to check the radio k-coloring condition for the pairs $\{x_s, x_{s-2}\}$ and $\{x_s, x_{s+2}\}$. Now,

$$\begin{aligned} f(x_s) - f(x_{s-2}) &= f(x_s) - f(x_{s-1}) + f(x_{s-1}) - f(x_{s-2}) \\ &\ge (1+k-q-2) + (1+k-q-1) \\ &= 2k - 2q - 1 \\ &\ge k. \end{aligned}$$

Similarly, $f(x_{s+2}) - f(x_s) \ge k$. Hence, f is a radio k-coloring of G. By the definition of f, it is clear that $\varepsilon_i = 0$, $2 \le i \le n$. Now by Lemma 1.4.2, we have

$$rc_k(f) = (n-1)(k+1) - ((q+2)n - 2q - 3) + 1.$$

Therefore,

$$rc_k(T) \le \begin{cases} (2p-q)n + 2(q-p) + 2 & if\ k = 2p+1, \\ (2p-q-1)n + 2(q-p) + 3 & if\ k = 2p. \end{cases}$$

□

Example 3.1.3. The tree in Figure 3.1 is a tree from $\mathscr{G}$ and is of diameter $d = 2(3)+1$, edge uv is its center. It is labeled as in the proof of Theorem 3.1.2. Note that $L_0^u = \{u\} = \{x_1\}$, $L_0^v = \{v\} = \{x_{30}\}$, $L_1^v = \{x_{20}, x_{22}, x_{24}, x_{26}, x_{28}\}$, $L_2^v = \{x_8, x_{10}, x_{12}, x_{14}, x_{16}, x_{18}\}$, $L_3^v = \{x_2, x_4, x_6\}$, $L_1^u = \{x_3, x_5, x_7\}$, $L_2^u = \{x_9, x_{11}, x_{13}, x_{15}, x_{17}, x_{19}\}$ and $L_3^u = \{x_{21}, x_{23}, x_{25}, x_{27}, x_{29}\}$. For the same tree, the radio 7-coloring in the proof of Theorem 3.1.2, is given in Figure 3.2.

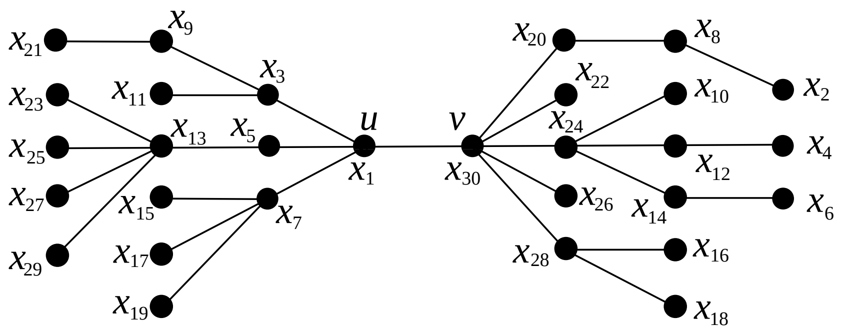

Figure 3.1 Labeling of a tree in $\mathscr{G}$ as in the proof of Theorem 3.1.2

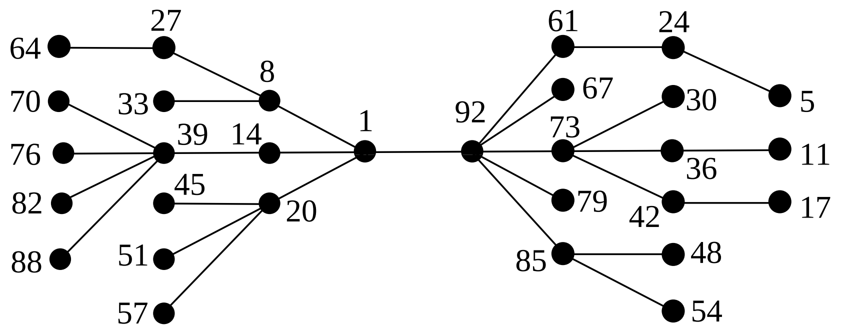

Figure 3.2 The radio 7-coloring in the proof of Theorem 3.1.2 for the tree in Figure 3.1

Theorem 3.1.4. *If $T \in \mathscr{G}'$ is a tree of order n, diameter $d = 2q$ and $k \geq d$, then*

$$rc_k(T) \leq \begin{cases} (2p-q)n - 2p + q + 2 & if\ k = 2p, \\ (2p-q+1)n - 2p + q + 1 & if\ k = 2p+1. \end{cases}$$

.

Proof: To define a radio k-coloring of T, we first order the vertices of T. Let u be the center of T. Let $x_1 = u$, and x_2 be a vertex of L^l_q chosen arbitrarily. We label the vertices of L^r_{q+1-i}, $i = 1,2,3,\ldots,q$, as x_{2j+1}, $j = 1,2,3,\ldots,\frac{n-1}{2}$, starting from the vertices of L^r_q and once all the vertices of L^r_q are labeled, we label the vertices of L^r_{q-1} and so on. Now, we choose x_4 from L^l_1 such that x_4 is not on $u - x_2$ path. We label the vertices of L^l_i, $i = 1,2,3,\ldots,q$, as x_{2j}, $j = 3,4,5,\ldots,\frac{n-1}{2}$, starting from the vertices of L^l_1 and once all the vertices of L^l_1 are labeled, we label the vertices of L^l_2 and so on. For $4 \leq s \leq n$, if $x_s \in L^r_{q+1-i}$, then $x_{s-1} \in L^l_i$ or L^l_{i-1}. If $x_s \in L^r_{q+1-i}$ and $x_{s-1} \in L^l_i$, then $d(x_s, x_{s-1}) = q+1$. If $x_s \in L^r_{q+1-i}$ and $x_{s-1} \in L^l_{i-1}$, then $d(x_s, x_{s-1}) = q$. For each

$i=2,3,4,\ldots,q$, $x_s \in L^r_{q+1-i}$ and $x_{s-1} \in L^l_{i-1}$ happens exactly once. For $4 \le s \le n$, if $x_s \in L^l_i$, then $x_{s-1} \in L^r_{q+1-i}$ and $d(x_s,x_{s-1}) = q+1$. Therefore,

$$\begin{aligned}\sum_{s=2}^{n} d(x_s,x_{s-1}) &= q+2q+q(q-1)+((n-3)-(q-1))(q+1) \\ &= (q+1)n-q-2.\end{aligned}$$

Now, we define a coloring f by $f(x_1)=1$ and $f(x_s) = f(x_{s-1}) + (1+k-d(x_s,x_{s-1}))$, $2 \le s \le n$. Since $d(x_2,x_3) = 2q$ and $d(x_3,x_4) = q+1$, $f(x_3) = f(x_2)+1+k-2q$ and $f(x_4) = f(x_3)+k-q$. Also, by the choice of x_4, $d(x_2,x_4) = q+1$. Therefore, $|f(x_4) - f(x_2)| = 1+2k-3q \ge 1+k-q > 1+k-d(x_2,x_4)$. Similar to Theorem 3.1.2, we can prove the radio k-coloring condition for the remaining pairs of vertices. From Lemma 1.4.2, $rc_k(f) = (2p-q)n-2p+q+2$ if $k=2p$ and $rc_k(f) = (2p-q+1)n-2p+q+1$ if $k=2p+1$. □

Example 3.1.5. The tree in Figure 3.3 is a tree from $\mathscr{G}'$ and is of diameter $d = 2(3)$, u is its center. It is labeled as in the proof of Theorem 3.1.4. Here $L_0 = \{u\} = \{x_1\}$, $L^l_1 = \{x_4,x_6\}$, $L^l_2 = \{x_8,x_{10},x_{12},x_{14},x_{16}\}$, $L^l_3 = \{x_2,x_{18},x_{20},x_{22},x_{24}\}$, $L^r_1 = \{x_{17},x_{19},x_{21},x_{23},x_{25}\}$, $L^r_2 = \{x_7,x_9,x_{11},x_{13},x_{15}\}$ and $L^r_3 = \{x_3,x_5\}$. For the same tree, in Figure 3.4, the radio 6-coloring in the proof of Theorem 3.1.4 is given.

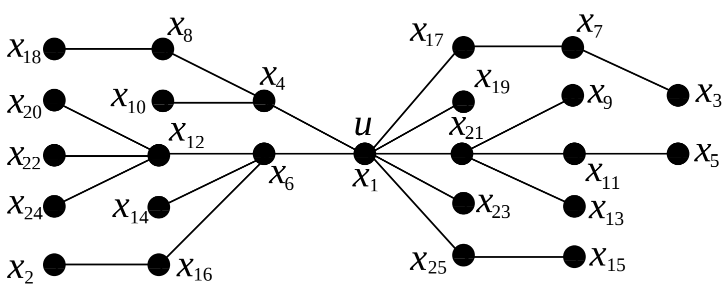

Figure 3.3 Labeling of a tree in $\mathscr{G}'$ as in the proof of Theorem 3.1.4

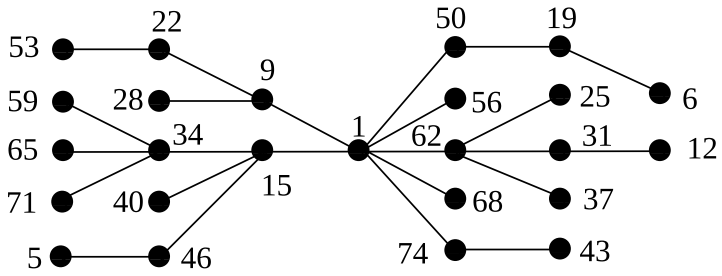

Figure 3.4 The radio 6-coloring in the proof of Theorem 3.1.4 for the tree in Figure 3.3

In the two theorems below, we use Theorem 3.1.1 to get lower bounds for $rc_k(T)$, $k \geq diam(T)$, of the trees in $\mathscr{G}$ and $\mathscr{G}'$.

Theorem 3.1.6. *If $T \in \mathscr{G}$ is a tree of order n, diameter $d = 2q+1$ and $k \geq d$, then*

$$rc_k(T) \geq \begin{cases} (2p-q)n+2(q-p)+2 & if\ k = 2p+1, \\ (2p-q-2)n+2(q-p)+4 & if\ k = 2p. \end{cases}$$

Proof: **Case 1:** $k = 2p+1$

Let uv be the center of T. We choose $L_0 = \{u, v\}$. So, we get $L_i = L_i^u \cup L_i^v$, $|L_0| = 2$, $|L_i| = |L_i^u| + |L_i^v| = |L_i^u| + |L_{q+1-(q+1-i)}^v| = |L_i^u| + |L_{q+1-i}^u|$, $i = 1, 2, 3, \ldots q$, $|L_i| = 0$ for $q < i \leq p$ and $|D_k| = |V(T)| = n$. Then by Theorem 3.1.1, we get

$$\begin{aligned} rc_k(T) &\geq n - 2p + 2p|L_0| + 2\sum_{i=1}^{p} |L_i|(p-i) \\ &\geq n + 2p + 2\sum_{i=1}^{q} (|L_i^u| + |L_{q+1-i}^u|)(p-i) \\ &= n + 2p + (2p-q-1)\sum_{i=1}^{q} 2|L_i^u| \\ &= n + 2p + (2p-q-1)(n-2) \\ &= (2p-q)n + 2(q-p) + 2. \end{aligned}$$

Case 2: $k = 2p$

It is easy to see that $rc_k(G) \geq rc_{k-1}(G)$. Therefore, $rc_k(T) \geq rc_{k-1}(T) = rc_{2p-1}(T) \geq (2(p-1)-q)n + 2(q-(p-1)) + 2 = (2p-q-2)n + 2(q-p+1) + 2.$ $\square$

Theorem 3.1.7. *If $T \in \mathscr{G}'$ is a tree of order n, diameter $d = 2q$ and $k \geq d$, then $rc_k(T) \geq (2p-q)n - 2p + q + 2$, where $k = 2p$ or $k = 2p+1$.*

Proof: Proof is similar to that of Theorem 3.1.6. □

The following theorems are the main results of this section which we get from the above theorems.

Theorem 3.1.8. *If $T \in \mathscr{G}$ is a tree of order n and $2q+1 = diam(T) \leq k = 2p+1$, then $rc_k(T) = (2p-q)n + 2(q-p) + 2$.*

Theorem 3.1.9. *If $T \in \mathscr{G}'$ is a tree of order n and $2q = diam(T) \leq k = 2p$, then $rc_k(T) = (2p-q)n - 2p + q + 2$.*

3.2 ON THE RADIO *k*-CHROMATIC NUMBER OF TREES AND GRAPHS CONSTRUCTED FROM SOME TREES IN $\mathscr{G}$ AND $\mathscr{G}'$

In this section, we determine the radio d-chromatic number of trees obtained by connecting some trees of diameter d in $\mathscr{G}$ ($\mathscr{G}'$). Also, we determine the radio d-chromatic number of graphs obtained from trees of diameter d in some subclasses of $\mathscr{G}$ ($\mathscr{G}'$).

3.2.1 Construction from Trees in $\mathscr{G}$

Let $\mathscr{H}$ be the collection of all the trees $T \in \mathscr{G}$ such that $|L_i^u| > 1$ for all $i = 1,2,3,\ldots,p$, where uv is the center of T and $diam(T) = 2p+1$. Let $T' \in \mathscr{H}$ be a tree of diameter $d = 2p+1$, center wz, $|L_i^w| > \ell$ for $i = 1,2,3,\ldots,p$ and L_p^w has ℓ vertices which are at distance $d-1$ from each other. Let $\mathscr{H}_{T',\ell}$ denotes the set of all trees $T \in \mathscr{H}$ such that $|L_i^u| = |L_i^w|$, $i = 1,2,3,\ldots,p$, where uv is the center of T and $diam(T) = diam(T') = 2p+1$. It is easy to see that $T' \in \mathscr{H}_{T',\ell}$.

In this subsection, we construct trees by connecting any tree T in $\mathscr{H}_{T',\ell}$ to one or more copies of T' in $\mathscr{H}$, and prove that the radio d-chromatic number of the constructed trees is same as the radio number of T' which is same as the radio number of T. Later, for given $d = 2p+1 \geq 3$, even integer $n \geq 2d$ and $n' \geq n$, we prove the existence of a tree of order n' and having the radio d-chromatic number $pn+2$. Similarly, for given $d = 2p+1 \geq 3$, even integer $n \geq 2d$ and $d' \geq d$, we prove the existence of a tree of diameter d' and having the radio d-chromatic number $pn+2$. Further, we construct graphs from any tree T in $\mathscr{H}$ such that the radio d-chromatic number of the constructed graphs is same as the radio number of T.

Theorem 3.2.1. *Let $T' \in \mathscr{H}$ be a tree of order n and diameter $d = 2p+1$, and let $T \in \mathscr{H}_{T',\ell}$. If $\ell \geq 2$, then there exist trees T_j^t of order $(j+1)n$, $j = 1,2,3,\ldots \ell$, $t = 1,2,3,\ldots,\binom{\ell}{j}$, such that $rc_d(T_j^t) = rn(T) = rn(T')$ and diameter of T_j^t is either $2d+1$ or $2d+2$. If $\ell = 0$, then there exists a tree T^* of order $2n$ and diameter $2d+1$ such that $rc_d(T^*) = rn(T) = rn(T')$.*

Proof: We label the vertices of T and T' with x_s and y_s, $1 \leq s \leq n$, as in Theorem 3.1.2 but with a variation explained as follows. In T, while labeling the vertices of L^v_{p+1-i}, $2 \leq i \leq p$, we first label the vertex on $x_2 - x_n$ path. If $\ell \geq 2$, then we label the ℓ vertices of L^w_p which are at distance $d-1$ from each other as $y_{n-1}, y_{n-3}, y_{n-5}, \ldots, y_{n-(2l-1)}$. Also, while labeling the vertices of L^w_i, $1 \leq i \leq p-1$, we use the last ℓ labels to the vertices on the paths $y_{n-1} - y_1, y_{n-3} - y_1, y_{n-5} - y_1, \ldots, y_{n-(2\ell-1)} - y_1$, in the order highest to lowest. If $\ell = 0$, then while labeling the vertices of L^w_i, $1 \leq i \leq p-1$, we use the last label to the vertex on the path $y_{n-1} - y_1$. Now, we consider the radio colorings f and g of T and T', respectively, defined by $f(x_1) = g(y_1) = 1$, $f(x_s) = f(x_{s-1}) + (1+d-d(x_s, x_{s-1}))$ and $g(y_s) = g(y_{s-1}) + (1+d-d(y_s, y_{s-1}))$, $2 \leq s \leq n$ (which are same as the radio coloring defined in Theorem 3.1.2). Suppose that $\ell \geq 2$. Let $T_1', T_2', T_3', \ldots, T_\ell'$ be ℓ copies of T'. We connect T and T_s' at x_2 and $y_{n-(2j-1)}$, $j = 1,2,3,\ldots,\ell$. For $j = 1,2,3,\ldots,\ell$, let T_j be a tree obtained by connecting j trees among $T_1', T_2', T_3', \ldots, T_\ell'$

to T. Since choosing j trees among $T_1', T_2', T_3', \ldots, T_\ell'$ has $\binom{\ell}{j}$ possibilities, we get the trees T_j^t, $t = 1, 2, 3, \ldots, \binom{\ell}{j}$. If $\ell = 0$, then we get the tree T^* by connecting T and T' at x_2 and y_{n-1}.

First, we show that the resultant coloring h of T_j^t is a radio d-coloring of T_j^t. The radio d-coloring condition is clearly satisfied among the copies of T'. It is remained to show the condition between the vertices of T and a copy of T'. We check the condition between T and T_1'. It is enough to check the radio d-coloring condition for x_s with y_{s-1}, y_s and y_{s+1} as the colors of the remaining y_rs differ by at least d from the color of x_s. If s is odd, then $d(x_s, y_{s-1}), d(x_s, y_s)$ and $d(x_s, y_{s+1})$ are at least d. Suppose s is even. Then clearly $d(x_s, y_s) > d$. If $x_s \in L_{p+1-i}^v$, then $x_{s-1} \in L_{i-1}^u$ or L_i^u and hence $y_{s-1} \in L_{i-1}^w$ or L_i^w. If $y_{s-1} \in L_{i-1}^w$, then both x_s and y_{s-1} are on $x_n - y_1$ path. Therefore, $d(x_s, y_{s-1}) = d(x_s, x_2) + 1 + d(y_{n-1}, y_{s-1}) = i - 1 + 1 + p - i - 1 = p + 1 = d(x_s, x_{s-1})$. If $y_{s-1} \in L_i^w$, then y_{s-1} cannot be on $y_{n-1} - y_1$ path. Therefore, $d(x_s, y_{s-1}) = d(x_s, x_2) + 1 + d(y_{n-1}, y_{s-1}) \geq i - 1 + 1 + p + 1 - i + 2 = p + 3 > d(x_s, x_{s-1})$. So, in any case, $d(x_s, y_{s-1}) \geq d(x_, x_{s-1})$. Hence $h(x_s) - h(y_{s-1}) = f(x_s) - g(y_{s-1}) = f(x_s) - f(x_{s-1}) \geq 1 + d - d(x_, x_{s-1}) \geq 1 + d - d(x_, y_{s-1})$. Therefore, radio d-coloring condition is satisfied for x_s and y_{s-1}. If $x_s \in L_{p+1-i}^v$, then $y_{s+1} \in L_i^w$. By the choice of vertices on $x_2 - x_n$ path and $y_{n-1} - y_1$ path, at most one of x_s and y_{s+1} can be on the $x_n - y_1$ path. Therefore, $d(x_s, y_{s+1}) = d(x_s, x_2) + 1 + d(y_{n-1}, y_{s+1}) \geq i - 1 + 1 + p - i + 2 = p + 2 = d(x_s, x_{s-1})$. Hence $h(y_{s+1}) - h(x_s) = g(y_{s+1}) - f(y_s) = f(x_{s+1}) - f(x_s) \geq 1 + d - d(x_s, x_{s+1}) \geq 1 + d - d(x_s, y_{s+1})$. Since T is a subtree of T_j^t, $rc_d(T_j^t) = rn(T) = rn(T')$. It is easy to see that the order of T_j^t is $(j+1)n$ and the diameter of T_j^t is $2d + 1$ if $j = 1$, else $2d + 2$. Similarly, it is easy to see that the resultant coloring of T^* is also a radio d-coloring of T^*. □

Example 3.2.2. In Figure 3.5, a tree T' in $\mathscr{H}$ and a tree T in $\mathscr{H}_{T',2}$ are given. The vertices of the trees T and T' are labeled as in the proof of Theorem 3.2.1. Also y_{19} and y_{21} are the two vertices in L_3^w such that $d(y_{19}, y_{21}) = 6 = diam(T') - 1$. In Figure 3.6, one copy of tree T is connected to two copies of T' at x_2, y_{21} (y_{n-1}) and x_2, y_{19} (y_{n-3}).

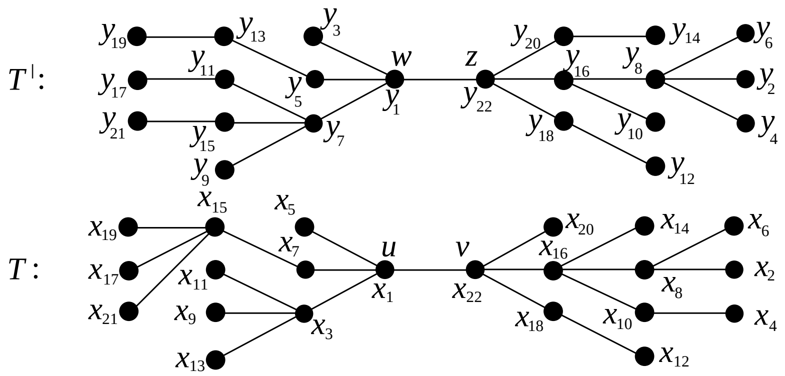

Figure 3.5 A tree $T' \in \mathscr{H}$ and a tree $T \in \mathscr{H}_{T',2}$ labeled as in the proof of Theorem 3.2.1

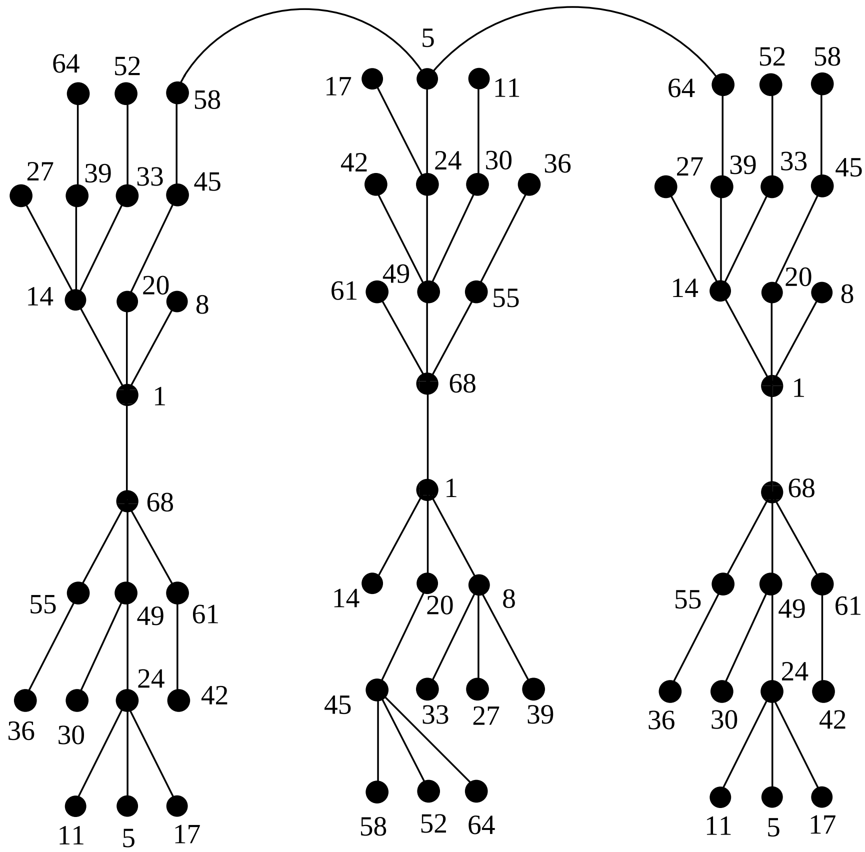

Figure 3.6 The trees in Figure 3.5 are connected as in the proof Theorem 3.2.1

Remark 3.2.3. Let $T_1', T_2', T_3', \ldots, T_\ell'$ be trees in $\mathscr{H}$ such that $T_i' \in \mathscr{H}_{T_j',\ell}$ for all i and j. Now, for any $T \in \bigcap_{i=1}^{\ell} \mathscr{H}_{T_i',\ell}$. Then, proof of Theorem 3.2.1 holds true if we replace l copies of T' by $T_1', T_2', T_3', \ldots, T_\ell'$ and labeling the vertices of $T_1', T_2', T_3', \ldots, T_\ell'$ similar to that of T', provided x_2 is not connected to two vertices of the same index. For the tree T in Theorem 3.2.1, suppose L_p^v has ℓ' number of vertices which are at distance $d-1$ from each other and $|L_i^v| > \ell'$, $i = 1,2,3,\ldots,p$. Then we label these ℓ' vertices as $x_2, x_4, x_6, \ldots, x_{2\ell'}$ and while labeling the vertices of L_{p+1-i}^v, $2 \leq i \leq p$, we use the first ℓ labels to the vertices on the paths $x_2 - x_n, x_4 - x_n, x_6 - x_n, \ldots, x_{2\ell'} - x_n$, in the order lowest to highest. We can connect T and T' at x_{2r} and $y_{n-(2s-1)}$ as in the proof of Theorem 3.2.1, provided the vertex of L_{p+1-i}^v on the path $x_{2r} - x_n$ and the vertex of L_i^w on the path $y_{n-(2s-1)} - y_1$ do not have consecutive indexed labels.

Theorem 3.2.4. *Let $T' \in \mathscr{H}$ be a tree of diameter $d = 2p+1$. If $T \in \mathscr{H}_{T',1}$, then there exist trees T_j of diameter $(j+1)d + j$, $j = 1,2,3,\ldots$, such that $rc_d(T_j) = rn(T) = rn(T')$.*

Proof: Let $x_1, x_2, x_3, \ldots, x_n$ be the labeling of vertices of T such that for each i, the labels of the vertices of L_i^v and L_i^u are same as that of L_i^v of T and L_i^w of T', respectively, as in the proof of Theorem 3.2.1 corresponding to $\ell = 0$ case. Let $y_1, y_2, y_3, \ldots, y_n$ be the labeling of vertices of T' such that for each i, the labels of the vertices of L_i^w and L_i^z are same as that of L_i^w of T' and L_i^v of T, respectively' as in the proof of Theorem 3.2.1 corresponding to $\ell = 0$ case. Let $T_1 = T^*$, where T^* is the tree in Theorem 3.2.1, obtained by connecting T and T' at x_2 and y_{n-1}. It is easy to see that the diameter and the order of T_1 are $2d+1$ and $2n$ respectively. Now to obtain T_j, $j = 2,3,4,\ldots$, we connect a copy of T (or T') to the last copy of tree which is connected in T_{j-1} at x_{n-1} (or y_{n-1}) and x_2 or y_2. It is easy to see that T_j is of order $(j+1)n$ and diameter $(j+1)d + j$. $\square$

For an odd integer $d = 2p+1 \geq 3$ and an even integer $n \geq 2d$, it is easy to see that there exists a tree $T \in \mathscr{H}$ of order n and diameter d.

Corollary 3.2.5. *Let $d = 2p+1$, $p \geq 1$, be an odd integer and $n \geq 2d$ be an even integer. Then for every integer $n' \geq n$, there exists a tree T^* of order n' such that $rc_d(T^*) = pn + 2$.*

Proof: Let $T \in \mathscr{H}$ be a tree of diameter d and order n. Among the trees T_j of order

$(j+1)n$ in Theorem 3.2.4, we consider the smallest tree T_t such that $n' \leq (t+1)n$. Now, we remove $(t+1)n - n'$ vertices from the last copy of the tree that is connected in T_t and get a tree T^* of order n' (removing pendant vertices recursively) with $rc_d(T^*) = rn(T) = pn+2$. $\square$

Corollary 3.2.6. *Let $d = 2p+1$, $p \geq 1$, be an odd integer and $n \geq 2d$ be an even integer. Then for every integer $d' \geq d$, there exists a tree T^* of diameter d' such that $rc_d(T^*) = pn+2$.*

Proof: Let $T \in \mathscr{H}$ be a tree of diameter d and order n. Among the trees T_j of diameter $(j+1)d+j$ in Theorem 3.2.4, we consider the smallest tree T_t such that $d' \leq (t+1)d+t$. Now, we remove all the pendant vertices of T_t in the last copy of tree that is connected in T_t to get a tree of diameter $(t+1)d+t-1$. We repeat this process $d' - (j+1)d - j + 1$ times to get T^*. $\square$

In the following theorem, we construct graphs from trees in $\mathscr{H}$.

Theorem 3.2.7. *Let $T \in \mathscr{H}$ be a tree of order n, diameter $d = 2p+1$, center uv, L_p^u has ℓ vertices which are at distance $d-1$ from each other and $|L_i^u| > \ell$ for all i. If $l \geq 2$, then there exist graphs G_j^t, $j = 1,2,3,\ldots,\ell$, $t = 1,2,3,\ldots,\binom{\ell}{j}$, of order n and size $n-1+j$, such that $rc_d(G_j^t) = rn(T)$. If $\ell = 0$ and $|L_i^u| > 1$ for all i, then there exists a graph G^* of order n and size n, such that $rc_d(G^*) = rn(T)$.*

Proof: Let $x_1, x_2, x_3, \ldots, x_n$ be the labeling of vertices of T such that for each i, the labels of the vertices of L_i^v and L_i^u are same as that of L_i^v of T and L_i^w of T', respectively, as in the proof of Theorem 3.2.1. We define a radio coloring f of T by $f(x_1) = 1$ and $f(x_s) = f(x_{s-1}) + (1 + d - d(x_s, x_{s-1}))$, $2 \leq s \leq n$. If $\ell \geq 2$, then we get a graph G_j by making j vertices among $x_{n-1}, x_{n-3}, \ldots, x_{n-(2\ell-1)}$ adjacent to x_2. Since choosing j vertices among $x_{n-1}, x_{n-3}, \ldots, x_{n-(2\ell-1)}$ has $\binom{\ell}{j}$ possibilities, we get the graphs G_j^t, $t = 1,2,3,\ldots,\binom{\ell}{j}$. Now, we prove that f is a radio d-coloring of G_j^t. If $\ell = 0$, then we get the graph G^* by making x_2 and x_{n-1} adjacent. First we prove that f is a radio d-coloring of G_j^t. It is enough to check the radio d-coloring condition for x_s with x_{s-1} and x_{s+1} as the colors of the remaining x_rs differ by at least d from the color of x_s. By the choice of vertices on the paths $x_n - x_2, x_{n-1} - x_1, x_{n-3} - x_1, x_{n-5} - x_1, \ldots, x_{n-(2\ell-1)} - x_1$ the distances of x_{s-1} and x_{s+1} from x_s in T and G_j^t are the same. Hence f is a radio d-coloring of G_j^t and $rc_d(G_j^t) = rn(T)$. Also, it is easy to see that G_j^t has $n-1+j$ edges. Similarly, we can prove for G^*. □

Example 3.2.8. For the tree T in Figure 3.7, $|L_i^u| > 2$. The graph in Figure 3.8 is the graph G_2^t constructed, as in the proof of Theorem 3.2.7, from the tree T.

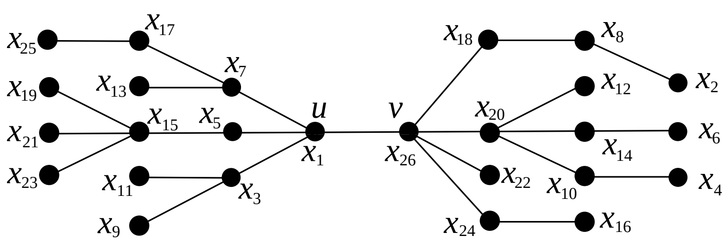

Figure 3.7 A tree $T \in \mathscr{H}$ labeled as in the proof of Theorem 3.2.7

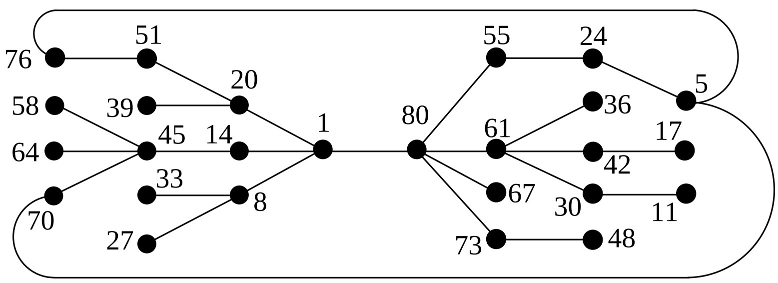

Figure 3.8 A graph obtained from the tree in Figure 3.7 as in the proof of Theorem 3.2.7

Remark 3.2.9. For the tree T in Theorem 3.2.7, suppose L_p^v has ℓ' number of vertices which are at distance $d-1$ from each other and $|L_i^v| > \ell$, $i = 1, 2, \ldots, p$. Then we label these ℓ' vertices as $x_2, x_4, x_6, \ldots, x_{2\ell'}$ and while labeling the vertices of L_{p+1-i}^v, $2 \le i \le p$, we use the first ℓ labels to the vertices on the paths $x_2 - x_n, x_4 - x_n, x_6 - x_n, \ldots, x_{2\ell'} - x_n$, in the order lowest to highest. We can make x_{2r}, $r = 2,3,4,\ldots,\ell'$, adjacent to $x_{n-(2s-1)}$, $s = 1,2,3,\ldots,\ell$, provided the vertex of L^v_{p+1-i} on the path $x_{2r} - x_n$ and the vertex of L^u_i on the path $x_{n-(2s-1)} - x_1$ do not have the consecutive indexed labels. Instead of $T \in \mathscr{G}$ taken in Theorem 3.2.7, we can consider any of the trees obtained in Theorem 3.2.1 or Theorem 3.2.4 to construct graphs as in Theorem 3.2.7.

3.2.2 Construction from Trees in $\mathscr{G}'$

To avoid ambiguity between trees T and T' in $\mathscr{G}'$, for the tree T', we use S_i, S^l_i and S^r_i in place of L_i, L^l_i and L^r_i respectively. Let $\mathscr{H}'$ be the collection of all the trees $T \in \mathscr{G}'$ such that L^l_p has at least two vertices at distance $diam(T) = 2p$ from each other and $|L^l_i| > 2$, $i = 1, p$. Let $T' \in \mathscr{H}'$ be a tree of diameter $d = 2p$, $|S^l_i| > \ell$ for $i = 1, p$ and S^l_p has $\ell \geq 2$ vertices which are at distance d from each other. Let $\mathscr{H}'_{T',\ell}$ denotes the set of all the trees $T \in \mathscr{G}'$ such that $diam(T) = diam(T')$ and $|L^l_i| = |S^l_i|$, $i = 1,2,3,\ldots,p$. It is easy to see that $T' \in \mathscr{H}'_{T',\ell}$.

In this subsection, we first give a construction of larger (in terms of diameter and order) trees from any tree T in $\mathscr{H}'$ such that the radio d-chromatic number of the constructed tree is same as the radio number of T. Later, for given $d = 2p \geq 2$, odd integer $n \geq 2d+5$ and $n' \geq n$, we prove the existence of a tree of order n' having the radio d-chromatic number $p(n-1)+2$. Similarly, for given $d = 2p \geq 2$, odd integer $n \geq 2d+5$ and $d' \geq d$, we prove the existence of a tree of diameter d' having the radio d-chromatic number $p(n-1)+2$. Further, we give a construction of graphs from any tree T in $\mathscr{H}$ such that the radio d-chromatic number of the constructed graphs is same as the radio number T.

Theorem 3.2.10. *Let $T' \in \mathscr{H}'$ be a tree of order n and diameter $d = 2p$. If $T \in \mathscr{H}'_{T',\ell}$, then there exist trees T^t_j of order $(j+1)n$, $j = 1,2,3,\ldots,\ell-1$, $t = 1,2,3,\ldots,\binom{\ell-1}{j}$, such that $rc_d(T^t_j) = rn(T) = rn(T')$ and diameter of T^t_j is either $2d+1$ or $2d+2$.*

Proof: We label the vertices of T with x_s, $1 \le s \le n$ as in Theorem 3.1.4 with a variation explained as follows. In T, while labeling the vertices of L^r_{p+1-i}, $2 \le i \le p$, we first label the vertices on $x_1 - x_3$ path. We label the ℓ vertices of S^l_p which are at distance d from each other as $y_2, y_{n-1}, y_{n-3}, y_{n-5}, \dots, y_{n-(2\ell-3)}$. Also, while labeling the vertices of S^l_i, $1 \le i \le p$, we use the last $\ell - 1$ labels to the vertices on the paths $y_{n-1} - y_1, y_{n-3} - y_1, y_{n-5} - y_1, \dots, y_{n-(2\ell-3)} - y_1$, in the order highest to lowest. Now, we consider the radio colorings f and g of T and T' respectively defined by $f(x_1) = g(y_1) = 1$, $f(x_s) = f(x_{s-1}) + (1 + d - d(x_s, x_{s-1}))$ and $g(y_s) = g(y_{s-1}) + (1 + d - d(y_s, y_{s-1}))$, $2 \le s \le n$ (which are same as the radio coloring defined in Theorem 3.1.4). Let $T'_1, T'_2, T'_3, \dots, T'_{\ell-1}$ be $\ell - 1$ copies of T'. We connect T and T'_s at x_3 and $y_{n-(2s-1)}$. For $j = 1, 2, 3, \dots, \ell - 1$, let T_j be a tree obtained by connecting j trees among $T'_1, T'_2, T'_3, \dots, T_{\ell-1}$ to T. Since choosing j trees among $T'_1, T'_2, \dots, T'_{\ell-1}$ has $\binom{\ell-1}{j}$ possibilities, we get trees T^t_j, $t = 1, 2, 3, \dots, \binom{\ell-1}{j}$. As in Theorem 3.2.1, we can prove that the coloring of T^t_j is a radio d-coloring. Therefore $rc_d(T^t_j) = rn(T) = rn(T')$ and the diameter of T^t_j is $2d + 1$ if $j = 1$, else $2d + 2$. □

Example 3.2.11. The tree $T' \in \mathscr{H}'$ in Figure 3.9 has three vertices (y_2, y_{24}, y_{26}) in S^l_3 which are at distance $6 = diam(T')$ from each other. Also, a tree $T \in \mathscr{H}'_{T',3}$ is given in Figure 3.9. The vertices of trees T and T' are labeled as in the proof of the Theorem 3.2.10. In Figure 3.10, a tree is constructed, as in the proof of Theorem 3.2.10, by connecting two copies of T' to T at y_{26} (y_{n-1}), x_3 and y_{24} (y_{n-3}), x_3.

Remark 3.2.12. Similar to Remark 3.2.3, here also, we get larger trees having the radio d-chromatic number same as the radio number of T.

Theorem 3.2.13. *Let $T' \in \mathscr{H}'$ be a tree of order n and diameter $d = 2p$. If $T \in \mathscr{H}'_{T',2}$, then there exist trees T_j of diameter $(j+1)d + j$, $j = 1, 2, 3, \dots,$ such that $rc_d(T_j) = rn(T) = rn(T')$.*

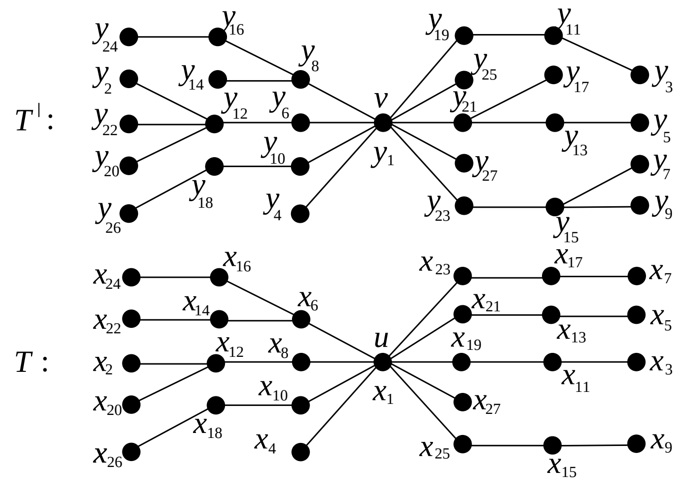

Figure 3.9 A tree $T' \in \mathscr{H}'$ and a tree $T \in \mathscr{H}'_{T',3}$

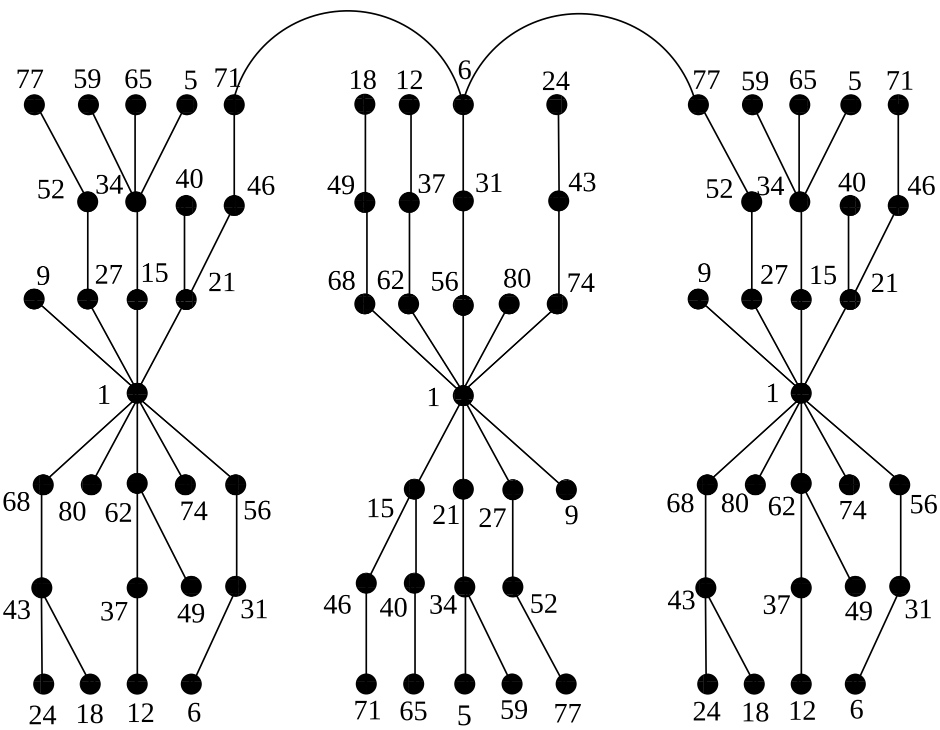

Figure 3.10 A tree constructed from the trees in Figure 3.9 as in the proof of Theorem 3.2.10

Proof: We consider the labeling of T as given in Theorem 3.2.10. Let $y_1, y_2, y_3, \ldots, y_n$ be the labeling of the vertices of T' such that for each i, the labels of the vertices of S_i^l and S_i^r are same as that of S_i^l of T' and L_i^r of T, respectively, as in the proof of Theorem 3.2.10. Let $T_1 = T_1^1$, be the tree in Theorem 3.2.10, obtained by connecting T and T' at x_3 and y_{n-1}. It is easy to see that the diameter and the order of T_1 are $2d+1$ and $2n$ respectively. Now to obtain $T_j,\ j = 2, 3, 4, \ldots,$ we connect a copy of T' to the last copy of T' which is connected in T_{j-1} at y_{n-1} and y_3. It is easy to see that T_j is of order $(j+1)n$ and with diameter $(j+1)d+j$. □

Remark 3.2.14. In addition to conditions in Theorem 3.2.13, suppose that $T \in \mathscr{H}'$. If we label the vertices of L_i^l similar to that of S_i^l in the proof of Theorem 3.2.13, then to get T_j, $j = 2,3,4,\ldots$, we connect a copy of T (or T') to the last copy of tree which is connected in T_{j-1} at x_{n-1} (or y_{n-1}) and x_3 or y_3.

Example 3.2.15. In Figure 3.11, a tree $T' \in \mathscr{H}'$ and a tree $T \in \mathscr{H}'_{T',1}$ are given. Also, $T \in \mathscr{H}'$. The vertices of T' are labeled as in Theorem 3.2.13 and the vertices of T are labeled as in Remark 3.2.14. Using the trees T and T', a tree T_3 of diameter 27 is constructed in Figure 3.12, as in Remark 3.2.14.

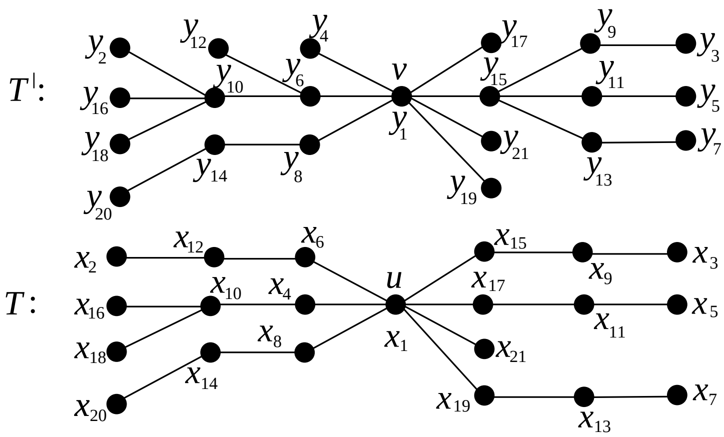

Figure 3.11 Two trees satisfying the condition in Remark 3.2.14 labeled as in the proof of Theorem 3.2.13

For an even integer $d = 2p \geq 3$ and an odd integer $n \geq 2d+5$, it is easy to see that there exists a tree $T \in \mathscr{H}'$ of order n and diameter d. The following corollaries are similar to Corollary 3.2.5 and Corollary 3.2.6 respectively.

Corollary 3.2.16. *Let $d = 2p$, $p \geq 1$, be an even integer and $n \geq 2d+5$ an odd integer. Then for every integer $n' \geq n$, there exists a tree T^* of order n' such that $rc_d(T^*) = p(n-1)+2$.*

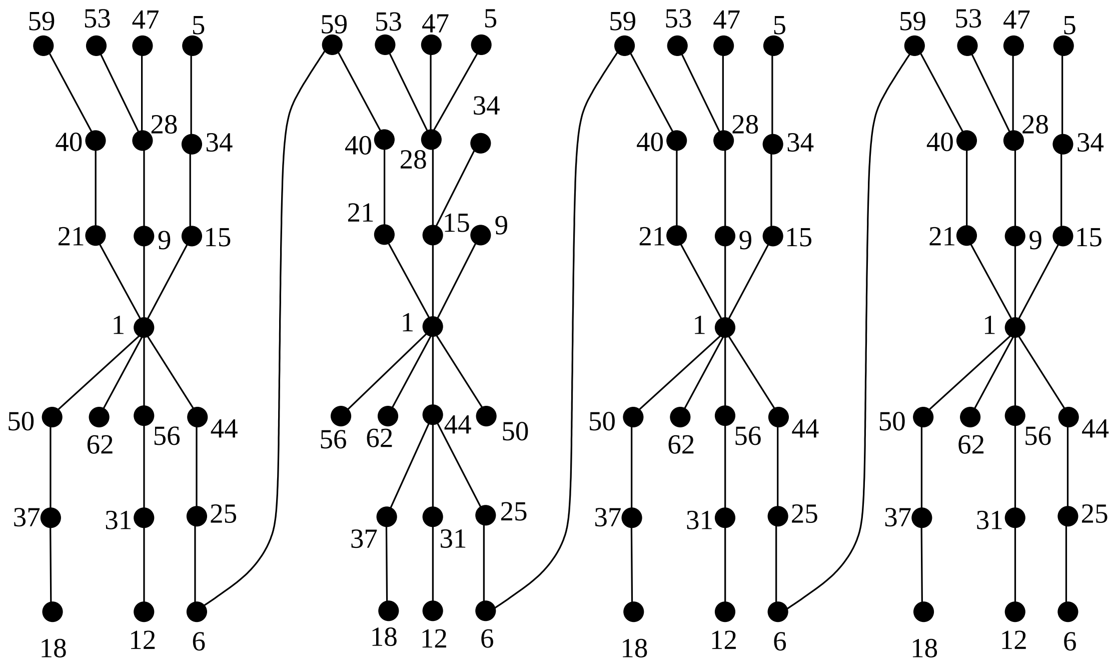

Figure 3.12 A tree is constructed as in Remark 3.2.14 from the trees in Figure 3.11

Proof: Let $T \in \mathscr{H}'$ be a tree of diameter d and order n. Among the trees T_j of order $(j+1)n$ in Theorem 3.2.13, we consider the smallest tree T_t such that $n' \leq (t+1)n$. Now, we remove $(t+1)n - n'$ vertices from the last copy of the tree that is connected in T_t and get a tree T^* of order n' (removing pendant vertices recursively) with $rc_d(T^*) = rn(T) = pn + 2$. □

Corollary 3.2.17. *Let $d = 2p$, $p \geq 1$, be an even integer and $n \geq 2d+5$ an odd integer. Then for every integer $d' \geq d$, there exists a tree T^* of diameter d' such that $rc_d(T^*) = p(n-1)+2$.*

Proof: Let $T \in \mathscr{H}'$ be a tree of diameter d and order n. Among the trees T_j of diameter $(j+1)d + j$ in Theorem 3.2.13, we consider the smallest tree T_t such that $d' \leq (t+1)d + t$. Now, we remove all the pendant vertices of T_t in the last copy of tree that is connected in T_t to get a tree of diameter $(t+1)d + t - 1$. We repeat this process $d' - (j+1)d - j + 1$ times to get T^*. □

Theorem 3.2.18. *Let $T \in \mathscr{H}'$ be a tree of order n, diameter $d = 2p$. If L_p^l has ℓ vertices at distance d from each other and $|L_i^l| > \ell$ for $i = 1, p$, then there exist graphs G_j^t, $j = 1,2,3,\ldots,\ell-1$, $t = 1,2,3,\ldots,\binom{\ell-1}{j}$, of order n and size $n-1+j$, such that $rc_d(G_j^t) = rn(T)$.*

Proof: Let $x_1, x_2, x_3, \ldots, x_n$ be the labeling of the vertices of T such that for each i, the labels of the vertices of L_i^r and L_i^l are same as that of L_i^r of T and S_i^l of T', respectively, as in the proof of Theorem 3.2.10. We define a radio coloring f of T by $f(x_1) = 1$ and $f(x_s) = f(x_{s-1}) + (1 + d - d(x_s, x_{s-1}))$, $2 \le s \le n$. We get a graph G_j by making j vertices among $x_{n-1}, x_{n-3}, \ldots, x_{n-(2\ell-3)}$ adjacent to x_3. Since choosing j vertices among $x_{n-1}, x_{n-3}, \ldots, x_{n-(2\ell-3)}$ has $\binom{\ell-1}{j}$ possibilities, we get the graphs G_j^t, $t = 1,2,3,\ldots,\binom{\ell-1}{j}$. Now, we prove that f is a radio d-coloring of G_j^t. It is enough to check the radio d-coloring condition for x_s with x_{s-1} and x_{s+1} as colors of remaining x_rs differ by at least d from the color of x_s. By the choice of vertices on paths $x_n - x_3, x_1 - x_{n-1}, x_1 - x_{n-3}, x_1 - x_{n-5}, \ldots, x_1 - x_{n-(2\ell-1)}$ distance of x_s from x_{s-1} and x_{s+1} in T and G_j^t are same. Hence f is a radio d-coloring of G_j^t and $rc_d(G_j^t) = rn(T)$. Also, it is easy to see that G_j^t has $n-1+j$ edges. □

Remark 3.2.19. For the tree T in Theorem 3.2.18, suppose that L_p^r has ℓ' number of vertices which are at distance d from each other. Then we label these ℓ' vertices as $x_3, x_5, x_7, \ldots, x_{2\ell'+1}$ and while labeling the vertices of L_{p+1-i}^v, $2 \le i \le p$, we use the first ℓ labels to the vertices on the paths $x_3 - x_1, x_5 - x_1, x_7 - x_1, \ldots, x_{2\ell'+1} - x_1$, in the order lowest to highest. We can make x_{2r+1}, $r = 2,3,4,\ldots,\ell'$, adjacent to x_{n-2s-1}, $s = 1,2,3,\ldots,\ell-1$, provided the vertex of L_{p+1-i}^v on the path $x_{2r+1} - x_1$ and vertex of L_i^u on the path $x_{n-(2s-1)} - x_1$ do not have the consecutive indexed labels. Instead of $T \in \mathscr{H}'$ taken in Theorem 3.2.18, we can consider any of the trees obtained in Theorem 3.2.10 or Theorem 3.2.13 to construct graphs as in Theorem 3.2.18.

3.3 SUMMARY

In this chapter, we have given an upper bound and a lower bound for the radio k-chromatic number of trees in $\mathscr{G}$ and trees in $\mathscr{G}'$, when k greater than or equal to the diameter of the tree. The upper bound matches with the lower bound when k and the diameter of the tree are of the same parity. Also, we have determined the radio d-chromatic number of the trees and graphs constructed from the trees in some subclasses of $\mathscr{G}$ and $\mathscr{G}'$.

CHAPTER 4

THE RADIO NUMBER FOR THE CARTESIAN PRODUCT OF COMPLETE GRAPH AND CYCLE

"The coloring of abstract graphs is a generalization of the coloring of maps, and the study of the coloring of abstract graphs opens a new chapter in the combinatorial part of mathematics."
- Gabriel Andrew Dirac (1951)

The Cartesian product, the direct product, and the strong product are the three fundamental products of graphs. These products have been widely investigated and have many significant applications. More details on products of graphs can be found in the handbook of Hammack et al. (2011). In the Cartesian products, radio k-coloring has been studied for $P_n \square P_m$, $C_n \square C_m$, $P_n \square K_m$ and $P_n \square C_m$. In this chapter, we determine the radio number for the Cartesian product of complete graph K_n and cycle C_m for the following values of m and n: (a) n even and m odd (b) any n and $m \equiv 6 \ (mod\ 8)$ (c) n odd and $m \equiv 5 \ (mod\ 8)$.

4.1 THE RADIO NUMBER OF $K_n \square C_m$ FOR n EVEN AND m ODD

In this section, we define a radio coloring of $K_n \square C_m$ for n even and m odd, whose span matches with the lower bound given in Theorem 1.4.4. To do this, we first order the vertices of $K_n \Box C_m$. The lemma below assures that such ordering exists.

Lemma 4.1.1. *If m odd and n even, then there exists an ordering* $x_1, x_2, x_3, \ldots, x_{mn}$ *of the vertices of* C_m*, which takes every vertex n times, such that* $\{d(x_i, x_{i-1})\}_{i=2}^{mn}$ *is an alternating sequence of* $\frac{m-1}{2}$ *and p, where*

$$p = \begin{cases} \frac{m+3}{4} & if\ m \equiv 1\ (mod\ 4), \\ \frac{m+1}{4} & if\ m \equiv 3\ mod\ 4, \end{cases}$$

and

$$d(x_i, x_{i-2}) = \begin{cases} \frac{m-1}{4} & if\ m \equiv 1\ (mod\ 4), \\ \frac{m+1}{4} & if\ m \equiv 3\ (mod\ 4), \end{cases} \qquad i = 3, 4, 5, \ldots, mn.$$

Proof:

Case I: $m \equiv 1\ (mod\ 4)$

Moving in the counter-clockwise direction on C_m, let $x_1, x_3, x_5, \ldots, x_{mn-1}$ be an ordering of the vertices of C_m such that the distance between any two consecutive vertices is $\frac{m-1}{4}$. Since m and $\frac{m-1}{4}$ are relatively prime, in this ordering each vertex of C_m appears $\frac{n}{2}$ times. We choose x_2 as that vertex of C_m which is at distance $\frac{m-1}{2}$ from x_1 in the clockwise direction. Now, again moving in the counter-clockwise direction on C_m, let $x_2, x_4, x_6, \ldots, x_{mn}$ be an ordering of the vertices of C_m such that the distance between any two consecutive vertices is $\frac{m-1}{4}$. It is easy to see that $d(x_{2i-1}, x_{2i}) = \frac{m-1}{2}, i = 1, 2, \ldots, \frac{mn}{2}$ and $d(x_{2i}, x_{2i+1}) = \frac{m+3}{4}, i = 1, 2, \ldots, \frac{mn}{2} - 1$.

Case II: $m \equiv 3\ (mod\ 4)$

Moving in the counter-clockwise direction on C_m, let $x_1, x_3, x_5, \ldots, x_{mn-1}$ be an ordering of vertices of C_m such that the distance between any two consecutive vertices is $\frac{m+1}{4}$. Since m and $\frac{m+1}{4}$ are relatively prime, in this ordering each vertex of C_m appears $\frac{n}{2}$ times. We choose x_2 as that vertex of C_m which is at distance $\frac{m-1}{2}$ from x_1 in the clockwise direction. Now, again moving in the counter-clockwise direction on C_m, let $x_2, x_4, x_6, \ldots, x_{mn}$ be an ordering of vertices of C_m such that the distance between any two consecutive vertices is $\frac{m+1}{4}$. It is easy to see that $d(x_{2i-1}, x_{2i}) = \frac{m-1}{2}$, $i = 1, 2, \ldots, \frac{mn}{2}$ and $d(x_{2i}, x_{2i+1}) = \frac{m+1}{4}$, $i = 1, 2, \ldots, \frac{mn}{2} - 1$. $\square$

The Cartesian product $K_n \Box C_m$ contains n copies of C_m and m copies of K_n. Now onwards, unless we mention, moving on a cycle, we mean clockwise.

Lemma 4.1.2. *For an even integer $n > 7$ and m odd, there exists an ordering $x_1, x_2, x_3, \ldots, x_{mn}$ of the vertices of $K_n \Box C_m$ such that $\{d(x_i, x_{i-1})\}_{i=2}^{mn}$ is an alternating sequence of $\frac{m-1}{2} + 1$ and p', where*

$$p' = \begin{cases} \frac{m+3}{4} + 1 & if\ m \equiv 1\ (mod\ 4), \\ \frac{m+1}{4} + 1 & if\ m \equiv 3\ (mod\ 4), \end{cases}$$

and

$$d(x_i, x_{i-2}) = \begin{cases} \frac{m-1}{4} + 1 & if\ m \equiv 1\ (mod\ 4), \\ \frac{m+1}{4} + 1 & if\ m \equiv 3\ (mod\ 4), \end{cases} \quad i = 3, 4, 5, \ldots, mn.$$

Proof:

Case I: $m \equiv 1\ (mod\ 4)$

If we treat each copy of K_n in $K_n \Box C_m$ as a single vertex, the ordering of vertices of $K_n \Box C_m$ that we need here is the ordering of C_m in Lemma 4.1.1. That is, to choose x_i in $K_n \Box C_m$, we move one less than the required distance on the cycle containing x_{i-1} and distance one across the cycles. To maintain the distance $d(x_i, x_{i-2}) = \frac{m-1}{4} + 1$, $i = 3, 4, 5, \ldots, mn$, we need to see that x_i is not on the cycle containing x_{i-2}. Now, we prove that this is possible. Suppose that $x_1, x_2, x_3, \ldots, x_l$, $l < mn$, are chosen. Let C^l and C^{l-1} be the copies of C_m on which x_l and x_{l-1} lies. Let u be the vertex of C^l at distance $\frac{m-1}{2}$ in the clockwise direction from x_l if l is odd and at distance $\frac{m+3}{4}$ in the clockwise direction from x_l if l is even. Let K^l be the copy of K_n containing u. By Lemma 4.1.1, at least one vertex of K^l is not chosen, say v. If v is not on C^{l-1} and $v \neq u$, then we choose the vertex v as x_{l+1}. Otherwise v is on C^{l-1} or $v = u$. Without loss of generality, we assume that $v = u$. Now, for a vertex labeled x_i in K^l, the possible positions for the vertices $x_{i-2}, x_{i-1}, x_{i+1}$ and x_{i+2} on C^l are the vertices of C^l at distance $\frac{m+3}{4}$, $\frac{m-1}{4}$ and $\frac{m-1}{2}$ from u (three positions in the clockwise direction and three positions in the counter-clockwise direction). Since $d(x_l, u) = \frac{m-1}{2}$ or $d(x_l, u) = \frac{m+3}{4}$, x_l is in one of the six positions. Since $n > 7$, there exists at least one vertex labeled x_j of K^l not on C^{l-1} such that none of the vertices x_{j-2}, x_{j-1}, x_{j+1} and x_{j+2} is on C^l. Now, relabel x_j as x_{l+1} and label u as x_j.

Case II: $m \equiv 3 \ (mod\ 4)$

Proof of this case is analogous to that of Case I by replacing $\frac{m+3}{4}$ with $\frac{m+1}{4}$. □

Example 4.1.3. In Figure 4.1, the vertices of $K_8 \square C_9$ are ordered as in Case I of Lemma 4.1.2. Here $\frac{m-1}{2}+1 = 5$, $\frac{m+3}{4}+1 = 4$ and $\frac{m-1}{4}+1 = 3$. In Figure 4.2, the vertices of $K_8 \square C_7$ are ordered as in Case II of Lemma 4.1.2. Here $\frac{m-1}{2}+1 = 4$, $\frac{m+1}{4}+1 = 3$ and $\frac{m+1}{4}+1 = 3$.

It is easy to see that $diam(K_n \square C_m) = \frac{m-1}{2}+1$ if m is odd and $diam(K_n \square C_m) = \frac{m}{2}+1$ if m is even.

Theorem 4.1.4. *For an even integer $n > 7$,*

$$rn(K_n \square C_m) \leq \begin{cases} \frac{1}{8}(m^2n + 3mn - 2m + 10) & if\ m \equiv 1 \ (mod\ 4), \\ \frac{1}{8}(m^2n + 5mn - 2m + 6) & if\ m \equiv 3 \ (mod\ 4). \end{cases}$$

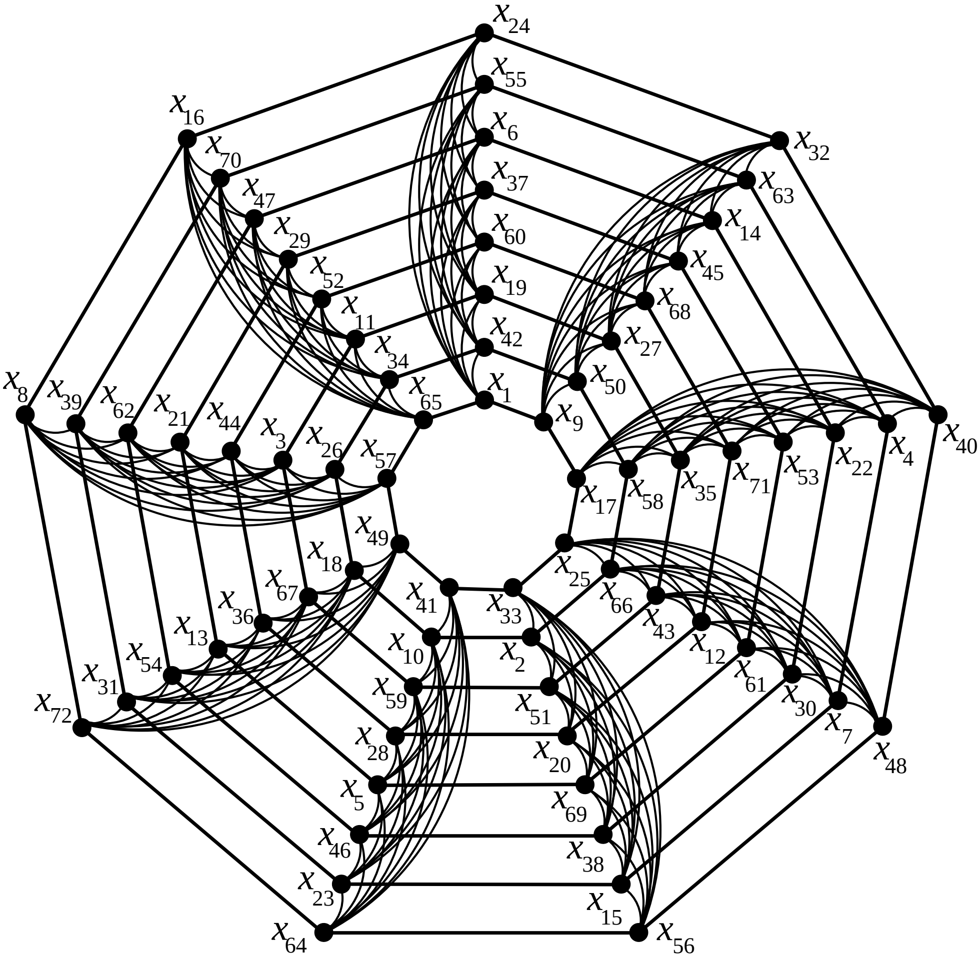

Figure 4.1 The ordering of the vertices of $K_8 \Box C_9$ as in Case I of the proof of Lemma 4.1.2

Proof: Let $x_1, x_2, x_3, \ldots, x_{mn}$ be the ordering of vertices in $K_n \Box C_m$ as in Lemma 4.1.2.

Case I: $m \equiv 1\ (mod\ 4)$

We define f by $f(x_1) = 1$ and $f(x_i) = f(x_{i-1}) + (1 + \frac{m-1}{2} + 1) - d(x_i, x_{i-1})$, $2 \leq i \leq mn$. We show that f is a radio coloring of $K_n \Box C_m$. Except x_i and x_{i-2}, $3 \leq i \leq mn$, and x_i and x_{i-3}, $4 \leq i \leq mn$ all other pairs of vertices satisfy the radio coloring condition clearly. So, we check the radio coloring condition for x_i and x_{i-2}. If i is odd, then $d(x_i, x_{i-1}) = \frac{m+3}{4} + 1$, $d(x_{i-1}, x_{i-2}) = \frac{m-1}{2} + 1$ and $d(x_i, x_{i-2}) = \frac{m-1}{4} + 1$. Therefore

$$\begin{aligned} f(x_i) - f(x_{i-2}) &= f(x_i) - f(x_{i-1}) + f(x_{i-1}) - f(x_{i-2}) \\ &= \left(1 + \frac{m-1}{2} + 1\right) - d(x_i, x_{i-1}) + \left(1 + \frac{m-1}{2} + 1\right) - d(x_{i-1}, x_{i-2}) \end{aligned}$$

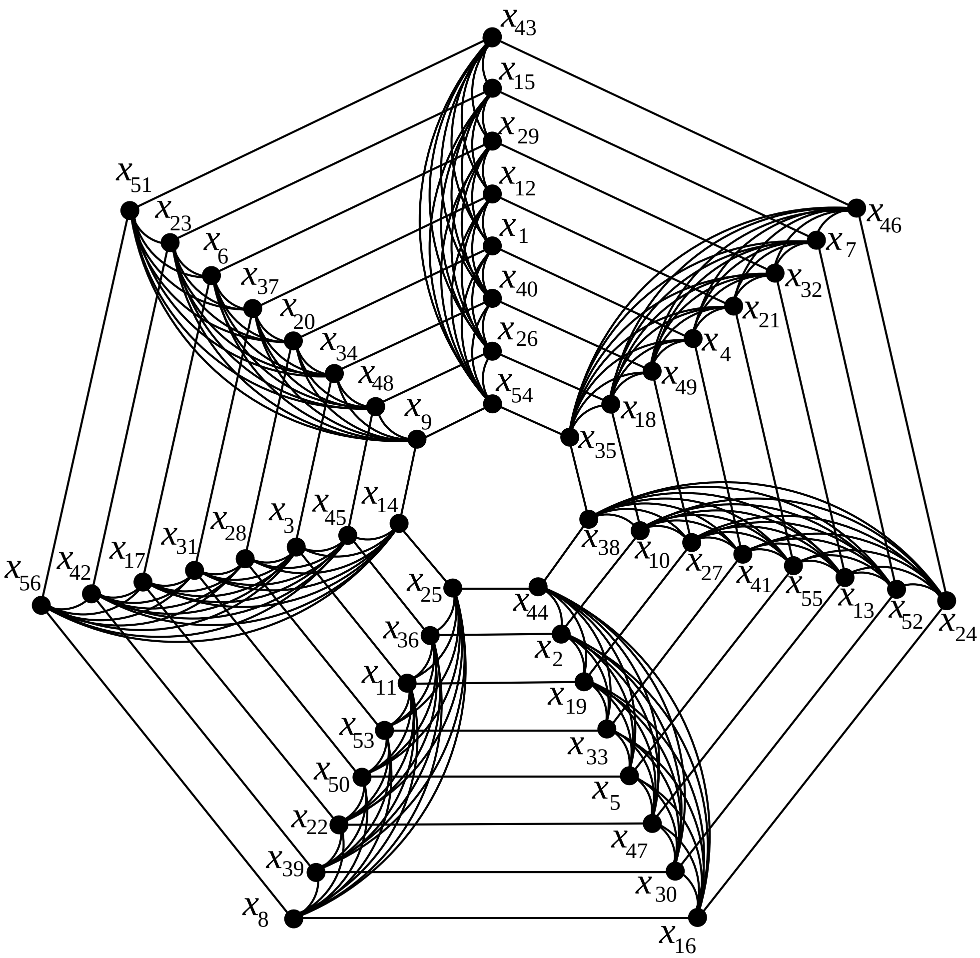

Figure 4.2 The ordering of the vertices of $K_8 \Box C_7$ as in Case II of the proof of Lemma 4.1.2

$$
\begin{aligned}
&= \frac{m-1}{4} + 1 \\
&= 1 + \frac{m-1}{2} + 1 - d(x_i, x_{i-2}).
\end{aligned}
$$

Since $d(x_i, x_{i-3}) \geq d(x_{i-2}, x_{i-3}) - d(x_{i-2}, x_i) = \frac{m-1}{2} + 1 - \frac{m-1}{4} - 1 = \frac{m-1}{4}$, we have

$$
\begin{aligned}
f(x_i) - f(x_{i-3}) &= f(x_i) - f(x_{i-2}) + f(x_{i-2}) - f(x_{i-3}) \\
&= \frac{m-1}{4} + 1 + \left(1 + \frac{m-1}{2} + 1\right) - d(x_{i-2}, x_{i-3}) \\
&= \frac{m-1}{4} + 1 + 1 \\
&= 1 + \frac{m-1}{2} + 1 - \frac{m-1}{4} \\
&\geq 1 + \frac{m-1}{2} + 1 - d(x_i, x_{i-3}).
\end{aligned}
$$

If i is even, then $d(x_i,x_{i-1}) = \frac{m-1}{2}+1$, $d(x_{i-1},x_{i-2}) = \frac{m+3}{4}+1$, $d(x_i,x_{i-2}) = \frac{m+3}{4}$. Therefore,

$$\begin{aligned} f(x_i)-f(x_{i-2}) &= f(x_i)-f(x_{i-1})+f(x_{i-1})-f(x_{i-2}) \\ &= 1+\frac{m-1}{4} \\ &= 1+\frac{m}{2}+1-d(x_i,x_{i-2}). \end{aligned}$$

Since $f(x_{i-2}) = f(x_{i-3})+\frac{m-1}{4}$, $f(x_{i-1}) = f(x_{i-2})+1$ and $f(x_i) = f(x_{i-1})+\frac{m-1}{4}$, $f(x_i)-f(x_{i-3}) = \frac{m-1}{2}+1 = diam(K_n\Box C_m)$. Hence, f is a radio coloring. From the choice of x_is and by the definition of f, it is easy to see that

$$\sum_{i=2}^{mn} d(x_i,x_{i-1}) = \frac{mn}{2}\left(\frac{m-1}{2}+1\right)+\left(\frac{mn}{2}-1\right)\left(\frac{m+3}{4}+1\right) \text{ and } \sum_{i=2}^{mn}\varepsilon_i = 0.$$

Now by Lemma 1.4.2,

$$\begin{aligned} rn(f) &= f(x_{mn}) \\ &= (mn-1)\left(\frac{m+1}{2}+1\right)-\frac{mn}{2}\left(\frac{m-1}{2}+1\right)-\left(\frac{mn}{2}-1\right)\left(\frac{m+3}{4}+1\right)+1 \\ &= \frac{1}{8}(m^2n+3mn-2m+10). \end{aligned}$$

Case II: $m \equiv 3\ (mod\ 4)$

We define a coloring g by $g(x_1) = 1$ and $g(x_i) = g(x_{i-1})+(1+\frac{m-1}{2}+1)-d(x_i,x_{i-1})$, $2 \le i \le mn$. Similar to the Case I, we can prove that g is a radio coloring and by using Lemma 1.4.2, we get $rn(g) = g(x_{mn}) = \frac{1}{8}(m^2n+5mn-2m+6)$. □

Theorem 4.1.5. *For an even integer $n > 7$, we have*

$$rn(K_n\Box C_m) = \begin{cases} \frac{1}{8}(m^2n+3mn-2m+10) & if\ m \equiv 1\ (mod\ 4), \\ \frac{1}{8}(m^2n+5mn-2m+6) & if\ m \equiv 3\ (mod\ 4). \end{cases}$$

Proof: Using Theorem 1.4.4, we prove that

$$rn(K_n\Box C_m) \geq \begin{cases} \frac{1}{8}(m^2n+3mn-2m+10) & if\ m\equiv 1\ (mod\ 4), \\ \frac{1}{8}(m^2n+5mn-2m+6) & if\ m\equiv 3\ (mod\ 4). \end{cases}$$

Here $k = diam(K_n\Box K_m) = \frac{m-1}{2}+1$. We choose $M = m+3$, the triameter of $K_n\Box C_m$.

Case I: $m\equiv 1$ $(mod\ 4)$

Since $m\equiv 1\ (mod\ 4)$, $(m+3)\not\equiv\left(\frac{m-1}{2}+1\right)\ (mod\ 2)$. Since mn is even and $(m+3)\not\equiv\left(\frac{m-1}{2}+1\right)\ (mod\ 2)$, by Theorem 1.4.4, we have

$$\begin{aligned} rn(K_n\Box C_m) &\geq \frac{(mn-2)\left(3(\frac{m-1}{2}+1+1)-(m+3)\right)}{4}+2 \\ &= \frac{1}{8}(m^2n+3mn-2m+10). \end{aligned}$$

Case II: $m\equiv 3$ $(mod\ 4)$

Since $m\equiv 3\ (mod\ 4)$, $(m+3)\equiv\left(\frac{m-1}{2}+1\right)\ (mod\ 2)$. Now, by Theorem 1.4.4, we have

$$\begin{aligned} rn(K_n\Box C_m) &\geq \frac{(mn-2)\left(3(\frac{m-1}{2}+1+1)-(m+3-1)\right)}{4}+2 \\ &= \frac{1}{8}(m^2n+5mn-2m+6). \end{aligned}$$

□

Example 4.1.6. The radio coloring f in Case I of the proof of Theorem 4.1.4 is given for $K_8\Box C_9$ in Figure 4.3. The span of f is 107. The radio coloring g in Case II of the proof of Theorem 4.1.4 is given for $K_8\Box C_7$ in Figure 4.4. The span of g is 83.

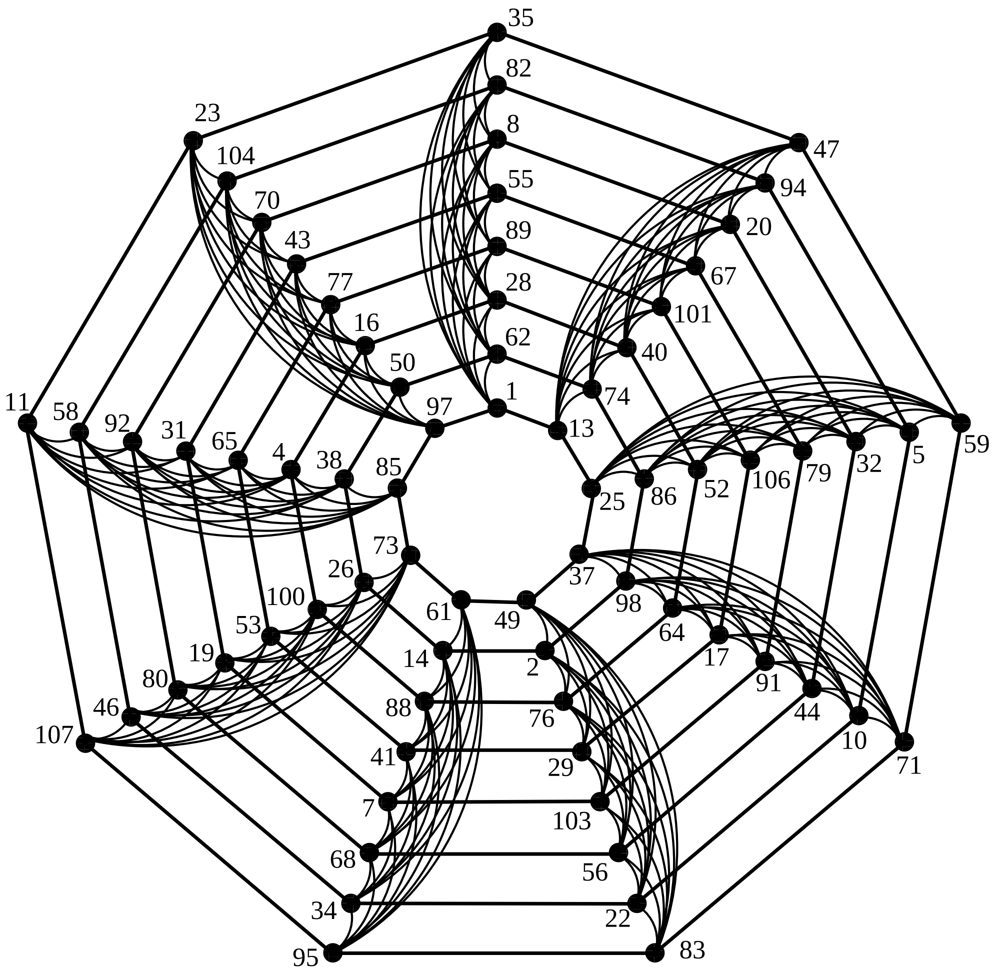

Figure 4.3 The radio coloring for $K_8 \square C_9$ as given in Case I of the proof of Theorem 4.1.4

4.2 THE RADIO NUMBER OF $K_n \square C_m$ FOR $m \equiv 6\ (mod\ 8)$

In this section, similar to the above section, we find the radio number of $K_n \square C_m$, $m \equiv 6\ (mod\ 8)$ and $n \geq 7$.

Lemma 4.2.1. *If $m \equiv 6\ (mod\ 8)$ and n is any positive integer, then there exists an ordering $x_1, x_2, x_3, \ldots, x_{mn}$ of the vertices of C_m, which takes every vertex n times, such that the sequence $\{d(x_i, x_{i-1})\}_{i=2}^{mn}$ is an alternating sequence of $\frac{m}{2}$ and $\frac{m+2}{4}$.*

Proof: Moving in the counter-clockwise direction on C_m, let $x_1, x_3, x_5, \ldots, x_{m-1}, x_2, x_4, x_6, \ldots, x_m, x_{m+1}, x_{m+3}, x_{m+5} \ldots, x_{2m-1}, x_{m+2}, x_{m+4}, \ldots, x_{mn-2}$, x_{mn} be an ordering of vertices of C_m such that the distance between any two consecutive vertices is $\frac{m-2}{4}$. Since m and $\frac{m-2}{4}$ are relatively prime, in the above ordering each vertex of C_m appears n times.

For $i = 1, 3, 5, \ldots, mn-1$,

$$\begin{aligned} d(x_i, x_{i+1}) &= \left[\left(\frac{m}{2}\right)\left(\frac{m-2}{4}\right)\right] \ (mod\ m) \\ &= \left[\left(\frac{m-6}{8}\right)m + \frac{m}{2}\right] \ (mod\ m) \\ &= \frac{m}{2}. \end{aligned}$$

□

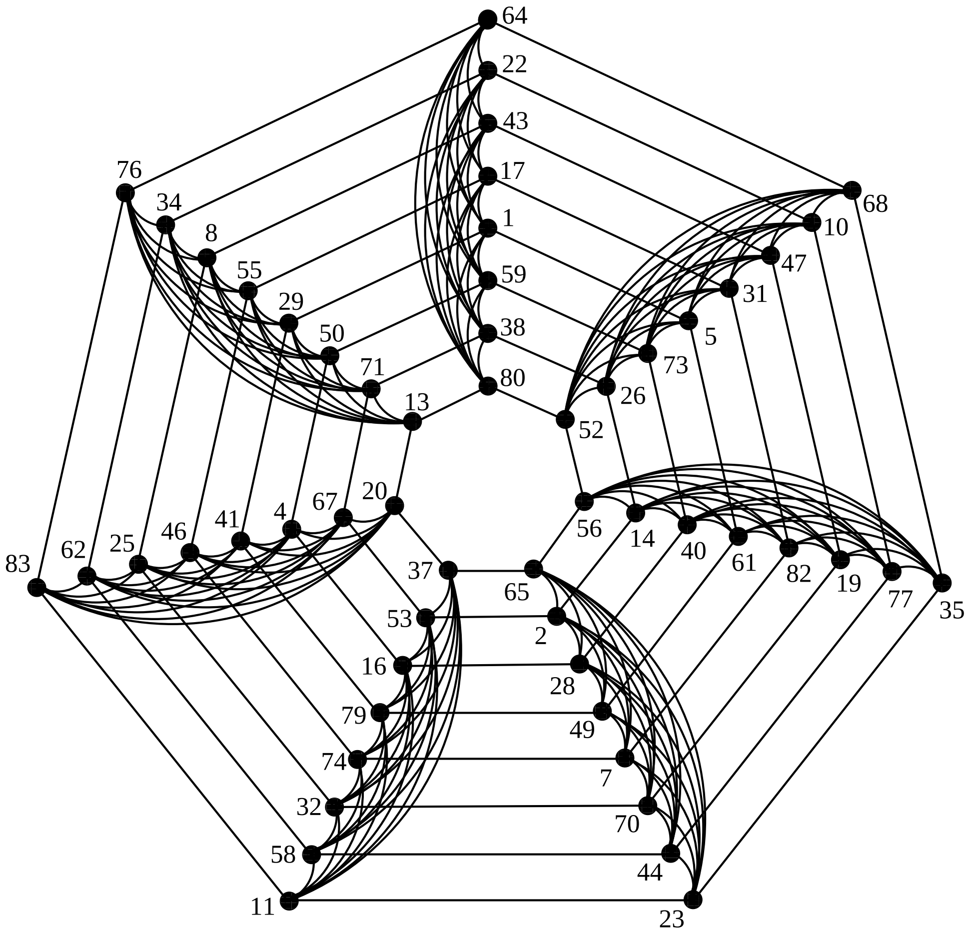

Figure 4.4 The radio coloring for $K_8 \square C_7$ as given in Case II of the proof of Theorem 4.1.4

Lemma 4.2.2. *If $m \equiv 6\ (mod\ 8)$ and $n \geq 7$, then there exists an ordering $x_1, x_2, x_3, \ldots, x_{mn}$ of the vertices of $K_n \square C_m$ such that the sequence $\{d(x_i, x_{i-1})\}_{i=2}^{mn}$ is an alternating sequence of $\frac{m}{2}+1$ and $\frac{m+2}{4}+1$, and $d(x_i, x_{i-2}) = \frac{m+2}{4}$, $i = 3, 4, 5, \ldots, mn$.*

Proof: Proof is similar to that of Lemma 4.1.2 with the following variations. The vertex u is at distance $\frac{m+2}{4}$ from x_l if l is even and at distance $\frac{m}{2}$ from x_l if l is odd. For a vertex labeled x_i in K^l, the possible positions for the vertices $x_{i-2}, x_{i-1}, x_{i+1}$ and x_{i+2} on C^l are the vertices of C^l at distance $\frac{m+2}{4}$, $\frac{m-2}{4}$ and $\frac{m}{2}$ from u (three positions in the clockwise direction and two positions in the counter-clockwise direction). □

Example 4.2.3. In Figure 4.5, the vertices of $K_7 \Box C_6$ are ordered as in Lemma 4.2.2. Here $\frac{m}{2}+1=3$, $\frac{m-2}{4}+1=2$ and $\frac{m-1}{4}+1=3$.

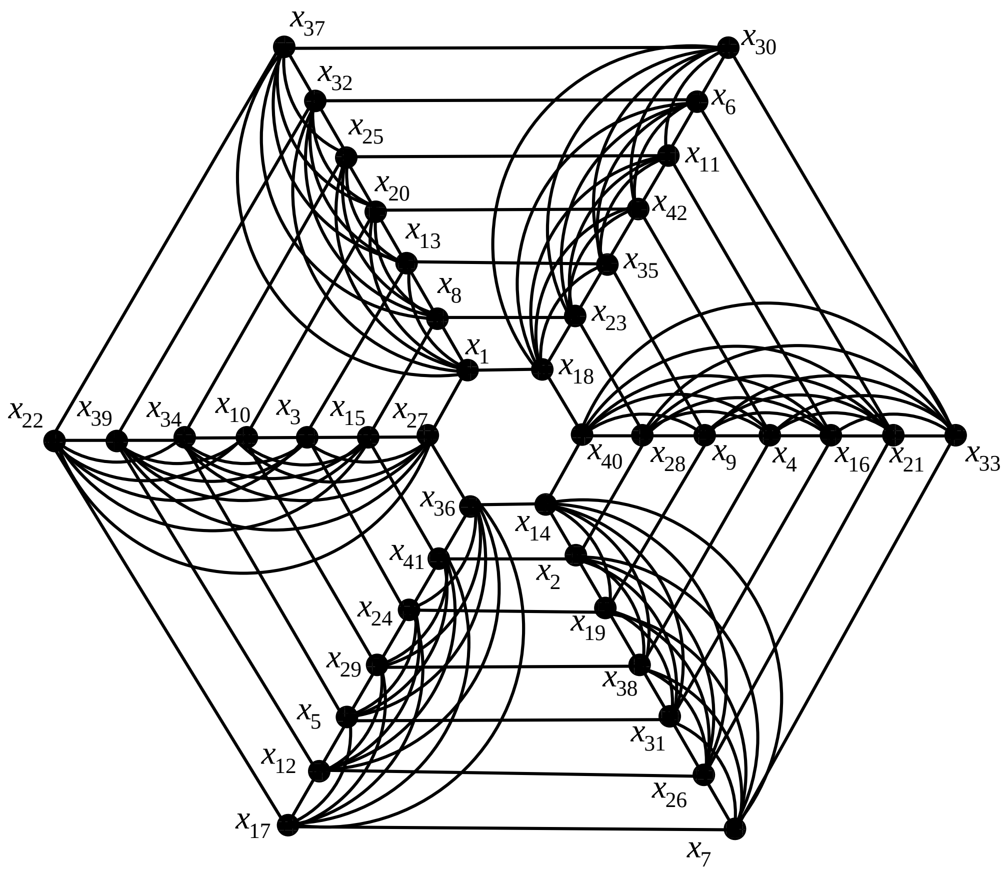

Figure 4.5 The ordering of the vertices of $K_7 \Box C_6$ as in the proof of Lemma 4.2.2

Theorem 4.2.4. *If $m \equiv 6 \ (mod\ 8)$ and $n \geq 7$, then $rn(K_n \Box C_m) = \frac{1}{8}(m^2n + 6mn - 2m + 4)$.*

Proof: Let $x_1, x_2, x_3, \ldots, x_{mn}$ be the ordering of vertices in $K_n \Box C_m$ as in Lemma 4.2.2. Now, we define f by $f(x_1) = 1$ and $f(x_i) = f(x_{i-1}) + (1 + \frac{m}{2} + 1) - d(x_i, x_{i-1})$, $2 \leq i \leq mn$. As in Theorem 4.1.4, we can prove that f is a radio coloring of $K_n \Box C_m$. By Lemma 1.4.2, we have

$$\begin{aligned} rn(f) &= f(x_{mn-1}) \\ &= (mn-1)\left(1+\frac{m}{2}+1\right) - \frac{mn}{2}\left(\frac{m}{2}+1\right) - \left(\frac{mn}{2}-1\right)\left(\frac{m+2}{4}+1\right)+1 \\ &= \frac{1}{8}(m^2n+6mn-2m+4). \end{aligned}$$

Next, we show that $rn(K_n \Box C_m) \geq \frac{1}{8}(mn^2+6mn-2m+4)$. To prove this we use Theorem 1.4.4. Since mn is even and $(m+3) \not\equiv \left(\frac{m}{2}+1\right)$ $(mod\ 2)$, by Theorem 1.4.4, we have

$$\begin{aligned} rn(K_n \Box C_m) &\geq \frac{(mn-2)\left(3(\frac{m}{2}+1+1)-(m+3)\right)}{4}+2 \\ &= \frac{1}{8}(m^2n+6mn-2m+4). \end{aligned}$$ □

Example 4.2.5. In Figure 4.6, for $K_7 \Box C_6$, using the vertex ordering in Figure 4.5, the minimal radio coloring in the proof of Theorem 4.2.4 is given.

4.3 THE RADIO NUMBER OF $K_n \Box C_m$ FOR n ODD AND $m \equiv 5$ $(mod\ 8)$

Similar to the above sections, here also, we order the vertices of $K_n \Box C_m$, using which we define a minimal radio coloring of $K_n \Box C_m$ for n odd and $m \equiv 5$ $(mod\ 8)$.

Lemma 4.3.1. *If $m \equiv 5$ $(mod\ 8)$ and n is any positive integer, then there exists an ordering $x_1, x_2, \ldots, x_{mn}$ of vertices of C_m, which takes every vertex n times, such that $d(x_i, x_{i-1}) = \frac{3m+1}{8}$, $i = 2, 3, \ldots, mn$, and $d(x_i, x_{i-2}) = \frac{m-1}{4}$, $i = 3, 4, 5, \ldots, mn$.*

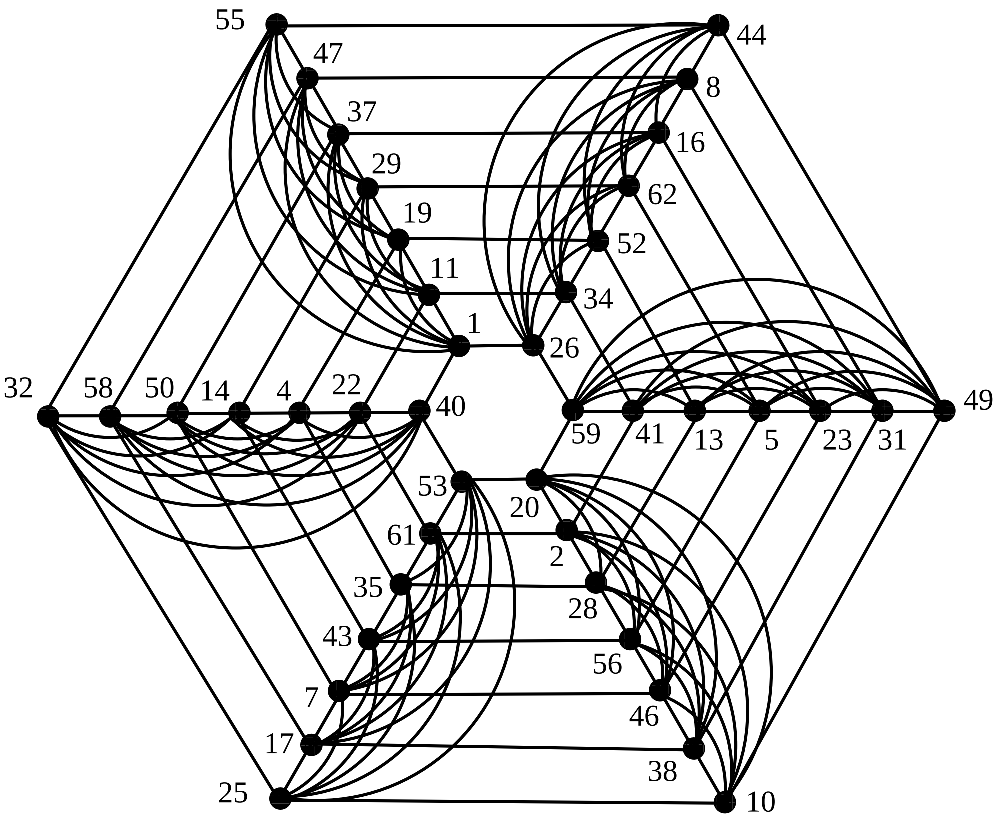

Figure 4.6 The minimal radio coloring of $K_7\Box C_6$, given in the proof of Theorem 4.2.4

Proof: Moving in the clockwise direction on C_m, let $x_1, x_2, x_3, \ldots, x_{mn}$ be an ordering of vertices of C_m such that the distance between any two consecutive vertices is $\frac{3m+1}{8}$. Since m and $\frac{3m+1}{8}$ are relatively prime, in this ordering, each vertex of C_m appears n times. It is easy to see that $d(x_i, x_{i-2}) = \frac{m-1}{4}$, $i = 3, 4, 5, \ldots, mn$. □

Lemma 4.3.2. *If $n \geq 7$ is odd and $m \equiv 5\ (mod\ 8)$, then there exists an ordering $x_1, x_2, x_3, \ldots, x_{mn}$ of vertices of $K_n\Box C_m$ such that $d(x_i, x_{i-1}) = \frac{3m+1}{8} + 1$, $i = 2, 3, 4, \ldots, mn$, and $d(x_i, x_{i-2}) = \frac{m-1}{4} + 1$, $i = 3, 4, 5, \ldots, mn$.*

Proof: Proof is similar to that of Lemma 4.1.2 with the following variation. Vertex u is at distance $\frac{3m+1}{8}$ from x_l. For a vertex labeled x_i in K^l, the possible positions for the vertices $x_{i-2}, x_{i-1}, x_{i+1}$ and x_{i+2} on C^l are the vertices of C^l at distance $\frac{3m+1}{8}$ and $\frac{3m+1}{4}$ from u (two positions in the clockwise direction and two positions in the counter-clockwise direction). □

Example 4.3.3. In Figure 4.7, the vertices of $K_7 \Box C_5$ are ordered as in Lemma 4.3.2. Here $d(x_i, x_{i-1}) = \frac{3m+1}{8} + 1 = 3$ and $d(x_i, x_{i-2}) = \frac{m-1}{4} + 1 = 2$.

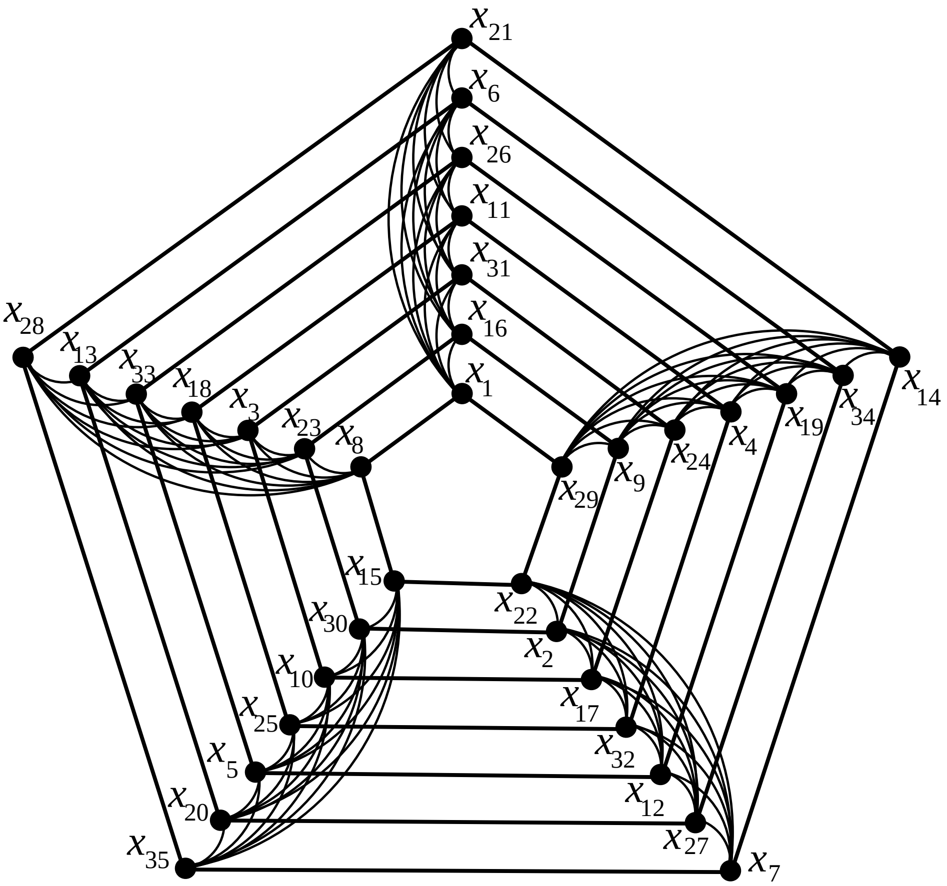

Figure 4.7 The ordering of the vertices of $K_7 \Box C_5$ as in the proof of Lemma 4.3.2

Theorem 4.3.4. *If $n \geq 7$ is odd and $m \equiv 5 \ (mod\ 8)$, then $rn(K_n \Box C_m) = \frac{1}{8}(m^2 n + 3mn - m + 5)$.*

Proof: Let $x_1, x_2, x_3, \ldots, x_{mn}$ be the ordering of vertices in $K_n \Box C_m$ as in Lemma 4.3.2. Now, we define f by $f(x_1) = 1$ and $f(x_i) = f(x_{i-1}) + (1 + \frac{m-1}{2} + 1) - d(x_i, x_{i-1})$, $2 \leq i \leq mn$. As in Theorem 4.1.4, we can prove that f is a radio coloring of $K_n \Box C_m$. By Lemma 1.4.2, we have

$$\begin{aligned} rn(f) = f(x_{mn}) &= (mn-1)\left(\frac{m+1}{2} + 1\right) - (mn-1)\left(\frac{3m+1}{8} + 1\right) + 1 \\ &= \frac{1}{8}(m^2 n + 3mn - m + 5). \end{aligned}$$

Next, we show that $rn(K_n \square C_m) \geq \frac{1}{8}(m^2n + 3mn - m + 5)$. To prove this, we use Theorem 1.4.4. Since mn is odd and $(m+3) \not\equiv \left(\frac{m-1}{2}+1\right)$ $(mod\ 2)$, by Theorem 1.4.4, we have

$$\begin{aligned} rn(K_n \square C_m) &\geq \frac{(mn-1)\left(3(\frac{m-1}{2}+1+1)-(m+3)\right)}{4}+1 \\ &= \frac{1}{8}(m^2n+3mn-m+5). \end{aligned}$$

□

Example 4.3.5. In Figure 4.8, for $K_7 \square C_5$, using the vertex ordering in Figure 4.7, the minimal radio coloring in the proof of Theorem 4.3.4 is given.

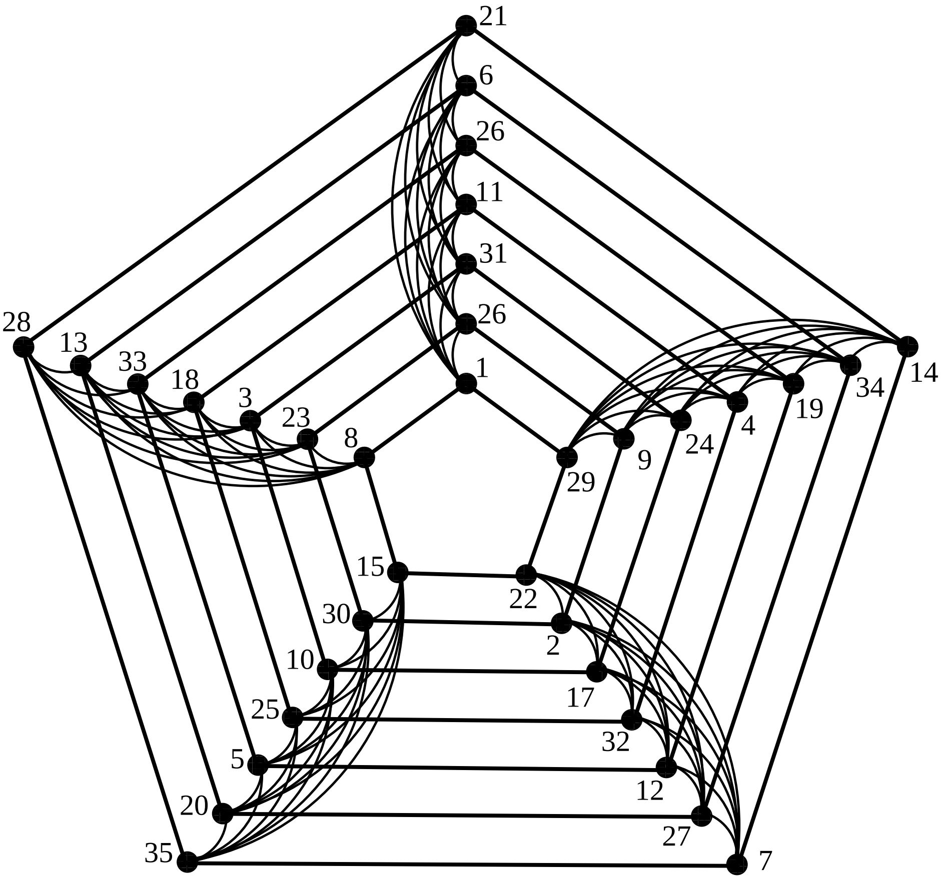

Figure 4.8 The minimal radio coloring of $K_7 \square C_6$, given in the proof of Theorem 4.3.4

4.4 SUMMARY

In this chapter, we have studied the radio number for the Cartesian product of complete graph and cycle. We have determined $rn(K_n \Box C_m)$ when n even and m odd; any n and $m \equiv 6 \ (mod\ 8)$; n is odd and $m \equiv 5 \ (mod\ 8)$.

CHAPTER 5

THE RADIO *k*-CHROMATIC NUMBER FOR CORONA OF ARBITRARY GRAPHS

"The older I get, the more I believe that at the bottom of most deep mathematical problems there is a combinatorial problem."
- Israil Moiseevich Gelfand (1990)

Products of graphs are often viewed as a convenient language to describe structures, but they are increasingly being applied in more substantial ways. Computer science is one of the many fields in which graph products are becoming commonplace. Let G and H be two graphs with vertex sets $\{v_1, v_2, v_3, \ldots, v_n\}$ and $\{u_1, u_2, u_3, \ldots, u_m\}$, respectively. Recall that, the corona $G \odot H$ of G and H is the graph with vertex set $V(G) \bigcup \Big(\bigcup_{i=1}^{n} \{v_i^j : 1 \le j \le m\} \Big)$ and edge set $E(G) \bigcup \Big(\bigcup_{i=1}^{n} \{v_i v_i^j : 1 \le j \le m\} \Big) \bigcup \Big(\bigcup_{i=1}^{n} \{v_i^l v_i^j : u_l u_j \in E(H)\} \Big)$. Equivalently, $G \odot H$ is the graph obtained by taking one copy of G and for each vertex v_i of G, one copy of H, say H_i, and joining v_i to each and every vertex of H_i by an edge. By the definition of corona of graphs, it is clear that $G \odot H \ncong H \odot G$ unless $G \cong H$. Many properties of $G \odot H$ mainly depend on G, but not on H. The corona $G \odot H$ is connected if and only if G is connected. Also, $diam(G \odot H) = diam(G) + 2$. Throughout this chapter, in corona $G \odot H$, the first graph G is connected, but not necessarily the second graph H.

In this chapter, we obtain a best possible upper bound for the radio k-chromatic number of corona of graphs. Also, for an arbitrary graph H, we improve the upper bound and obtain lower bounds for the radio numbers of $P_{2p+1} \odot H$ and $Q_n \odot H$.

5.1 AN UPPER BOUND FOR THE RADIO k-CHROMATIC NUMBER FOR CORONA OF ARBITRARY GRAPHS

In this section, we first give an upper bound for the radio k-chromatic number of corona of two arbitrary graphs. Later, we show that this upper bound is exact for $P_{2p} \odot H$ and $K_n \odot H$.

Theorem 5.1.1. *If G is a connected graph and H is a graph, of orders $n > 1$ and m respectively, then for $k > 3$, $rc_k(G \odot H) \leq (m+1)rc_{k-2}(G) + (k-3)m + 2(n-1)$.*

Proof: Let f be a minimal radio $(k-2)$-coloring of G. Let $y_1, y_2, y_3, \ldots, y_n$ be an ordering of vertices of G such that $f(y_i) \leq f(y_{i+1})$ for all i. Let H_i be the copy of H in $G \odot H$ corresponding to the vertex y_i of G. We order the vertices of $G \odot H$ as follows. Let $x_1 = y_1$ and for $j = 0, 1, 2, \ldots, m-1$, we choose x_{jn+i}, $jn + i \neq 1$, from H_i, $1 \leq i \leq n$. We choose x_{mn+1} from H_1 and x_{mn+i} as y_i, $2 \leq i \leq n$. Now, we define a coloring g of $G \odot H$ by $g(x_1) = f(y_1) = 1$,

$$g(x_i) = \begin{cases} g(x_{i-1}) + f(y_2) - f(y_1) + 1 & if\ i = 2,\ mn+2, \\ g(x_{i-1}) + f(y_l) - f(y_{l-1}) & if\ 3 \leq i \leq mn,\ i \not\equiv 1\ (mod\ n)\ and\ l \equiv i\ (mod\ n), \\ g(x_{i-1}) + k - 2 & if\ i \equiv 1\ (mod\ n), \\ g(x_{i-1}) + f(y_l) - f(y_{l-1}) + 2 & if\ mn+2 < i \leq mn+n\ and\ l \equiv i\ (mod\ n). \end{cases}$$

It is clear that $d(x_1, x_2) = d(x_{mn+1}, x_{mn+2}) = d(y_1, y_2) + 1$; $d(x_i, x_{i-1}) = d(y_l, y_l) + 2$, $3 \leq i \leq mn$, $i \not\equiv 1\ (mod\ n)$, $l \equiv i\ (mod\ n)$; $d(x_i, x_{i-1}) \geq 3$ for $i \equiv 1\ (mod\ n)$; and $d(x_i, x_{i-1}) = d(y_l, y_{l-1})$, $mn + 2 < i \leq mn + n$, $l \equiv i\ (mod\ n)$. Since f is a radio $(k-2)$-coloring of G, we have g is a radio k-coloring of $G \odot H$. Now,

$$\begin{aligned} rc_k(g) &= g(x_{mn+n}) = g(x_1) + \sum_{i=2}^{mn+n} [g(x_i) - g(x_{i-1})] \\ &= 1 + (1 + rc_{k-2}(G) - 1) + (m-1)(rc_{k-2}(G) - 1) + m(k-2) \\ &\qquad + (1 + rc_{k-2}(G) - 1 + 2(n-2)) \end{aligned}$$

$$= \quad (m+1)rc_{k-2}(G)+(k-3)m+2(n-1).$$

Therefore $rc_k(G\odot H) \leq (m+1)rc_{k-2}(G)+(k-3)m+2(n-1)$. $\square$

Let $v_1, v_2, v_3, \ldots, v_m$ be a vertex ordering of a graph H. Let $\alpha'(v_1, v_2, v_3, \ldots, v_m)$ denotes the number of pairs of adjacent consecutive vertices in the ordering. Let $\alpha'(H)$ be the minimum of $\alpha'(v_1, v_2, v_3, \ldots, v_m)$ over all the vertex orderings of H.

Theorem 5.1.2. *If G is a connected graph and H is a graph, of orders $n > 1$ and m respectively, then*

$$rc_3(G\odot H) \leq \begin{cases} (m+1)\chi(G)+2(n-1) & if\ G\ is\ not\ bipartite, \\ 2(m+n)+\alpha'(H) & if\ G\ is\ bipartite. \end{cases}$$

Proof: Theorem 5.1.1 holds good for $k=3$ if G is not a bipartite graph. Hence, $rc_3(G\odot H) \leq (m+1)\chi(G)+2(n-1)$ if G is not bipartite. Let G be a bipartite graph and f be a minimal radio 1-coloring of G. Let $y_1, y_2, y_3, \ldots, y_n$ be an ordering of the vertices of G such that $f(y_i) \leq f(y_{i+1})$ for all i. Let H_i be the copy of H in $G\odot H$ corresponding to the vertex y_i of G. Let $v_1, v_2, v_3, \ldots, v_m$ be a vertex ordering of H such that $\alpha'(v_1, v_2, v_3, \ldots, v_m) = \alpha'(H)$. We consider the vertex ordering $x_1, x_2, x_3, \ldots, x_{mn+n}$ of $G\odot H$ as in the proof of Theorem 5.1.1 with the following modification. The vertex x_{jn+1} from H_1 is chosen such that $x_{jn+1} = v_j$, $j = 1,2,3,\ldots,m$. For $j = 1,2,3,\ldots,m$, the vertex $x_{(j-1)n+i}$ is chosen from the copy of H_i such that $x_{(j-1)n+i} = v_j$, $i = 2,3,4,\ldots,n$. We modify the coloring g defined in the proof of Theorem 5.1.1 for the vertices x_{jn+2}, $j = 1,2,3,\ldots,m-1$ as follows. For $j = 1,2,3,\ldots,m-1$,

$$g(x_{jm+2}) = \begin{cases} g(x_{jm+1})+f(y_2)-f(y_1)+1 & if\ v_{j-1}\ and\ v_j\ are\ adjacent, \\ g(x_{jm+1})+f(y_2)-f(y_1) & if\ v_{j-1}\ and\ v_j\ are\ not\ adjacent. \end{cases}$$

It is easy to see that g is a radio 3-coloring of $G\odot H$ and $rc_3(g) = (m+1)rc_1(G) + 2(n-1)+\alpha'(H) = 2(m+n)+\alpha'(H)$. $\square$

Example 5.1.3. A minimal radio coloring and the ordering of vertices in the increasing order of their colors for P_6 and P_5 are considered in Figure 5.1 and Figure 5.4, respectively. The vertex orderings, as in the proof of Theorem 5.1.1, for $P_6 \odot C_4$ and $P_5 \odot C_4$ are given in Figure 5.2 and Figure 5.5, respectively. The radio coloring in the proof of Theorem 5.1.1 for $P_6 \odot C_4$ and $P_5 \odot C_4$ is given in Figure 5.3 and Figure 5.6, respectively.

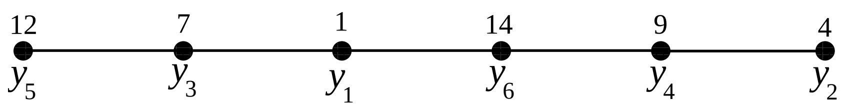

Figure 5.1 A minimal radio coloring of P_6 and the ordering of vertices in the increasing order of their colors

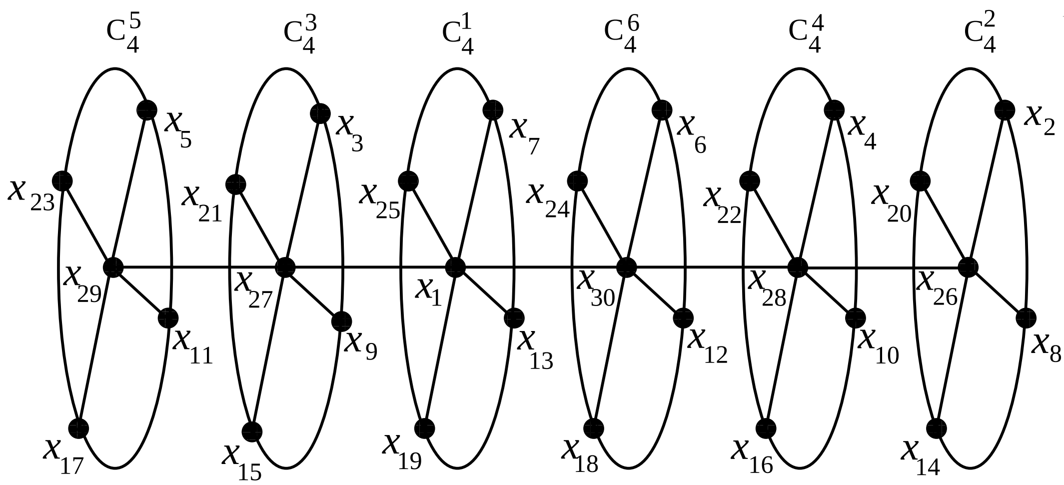

Figure 5.2 The vertex ordering of $P_6 \odot C_4$ as in the proof of Theorem 5.1.1

Next, we give lower bounds for $rc_k(K_n \odot H)$ and $rn(P_{2p} \odot H)$ which match with the upper bound given in Theorem 5.1.1.

Theorem 5.1.4. *If H is a graph of order m, $k > 2$ and $n > 1$ are integers, then* $rc_k(K_n \odot H) = (k-2)mn + k(n-1) + 1$.

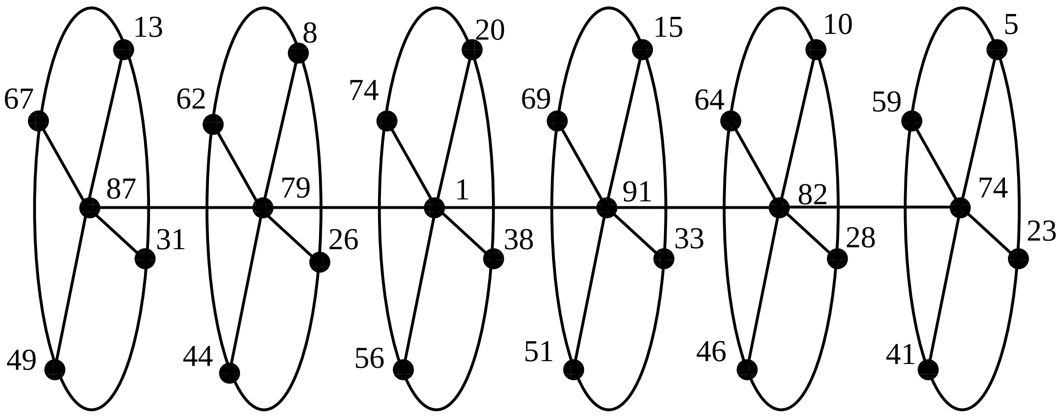

Figure 5.3 The radio coloring of $P_6 \odot C_4$ as in the proof of Theorem 5.1.1

Proof: It is easy to see that $rc_{k-2}(K_n) = (k-2)(n-1)+1$. From Theorem 5.1.1, we have $rc_k(K_n \odot H) \leq (m+1)rc_{k-2}(K_n) + (k-3)m + 2(n-1) = (k-2)mn + k(n-1) + 1$. To get the lower bound that matches with the upper bound, we use Lemma 1.4.2. It is easy to see that the maximum distance sum for $K_n \odot H$ is $2+3(mn-1)+2+n-2 = 3mn+n-1$. Now, by Lemma 1.4.2, we have $rc_k(K_n \odot H) \geq (mn+n-1)(1+k) - (3mn+n-1)+1 = (k-2)mn+k(n-1)+1$. □

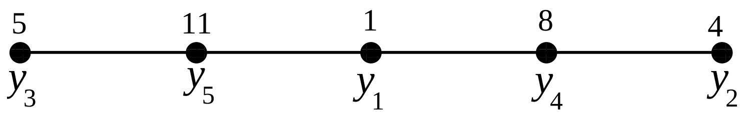

Figure 5.4 A minimal radio coloring of P_5 and the ordering of vertices in the increasing order of their colors

Theorem 5.1.5. *If H is a graph of order m, then* $rn(P_{2p} \odot H) = (2m+2)p^2 + 2p$.

Proof: From Theorem 5.1.1, $rn(P_{2p} \odot H) \leq (m+1)rn(P_{2p}) + (k-3)m + 2(2p-1) = (m+1)(2p^2-2p+2) + ((2p+1)-3)m + 2(2p-1) = (2m+2)p^2 + 2p$. To get the lower bound that matches with the upper bound, we use Theorem 1.4.3. Let P_{2p} : $v_1v_2v_3\ldots v_{2p}$ be the path. We choose $L_0 = \{v_p, v_{p+1}\}$. Then for $i = 1,2,3,\ldots,p-1$, $|L_i| = 2(m+1)$ and $|L_p| = 2m$. Now, by Theorem 1.4.3, $rn(P_{2p} \odot H) \geq (2pm+2p) - 2p + 2\times 2 + 2\sum_{i=1}^{p-1} 2(m+1)(p-i) = (2m+2)p^2 + 2p$. □

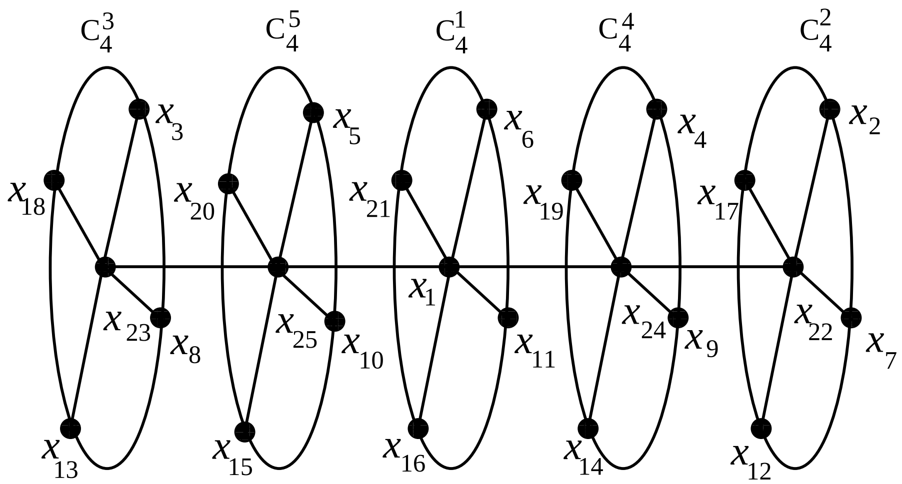

Figure 5.5 The vertex ordering of $P_5 \odot C_4$ as in the proof of Theorem 5.1.1

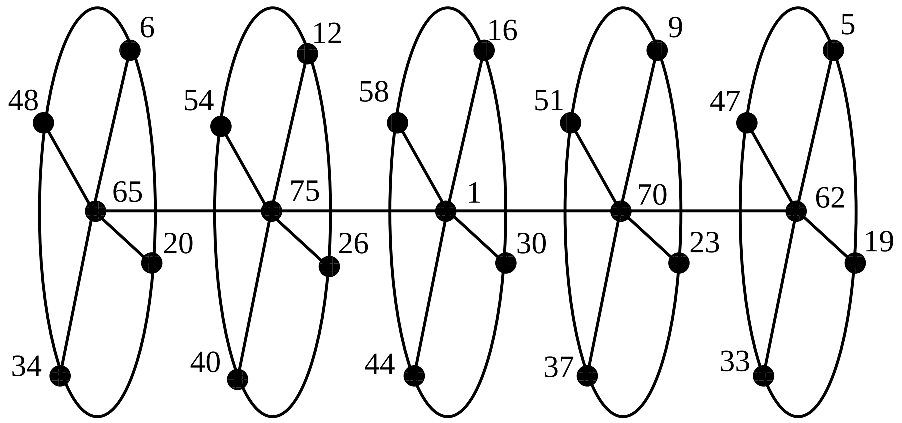

Figure 5.6 The radio coloring of $P_5 \odot C_4$ as in the proof of Theorem 5.1.1

5.2 AN IMPROVED UPPER BOUND FOR THE RADIO NUMBER OF $P_{2p+1} \odot H$

In this section, we improve the upper bound for the radio number of $P_n \odot H$ when n is odd. Also, we show that the improved upper bound is at most 1 more than the exact number.

Theorem 5.2.1. *If H is a graph of order m, then for $n = 2p+1 > 4$, $rn(P_n \odot H) \leq (2m+2)p^2 + (2m+4)p + m + 3$.*

Proof: Let $v_1 v_2 v_3 \ldots v_n$ be the path P_n and in $P_n \odot H$, H_i denotes the copy of H, corresponding to the vertex v_i. To obtain an upper bound for $rn(P_n \odot H)$, we first order the vertices of $P_n \odot H$ using which we define a radio coloring. For $j = 0,1,2,\ldots,m-1$, we choose x_{jn+i} from $H_{\frac{i}{2}}$ if i is even and from $H_{p+\frac{i+1}{2}}$ if i is odd, $i = 1,2,3,\ldots,n$. Next, we label the vertices $v_1, v_2, v_3, \ldots, v_{p+1}$ and $v_{p+2}, v_{p+3}, v_{p+4}, \ldots, v_n$ as $x_{mn+1}, x_{mn+3}, x_{mn+5}, \ldots, x_{mn+n}$ and $x_{mn+2}, x_{mn+4}, x_{mn+6}, \ldots, x_{mn+n-1}$ respectively.

We define a coloring f of $P_n \odot H$ by $f(x_1) = 1$, $f(x_i) = f(x_{i-1}) + (1+2p+2) - d(x_i, x_{i-1})$, $i = 2,3,4,\ldots,mn, mn+2, mn+3, mn+4, \ldots, mn+n$, and $f(x_{mn+1}) = f(x_{mn}) + (1+2p+2) - d(x_{mn+1}, x_{mn}) + 1$. By definition of f, it is easy to see that $|f(x_{i+1}) - f(x_i)| \geqslant (1+2p+2) - d(x_{i+1}, x_i)$ for all $i = 1,2,3,\ldots,mn+n-1$. For any i, $1 \leq i \leq mn+n-4$, it is easy to see that $f(x_{i+4}) - f(x_i) \geq 2p+2 = n+1 = diam(P_n \odot H)$. So, it remains to check the radio coloring condition for x_i with x_{i+2} and x_{i+3}.

For any $1 \leq i \leq mn+n-3$ and $i \notin \{mn-2, mn-1, mn\}$, we have

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= (f(x_{i+3}) - f(x_{i+2})) + (f(x_{i+2}) - f(x_{i+1})) + (f(x_{i+1}) - f(x_i)) \\ &\geq p+p+p \\ &\geq 2p+2 \end{aligned}$$

as $d(x_{i+1}, x_i)$, $d(x_{i+2}, x_{i+1})$ and $d(x_{i+3}, x_{i+2})$ are at the most $p+3$. If $i \in \{mn-2, mn-1, mn\}$, then one of $i+1, i+2$ and $i+3$ is $mn+1$. So, one of $d(x_{i+1}, x_i)$, $d(x_{i+2}, x_{i+1})$ and $d(x_{i+3}, x_{i+2})$ is $2p+1$ and the other two are at most $p+3$. Therefore,

$$\begin{aligned} f(x_{i+3}) - f(x_i) &= (f(x_{i+3}) - f(x_{i+2})) + (f(x_{i+2}) - f(x_{i+1})) + (f(x_{i+1}) - f(x_i)) \\ &\geq 3+p+p \\ &> 2p+2 \end{aligned}$$

If both x_i and x_{i+2} are on copies of H, then $d(x_{i+1}, x_i)$ and $d(x_{i+2}, x_{i+1})$ are at the most $p+3$, and $d(x_i, x_{i+2}) \geq 3$. So, $f(x_{i+2}) - f(x_i) \geq (2p+3-p-3) + (2p+3-p-3) = 2p \geq 1+2p+2-d(x_i, x_{i+2})$. If both x_i and x_{i+2} are not on the copies of H, then

$d(x_{i+1},x_i)$ and $d(x_{i+2},x_{i+1})$ are at the most $p+1$, and hence $f(x_{i+2})-f(x_i) > 2p+2$. If one of x_i and x_{i+2} is on a copy of H and the other is on P_n, then $i = mn-1$ or $i = mn$. Suppose that $i = mn-1$. Then $d(x_{i+1},x_i) = p+3$, $d(x_{i+2},x_{i+1}) = 2p+1$ and $d(x_{i+2},x_i) = p$ which implies $f(x_{i+2})-f(x_i) = 3+p = (1+2p+2)-d(x_{i+2},x_i)$. Suppose that $i = mn$. Then $d(x_{i+1},x_i) = 2p+1$, $d(x_{i+2},x_{i+1}) = p+1$ and $d(x_{i+2},x_i) = p$ which implies $f(x_{i+2})-f(x_i) = (p+3)+3 > (1+2p+2)-d(x_{i+2},x_i)$. Therefore f is a radio coloring of $P_n \odot H$.

By the ordering of vertices, the distance sum is as follows. For $j = 0,1,2,\ldots,m-1$, the sum $\sum_{i=2}^{n} d(x_{jn+i},x_{jn+i-1})$ is an alternating series of $p+2$ and $p+3$, $d(x_{jn+1},x_{jn}) = p+2$ $(j \neq 0)$, $d(x_{mn+1},x_{mn}) = 2p+1$ and $\sum_{i=2}^{n} d(x_{mn+i},x_{mn+i-1})$ is an alternating series of $p+1$ and p. That is,

$$\begin{aligned}\sum_{i=2}^{mn+n} d(x_i,x_{i-1}) &= m(p(p+2)+p(p+3))+(m-1)(p+2)\\ &\quad +2p+1+p(p+1)+p(p)\\ &= (2m+2)p^2+(6m+2)p+2m-1.\end{aligned}$$

By the definition of f, $\sum_{i=2}^{mn+n} \varepsilon_i = 1$. Now, by Lemma 1.4.2,

$$\begin{aligned} rn(f) = f(x_{mn+n}) &= (mn+n-1)(1+2p+2)\\ &\quad -((2m+2)p^2+(6m+2)p+2m-1)+1+1\\ &= (2m+2)p^2+(2m+4)p+m+3.\end{aligned}$$

□

Remark 5.2.2. Liu and Zhu (2005) have proved that $rn(P_{2p+1}) = 2p^2+3$. For a graph H of order m, by Theorem 5.1.1, $rn(P_{2p+1} \odot H) \leq (m+1)rn(P_{2p+1})+(2p+2-3)m+2(2p+1-1) = (2m+2)p^2+(2m+4)p+2m+3$ which is m more than the upper bound given in Theorem 5.2.1.

Example 5.2.3. The vertex ordering and the radio coloring, as in the proof of Theorem 5.2.1, of $P_5 \odot C_4$ are given in Figure 5.7 and Figure 5.8, respectively.

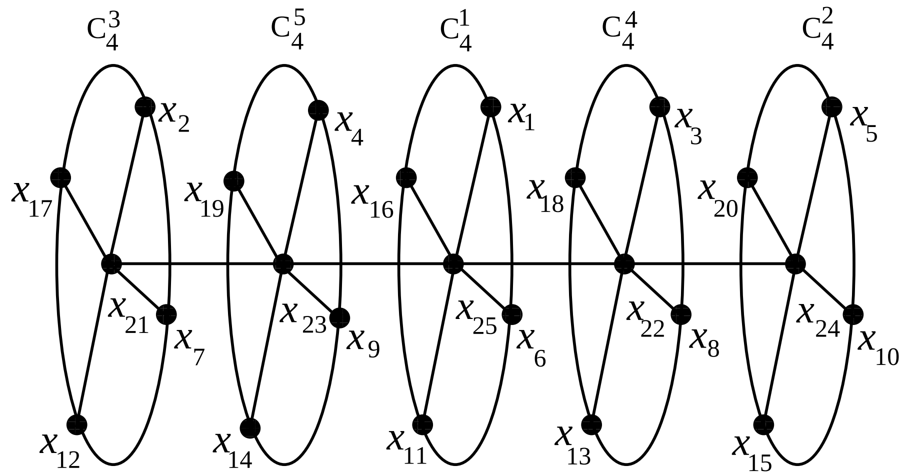

Figure 5.7 The vertex ordering of $P_5 \odot C_4$ as in the proof of Theorem 5.2.1

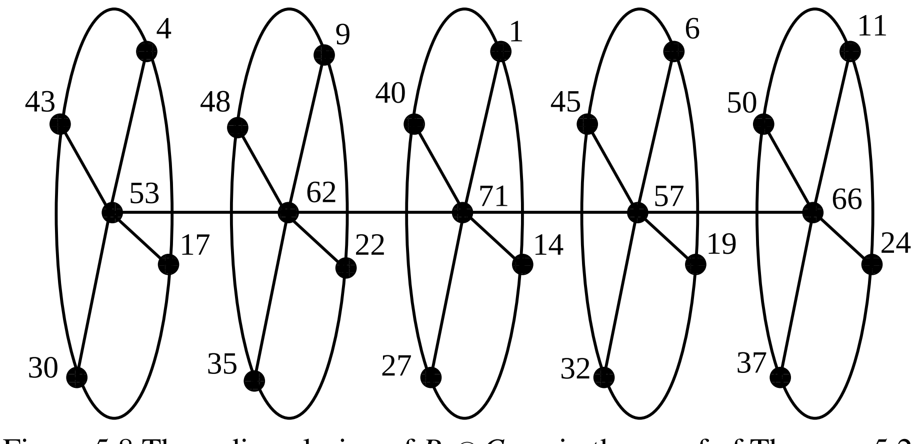

Figure 5.8 The radio coloring of $P_5 \odot C_4$ as in the proof of Theorem 5.2.1

Theorem 5.2.4. *If H is a graph of order m, then for* $n = 2p+1 > 4$, $rn(P_n \odot H)$ *is either* $(2m+2)p^2 + (2m+4)p + m + 2$ *or* $(2m+2)p^2 + (2m+4)p + m + 3$.

Proof: Let $P_n : v_1v_2v_3 \ldots v_n$ be the path. We choose $L_0 = \{v_{p+1}\}$. By Theorem 1.4.3, we get $|L_1| = m+2$, $|L_{p+1}| = 2m$ and $|L_i| = 2(m+1)$, $i = 2,3,4,\ldots p$ and $rn(P_n \odot H) \geq (2m+2)p^2 + (2m+4)p + m + 2$. Hence, by Theorem 5.2.1, the radio number of $P_n \odot H$ is either $(2m+2)p^2 + (2m+4)p + m + 2$ or $(2m+2)p^2 + (2m+4)p + m + 3$. $\square$

5.3 BOUNDS FOR THE RADIO NUMBER OF $Q_n \odot H$

Kola and Panigrahi (2010) have determined the radio number of hypercube by maximizing the distance sum simultaneously minimizing the epsilon sum (see Lemma 1.4.2). For hypercube Q_n, n odd, a minimal radio coloring is obtained using a vertex ordering $y_1, y_2, y_3, \ldots, y_{2^n}$ of Q_n such that $\{d(y_i, y_{i-1})\}_{i=2}^{2^n}$ is an alternating sequence of n and $\frac{n+1}{2}$ starting and ending with n, and all ε_is are zero. For n even, the same is obtained by a vertex ordering $y_1, y_2, y_3, \ldots, y_{2^n}$ of Q_n such that $\{d(y_i, y_{i-1})\}_{i=2}^{2^{n-1}}$ and $\{d(y_i, y_{i-1})\}_{i=2^{n-1}+2}^{2^n}$ are alternating sequences of n and $\frac{n}{2}$ starting with n and $d(y_{2^{n-1}+1}, y_{2^{n-1}}) = \frac{n+2}{2}$ with epsilon sum 1 ($\varepsilon_{2^{n-1}+1} = 1$). The vertices y_{2j+1}, $1 \leq j < 2^{n-1}$, are chosen at distance $\frac{n+1}{2}$ or $\frac{n}{2}$ to maximize the distance sum and minimize the epsilon sum simultaneously. Also, for any vertex u of Q_n, there exists exactly one vertex v of Q_n such that $d(u,v) = diam(Q_n) = n$. Since Q_n is a vertex-transitive graph, the ordering of the vertices of Q_n can be started with any vertex. In this section, we provide an upper bound for the radio number of $Q_n \odot H$, which is improved to that given in Theorem 5.1.1. Later, we obtain a lower bound for the same.

Theorem 5.3.1. *If H is any graph of order m, then for $n > 3$,*

$$rn(Q_n \odot H) \leq \begin{cases} (mn+n+3m+11)2^{n-2} - \frac{n-1}{2} - 1 & if\ n\ is\ odd, \\ (mn+n+4m+12)2^{n-2} - \frac{n}{2} - 2 & if\ n\ is\ even. \end{cases}$$

Proof: For n odd, we first obtain an ordering of vertices of $Q_n \odot H$, which we use to obtain a radio coloring of $Q_n \odot H$. We choose $x_1, x_2, x_3, \ldots, x_{m2^n}$ from the copies of H as follows.

1. For $j = 0, 1, 2, \ldots, m-1$, $x_{j2^n+1}, x_{j2^n+2}, x_{j2^n+3}, \ldots, x_{(j+1)2^n}$ are on different copies of H such that $\{d(x_{j2^n+i}, x_{j2^n+i-1})\}_{i=2}^{2^n}$ is an alternating sequence of $n+2$ and $\frac{n+1}{2}+2$ starting and ending with $n+2$.

2. For $j = 1, 2, 3, \ldots, m-1$, $d(x_{j2^n+1}, x_{j2^n}) = \frac{n+1}{2} + 2$.

Now, we choose $x_{m2^n+1}, x_{m2^n+2}, x_{m2^n+3}, \ldots, x_{(m+1)2^n}$ on Q_n such that

1. $d(x_{m2^n+1}, x_{m2^n}) = \frac{n+1}{2} + 1$.

2. $\{d(x_{m2^n+i}, x_{m2^n+i-1})\}_{i=2}^{2^n}$ is an alternating sequence of n and $\frac{n+1}{2}$ starting and ending with n.

We define a coloring f of $Q_n \odot H$ as $f(x_1) = 1$ and for $2 \le i \le (m+1)2^n$, $f(x_i) = f(x_{i-1}) + (1+n+2) - d(x_i, x_{i-1})$. For the vertices on the copies of H, $d(x_i, x_{i+2}) = \frac{n+1}{2}$; $d(x_i, x_{i+3}) = \frac{n+1}{2} + 2$ when i is odd; and $f(x_{i+3}) - f(x_i) = n+2$ when i is even. For the vertices on Q_n, $d(x_i, x_{i+2}) = \frac{n-1}{2}$; $d(x_i, x_{i+3}) = \frac{n+1}{2}$ when i is odd; and $f(x_{i+3}) - f(x_i) = n+8$ when i is even. Now, it is easy to verify that f is a radio coloring of $Q_n \odot H$. By the ordering of the vertices of $Q_n \odot H$, we have

$$\begin{aligned}\sum_{i=2}^{(m+1)2^n} d(x_i, x_{i-1}) &= \left(\left(n+2+\frac{n+1}{2}+2\right)(2^{n-1}-1)+n+2\right)m \\ &\quad + \left(\left(\frac{n+1}{2}+2\right)m-1\right) + \left(\left(n+\frac{n+1}{2}\right)(2^{n-1}-1)+n\right) \\ &= (3mn+9m+3n+1)2^{n-2} - \frac{n+3}{2}.\end{aligned}$$

Also, by Lemma 1.4.2,

$$\begin{aligned}rn(f) = f(x_{(m+1)2^n}) &= ((m+1)2^n - 1)(n+3) \\ &\quad - \left((3mn+9m+3n+1)2^{n-2} - \frac{n+3}{2}\right) + 1 \\ &= (mn+n+3m+11)2^{n-2} - \frac{n-1}{2} - 1.\end{aligned}$$

Let n be even. We choose $x_1, x_2, x_3, \ldots, x_{m2^n}$ from the copies of H as follows.

1. For $j = 0, 1, 2, \ldots, m-1$, $x_{j2^n+1}, x_{j2^n+2}, x_{j2^n+3}, \ldots, x_{(j+1)2^n}$ are on different copies of H such that $\{d(x_{j2^n+i}, x_{j2^n+i-1})\}_{i=2}^{2^{n-1}}$ and $\{d(x_{j2^n+i}, x_{j2^n+i-1})\}_{i=2^{n-1}+2}^{2^n}$ are alternating sequences of $n+2$ and $\frac{n}{2}+2$, starting and ending with $n+2$, and $d(x_{j2^n+2^{n-1}+1}, x_{j2^n+2^{n-1}}) = \frac{n+2}{2} + 2$.

2. For $j = 1, 2, 3, \ldots, m-1$, $d(x_{j2^n+1}, x_{j2^n}) = \frac{n}{2} + 2$.

Now, we choose $x_{m2^n+1}, x_{m2^n+2}, x_{m2^n+3}, \ldots, x_{(m+1)2^n}$ on Q_n such that

1. $d(x_{m2^n+1}, x_{m2^n}) = \frac{n}{2} + 1$.

2. $\{d(x_{m2^n+i}, x_{m2^n+i-1})\}_{i=2}^{2^{n-1}}$ and $\{d(x_{m2^n+i}, x_{m2^n+i-1})\}_{i=2^{n-1}+2}^{2^n}$ are alternating sequences of n and $\frac{n}{2}$, starting and ending with n, and $d(x_{m2^n+2^{n-1}+1}, x_{m2^n+2^{n-1}}) = \frac{n+2}{2}$.

We define a coloring g of $Q_n \odot H$ as

$$g(x_i) = \begin{cases} 1 & if\ i = 1, \\ g(x_{i-1}) + (1+n+2) - d(x_i, x_{i-1}) + 1 & if\ i = j2^n + 2^{n-1} + 1\ and \\ & \quad 0 \leq j \leq m-1, \\ g(x_{i-1}) + (1+n+2) - d(x_i, x_{i-1}) & otherwise. \end{cases}$$

Similar to n odd case, we can verify that g is a radio coloring of $Q_n \odot H$. By the ordering of the vertices of $Q_n \odot H$, we have

$$\begin{aligned} \sum_{i=2}^{(m+1)2^n} d(x_i, x_{i-1}) &= \left(\left(n+2+\frac{n}{2}+2\right)(2^{n-1}-2) + 2(n+2) + \frac{n+2}{2} + 2\right)m \\ &\quad + \left(\left(\frac{n}{2}+2\right)m - 1\right) + \left(\left(n+\frac{n}{2}\right)(2^{n-1}-2) + 2n + \frac{n+2}{2}\right) \\ &= (3mn + 8m + 3n)2^{n-2} - \frac{n}{2} + m. \end{aligned}$$

Now, by the coloring g, we have $\sum_{i=2}^{(m+1)2^n} \varepsilon_i = m$. By Lemma 1.4.2, we get

$$\begin{aligned} rn(g) = g(x_{(m+1)2^n}) &= ((m+1)2^n - 1)(n+3) \\ &\quad - \left((3mn + 8m + 3n)2^{n-2} - \frac{n}{2} + m\right) + m + 1 \\ &= (mn + n + 4m + 12)2^{n-2} - \frac{n}{2} - 2. \end{aligned}$$ □

Remark 5.3.2. Kola and Panigrahi (2010) have proved that $rn(Q_n)$ is $(\frac{n+3}{2})2^{n-1} - \frac{n-1}{2}$ if n is odd and $(\frac{n+4}{2})2^{n-1} - \frac{n}{2}$ if n is even. For odd $n > 4$ and a graph H of order m, by Theorem 5.1.1, $rn(Q_n \odot H) \leq (m+1)rn(Q_n) + (n+2-3)m + 2(2^n - 1) = (m+1)\left((\frac{n+3}{2})2^{n-1} - \frac{n-1}{2}\right) + (n+2-3)m + 2(2^n-1) = (mn+n+3m+11)2^{n-2} - \frac{n-1}{2} - 1 + \frac{m(n-1)}{2} - 1$ which is $\frac{m(n-1)}{2} - 1$ more than the upper bound given in Theorem 5.3.1. For even $n > 3$ and a graph H of order m, by Theorem 5.1.1, $rn(Q_n \odot H) \leq (mn+n+4m+12)2^{n-2} - \frac{n}{2} - 2 - \frac{m(n+2)}{2}$ which is $\frac{m(n+2)}{2} - 1$ more than the upper bound given in Theorem 5.3.1.

For any three vertices x, y and z of $Q_n \odot H$, we have Table 5.1. The following

Positions of x, y and z in $Q_n \odot H$	$\max\{d(x,y)+d(y,z)+d(z,x)\}$
All the three are on the copies of H	$2n+6$
Only one of x, y and z is on Q_n	$2n+4$
Only two of x, y and z are on Q_n	$2n+2$
All the three are on Q_n	$2n$

Table 5.1 The maximum of $d(x,y)+d(y,z)+d(z,x)$ for any three vertices of $Q_n \odot H$ depending on their positions

theorem gives a lower bound for the radio number of $Q_n \odot H$, where H is an arbitrary graph.

Theorem 5.3.3. *If H is a graph of order m, then*

$$rn(Q_n \odot H) \geq \begin{cases} (mn+n+3m+7)2^{n-2} - \frac{n-1}{2} + 2 & if\ n\ is\ odd, \\ (mn+n+4m+8)2^{n-2} - \frac{n}{2} & if\ n\ is\ even. \end{cases}$$

Proof: Let f be a radio coloring of $Q_n \odot H$ and $x_1, x_2, x_3, \ldots, x_{(m+1)2^n}$ be an ordering of vertices of $Q_n \odot H$ such that $f(x_i) < f(x_{i+1})$ for all i. For any $2 \leq i \leq (m+1)2^n$, let $d_i = d(x_i, x_{i-1})$ and $\varepsilon_i = f(x_i) - f(x_{i-1}) - ((1+n+2) - d(x_i, x_{i-1}))$. Let $n > 3$ be odd. First, we show that $\sum_{i=2}^{(m+1)2^n} d_i - \sum_{i=2}^{(m+1)2^n} \varepsilon_i$ is at most $\left(\frac{3n+9}{2}\right)(m+1)2^{n-1} - 2^n - \frac{n+1}{2} - 4$.

Now, for any $2 \le i \le (m+1)2^n - 1$, depending on the positions of x_{i-1}, x_i and x_{i+1}, bound for $d_i + d_{i+1} - (\varepsilon_i + \varepsilon_{i+1})$ is given in Table 5.2.

Positions of x_{i-1}, x_i and x_{i+1}	$d_i + d_{i+1} - (\varepsilon_i + \varepsilon_{i+1})$
All the three are on the copies of H	$\le n+2+\frac{n+1}{2}+2-(0)$
Only one of x_{i-1} and x_{i+1} is on Q_n and x_i is on a copy of H	$\le n+2+\frac{n+1}{2}+1-(0)$
Only x_i is on Q_n	$\le n+1+\frac{n+1}{2}+2-(0)$
Only x_i is on H	$\le n+1+\frac{n+1}{2}+1-(0)$
Only one of x_{i-1} and x_{i+1} is on H and x_i is on Q_n	$\le n+1+\frac{n+1}{2}+1-(0)$
All the three are on Q_n	$\le n+\frac{n+1}{2}+1-(0)$

Table 5.2 Upper bounds for $d_i + d_{i+1} - (\varepsilon_i + \varepsilon_{i+1})$ depending on the positions of x_{i-1}, x_i and x_{i+1} in a radio coloring of $Q_n \odot H$, n odd

From Table 5.2, it is easy to see that

$$\begin{aligned}\sum_{i=2}^{(m+1)2^n} (d_i - \varepsilon_i) &\le \left(n+1+\frac{n+1}{2}+2\right)2^n \\ &\quad + \left(n+2+\frac{n+1}{2}+2\right)((m-1)2^{n-1}-1)+n+2 \\ &= (3n+9)(m+1)2^{n-2} - 2^n - \frac{n+1}{2} - 4\end{aligned}$$

Now, by Lemma 1.4.2, we have

$$\begin{aligned}rn(Q_n \odot H) &\ge ((m+1)2^n - 1)(1+n+2) \\ &\quad - \left((3n+9)(m+1)2^{n-2} - 2^n - \frac{n+1}{2} - 4\right) + 1 \\ &= (mn+n+3m+7)2^{n-2} - \frac{n-1}{2} + 2\end{aligned}$$

Let $n \ge 4$ be even. Now, for any $2 \le i \le (m+1)2^n - 1$, depending on the positions of x_{i-1}, x_i and x_{i+1}, bound for $d_i + d_{i+1} - (\varepsilon_i + \varepsilon_{i+1})$ is given in Table 5.3.

From Table 5.2, it is easy to see that $\sum\limits_{i=2}^{(m+1)2^n} (d_i - \varepsilon_i) \le (3mn+3n+8m+4)2^{n-2} -$

Positions of x_{i-1}, x_i and x_{i+1}	$d_i + d_{i+1} - (\varepsilon_i + \varepsilon_{i+1})$
All the three are on the copies of H	$\leq n+2+\frac{n}{2}+2-(0)$
Only one of x_{i-1} and x_{i+1} is on Q_n and x_i is on a copy of H	$\leq n+2+\frac{n}{2}+1-(0)$
Only x_i is on Q_n	$\leq n+1+\frac{n}{2}+2-(0)$
Only x_i is on H	$\leq n+1+\frac{n}{2}+1-(0)$
Only one of x_{i-1} and x_{i+1} is on H and x_i is on Q_n	$\leq n+1+\frac{n}{2}+1-(0)$
All the three are on Q_n	$\leq n+\frac{n}{2}+1-(0)$

Table 5.3 Upper bounds for $d_i + d_{i+1} - (\varepsilon_i + \varepsilon_{i+1})$ depending on the positions of x_{i-1}, x_i and x_{i+1} in a radio coloring of $Q_n \odot H$, n even

$\frac{n}{2} - 2$. Now, by Lemma 1.4.2, we have $rn(Q_n \odot H) \geq (mn+n+4m+8)2^{n-2} - \frac{n}{2}$. □

5.4 SUMMARY

In this chapter, we have studied radio k-coloring for the corona $G \odot H$ of arbitrary graphs G and H. We have obtained an upper bound for $rc_k(G \odot H)$. Also, we have proved that this bound is sharp by determining the radio k-chromatic number of $K_n \odot H$ and the radio number of $P_{2p+1} \odot H$. Further, we have improved the upper bounds and obtained lower bounds for the radio numbers of $P_{2p+1} \odot H$ and $Q_n \odot H$.

CHAPTER 6

THE *k*-DISTANCE CHROMATIC NUMBER OF TREES AND CYCLES

> "*Many of the concepts, theorems, and problems of Graph Theory lie in the shadows of the Four Color Problem.*"
>
> - Gary Chartrand (2009)

Proper coloring of graphs is motivated by coloring regions of a map. The famous Four Color Problem was first posed by Francis Guthrie, a student of Augustus De Morgan, in 1852 as "The regions of every map can be colored with four or fewer colors in such a way that every two regions sharing a boundary are colored differently". In Graph Theory, it was famous as Four Color Conjecture stated as "The vertices of every planar graph can be colored with four or fewer colors in such a way that no two adjacent vertices receive the same color". The conjecture remained unsolved until 1977. In the process of solving the conjecture many false proofs were given. Finally, the conjecture was proved by Appel and Haken (1977) with the aid of computer. Kramer and Kramer (1969b) have introduced k-distance coloring of graphs as a generalization of proper coloring. In the recent times, few authors have studied k-distance coloring as a variation of FAP. We recall the definition and the theorem below. The theorem gives a lower bound for the k-distance chromatic number of a graph.

Theorem 1.3.10. *(Sharp, 2007) For any graph G and a positive integer k,*

$$\chi_k(G) \geq \begin{cases} \max\limits_{v \in V(G)} \left| V\left(G_v^{\frac{k}{2}}\right)\right| & if\ k\ is\ even, \\ \max\limits_{v \in V(G)} \left| V\left(G_v^{\frac{k-1}{2}}\right)\right| + 1 & if\ k\ is\ odd. \end{cases}$$

In this chapter, we study k-distance coloring of graphs. We improve the lower bound given in Theorem 1.3.10 for the k-distance chromatic number of arbitrary graphs, when k is odd. We prove that the trees achieve the lower bound. Also, we determine the k-distance chromatic number of cycles. Although, k-distance coloring is defined for all positive integers k, it is mostly studied for $k = 2$ and that too for planar graphs, as it is an immediate generalization of the Four Color Theorem and due to Conjecture 1.3.11 by Wegner (1977). Motivated by this, we determine the 2-distance chromatic number of cactus graphs.

6.1 THE *k*-DISTANCE CHROMATIC NUMBER OF TREES

In this section, for k odd, we improve the lower bound for the k-distance chromatic number of an arbitrary graph given by Sharp (2007). Later, we prove that trees attain the lower bound. Before we give a lower bound for $\chi_k(G)$, similar to G_v^r, we define G_S^r for any subset S of $V(G)$. For a subset S of $V(G)$ and a vertex $v \in V(G)$, the distance from v to S, denoted by $d(S,v)$, is $\min\{d(u,v) : u \in S\}$.

Definition 6.1.1. *For any non-negative integer r and a subset S of the vertex set of a graph G, the graph G_S^r denotes the subgraph of G induced by the vertices of G which are at distance less than or equal to r from S.*

Theorem 6.1.2. *If G is a graph and k is any positive integer, then*

$$\chi_k(G) \geq \begin{cases} \max\left\{ \left| V\left(G_v^{\frac{k}{2}}\right)\right| : v \in V(G)\right\} & if\ k\ is\ even, \\ \max\left\{ \left| V\left(G_S^{\frac{k-1}{2}}\right)\right| : S\ is\ a\ maximal\ clique\ in\ G\right\} & if\ k\ is\ odd. \end{cases}$$

Proof: If k is even, the result follows from Theorem 1.3.10. Suppose that k is odd. Let S be a maximal clique in G and w, w' be any two vertices of $G_S^{\frac{k}{2}}$. Then $d(S,w) \leq \frac{k-1}{2}$ and $d(S,w') \leq \frac{k-1}{2}$. Therefore, $d(u,w) \leq \frac{k-1}{2}$ and $d(v,w') \leq \frac{k-1}{2}$ for some $u,v \in S$. Since u and v are adjacent, $d(w,w') \leq d(u,w)+d(v,w')+d(u,v) \leq k$. Therefore, in any k-distance coloring, w and w' should receive different colors. Hence $\chi_k(G) \geq \max\left\{\left|V\left(G_S^{\frac{k-1}{2}}\right)\right| : S \text{ is a maximal clique in } G\right\}$. □

In the introductory paper on $L(2,1)$-coloring, Griggs and Yeh (1992) have proved that the $L(2,1)$-span of a tree with the maximum degree Δ is either $\Delta+1$ or $\Delta+2$ by giving an $L(2,1)$-coloring. Motivated by this, we give a k-distance coloring of a tree and determine $\chi_k(T)$. For this, we use the lemma below. Recall that, in a graph G, $e_G(u)$ (or simply $e(u)$) denotes the eccentricity of a vertex u in G, $diam(G)$ and $rad(G)$ are the diameter and the radius of G, respectively.

Lemma 6.1.3. *Let T be a tree with n vertices. Let $T_{i-1} = T_i - v_i, i = n, n-1, \ldots, 2, 1$, where $T_n = T$ and v_i is a vertex of T_i such that $e_{T_i}(v_i) = diam(T_i)$. If $T' = (T_i)^r_{v_i}$, then $diam(T') \leq r$.*

Proof: On the contrary, suppose $diam(T') > r$. Since $e_{T_i}(v_i) = diam(T_i)$, v_i is a leaf of T_i and so in T'. Let v_j and v_l be antipodal vertices of T' (that is, $d(v_j,v_l) = diam(T')$). Let P_j and P_l be the v_j,v_i-path and v_l,v_i-path in T' respectively. Since v_i is a leaf of T' and the paths P_j and P_l end at v_i, there must be a vertex common to them other than v_i. Let u be the first vertex of P_j which is in P_l. Then v_j,u-subpath of P_j followed by u,v_l-subpath of P_l give the v_j,v_l-path in T'. If $d(u,v_i) \geq d(u,v_l)$, then $d(v_j,v_l) = d(v_j,u)+d(u,v_l) \leq d(v_j,u)+d(u,v_i) \leq r$, a contradiction. So $d(u,v_i) < d(u,v_l)$.

Using the above inequality $d(u,v_i) < d(u,v_l)$, we show that there is a path in T_i whose length is greater than the $diam(T_i)$ which is a contradiction. Let v be a vertex of T_i such that $d(v,v_i) = diam(T_i)$. Let P be the v_i,v-path in T_i. Since v_i is a leaf and the paths P_j and P end at v_i, there must be a vertex common to them other than v_i. Let w be the first vertex of P which is also in P_j. If w is in v_j,u-subpath of P_j, then v,w-subpath of P followed by w,u-subpath of P_j, followed by u,v_l-subpath of P_l is the v,v_l-path. So $d(v,v_l) = d(v,w) + d(w,u) + d(u,v_l) = d(v,u) + d(u,v_l) > d(v,u) + d(u,v_i) = d(v,v_i) = diam(T')$, a contradiction. Therefore w cannot be in v_j,u-subpath of P_j. If w is in u,v_i-subpath of P_j, then v,w-subpath of P followed by w,u-subpath of P_j, followed by u,v_l-subpath of P_l is the v,v_l-path. So $d(v,v_l) = d(v,w) + d(w,u) + d(u,v_l) = d(v,u) + d(u,v_l) > d(v,u) + d(u,v_i) \geq d(v,v_i) = diam(T')$, a contradiction. Therefore $diam(T') \leq r$. □

Theorem 6.1.4. *For any tree T,*

$$\chi_k(T) = \begin{cases} \max\limits_{v\in V(T)} \left|V\left(T_v^{\frac{k}{2}}\right)\right| & if\ k\ is\ even, \\ \max\limits_{uv\in E(T)} \left|V\left(T_{uv}^{\frac{k-1}{2}}\right)\right| & if\ k\ is\ odd. \end{cases}$$

Proof: Since in a tree any maximal clique is an edge, from Theorem 6.1.2 it is clear that

$$\chi_k(T) \geq \begin{cases} \max\limits_{v\in V(T)} \left|V\left(T_v^{\frac{k}{2}}\right)\right| & \text{if } k \text{ is even,} \\ \max\limits_{uv\in E(T)} \left|V\left(T_{uv}^{\frac{k-1}{2}}\right)\right| & \text{if } k \text{ is odd.} \end{cases}$$

Now, we give a k-distance α-coloring for the tree T, where $\alpha = \max\limits_{v\in V(T)} \left|V\left(T_v^{\frac{k}{2}}\right)\right|$, if k is even and $\alpha = \max\limits_{uv\in E(T)} \left|V\left(T_{uv}^{\frac{k-1}{2}}\right)\right|$, if k is odd. Let T_i and v_i, $i = 1,2,3,\ldots,n$ are subtrees and vertices of T, respectively, as given in Lemma 6.1.3. We assign the color 1 to v_1 and 2 to v_2. Suppose $v_1, v_2, \ldots, v_{i-1}$ are colored. Now, to color v_i, we consider the tree $T' = (T_i)_{v_i}^k$. By Lemma 6.1.3, $diam(T') \leq k$.

Case I: k is even

Let x be a vertex of T' such that $e_{T'}(x) = rad(T') = \left\lceil \frac{diam(T')}{2} \right\rceil \leq \left\lceil \frac{k}{2} \right\rceil = \frac{k}{2}$. Then every vertex of T' is distance at most $\frac{k}{2}$ from x. So T' is a subgraph of $T_x^{\frac{k}{2}}$. Since $|V(T') - \{v_i\}| < |V(T')| \leq \left| V\left(T_x^{\frac{k}{2}}\right) \right| \leq \max_{v \in V(T)} \left| V\left(T_v^{\frac{k}{2}}\right) \right| = \alpha$, we have at least on color not used in T' to color v_i.

Case II: k is odd.

Subcase (i): $diam(T')$ is odd.

Since the diameter of T' is odd, the center of T' is an edge, say xy. So $e_{T'}(x) = e_{T'}(y) = rad(T') = \left\lceil \frac{diam(T')}{2} \right\rceil \leq \left\lceil \frac{k}{2} \right\rceil = \frac{k+1}{2}$. Therefore, every vertex of T' is at distance at most $\frac{k+1}{2}$ from x and y. If w is a vertex of T' with $d(w,x) = \frac{k+1}{2}$, then $d(w,y) \leq \frac{k-1}{2}$ and vice versa. So every vertex of T' is at distance less than or equal to $\frac{k-1}{2}$ from x or y. So T' is a subgraph of $T_{xy}^{\frac{k-1}{2}}$. Since $|V(T') - \{v_i\}| < |V(T')| \leq \left| V\left(T_{xy}^{\frac{k-1}{2}}\right) \right| \leq \max_{uv \in E(T)} \left| V\left(T_{uv}^{\frac{k-1}{2}}\right) \right| = \alpha$, we have a color not used in T' to color v_i.

Subcase (ii): $diam(T')$ is even.

Let x be the center of T'. Then $e_{T'}(x) = rad(T') = \left\lceil \frac{diam(T')}{2} \right\rceil \leq \left\lceil \frac{k-1}{2} \right\rceil = \frac{k-1}{2}$. Therefore, every vertex of T' is at distance less than or equal to $\frac{k-1}{2}$ from x. Since $|V(T') - \{v_i\}| < |V(T')| \leq \left| V\left(T_x^{\frac{k-1}{2}}\right) \right| \leq \left| V\left(T_{xy}^{\frac{k-1}{2}}\right) \right| \leq \max_{uv \in E(T)} \left| V\left(T_{uv}^{\frac{k-1}{2}}\right) \right| = \alpha$, where y is any neighbor of x in T', we have a color not used in T' to color v_i. □

Example 6.1.5. In Figure 6.1, a 4-distance coloring, as in the proof of Theorem 6.1.4, for a tree T with 42 vertices is given. A 3-distance coloring for the same tree is given in Figure 6.2.

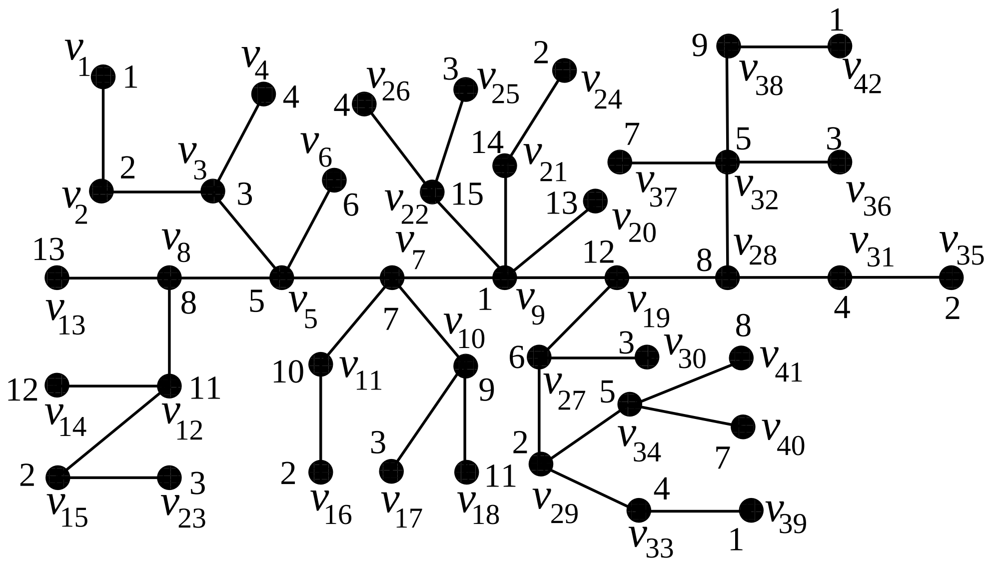

Figure 6.1 A 4-distance coloring of a tree as in the proof of Theorem 6.1.4

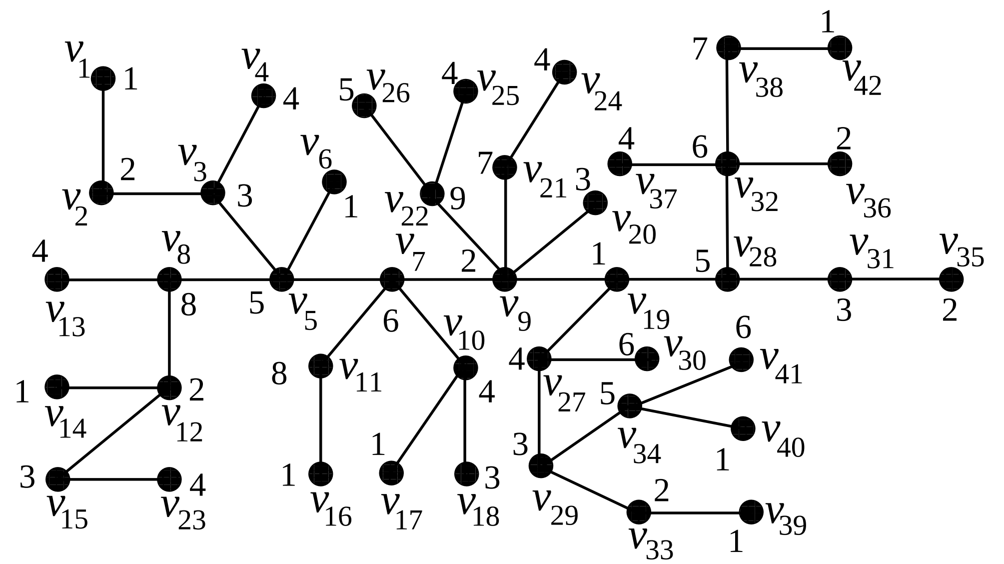

Figure 6.2 A 3-distance coloring of a tree as in the proof of Theorem 6.1.4

6.2 THE *k*-DISTANCE CHROMATIC NUMBER OF CYCLES

Kramer and Kramer (1969a,b) have determined $\chi_k\left(C_{l(k+1)}\right)$ as $k+1$. Now, we find the same for any cycle C_n.

Theorem 6.2.1. *For any cycle C_n, $\chi_k(C_n) = k+1+\left\lceil \frac{r}{l} \right\rceil$, where r and l are integers such that $n = l(k+1)+r$, $0 \le r < k+1$.*

Proof: In any k-distance coloring of a graph G if two vertices u and v receive the same color, then $d(u,v)$ is at least $k+1$. Since $n = l(k+1)+r$, any k-distance coloring of

C_n assign a color c to at most l vertices of C_n. Therefore, any k-distance coloring of C_n needs at least $\lceil \frac{n}{l} \rceil = k+1+\left\lceil \frac{r}{l} \right\rceil$ colors. Now, we show that $\chi_k(C_n) \leq k+1+\left\lceil \frac{r}{l} \right\rceil$ by defining a k-distance coloring of C_n. Let $C_n : v_1, v_2, \ldots, v_n, v_1$ be the cycle and $\alpha = k+1+\left\lceil \frac{r}{l} \right\rceil$.

Now,

$$\begin{aligned} n &= l(k+1)+r \\ &= l(k+1)+l'\left\lceil \frac{r}{l} \right\rceil + r', \text{where } r' \text{ and } l' \text{ are integers such that } 0 \leq r' < \left\lceil \frac{r}{l} \right\rceil \\ &= l'\left(k+1+\left\lceil \frac{r}{l} \right\rceil\right) + \left(l-l'\right)(k+1)+r' \\ &= l'\alpha + \left(l-l'\right)(k+1)+r'. \end{aligned}$$

It is easy to see that $l' \leq l$. Define a map f from $\{v_1, v_2, \ldots, v_n\}$ to $\{1, 2, \ldots, \alpha\}$ by

$$\begin{aligned} f(v_i) &= t \text{ if } i \equiv t \pmod{\alpha}, \ 1 \leq i \leq l'\alpha, \\ f\left(v_{l'\alpha+j}\right) &= t \text{ if } j \equiv t \pmod{k+1}, \ 1 \leq j \leq \left(l-l'\right)(k+1) = n-r', \\ f(v_{n-s}) &= \alpha - s, \ 0 \leq s < r'. \end{aligned}$$

It is easy to verify k-distance coloring condition for v_i and v_j, $1 \leq i < j \leq n-r'$. Since the color given to $v_{n-(r'-1)}$ is $\alpha-(r'-1) > \alpha - \lceil \frac{r}{l} \rceil + 1 \geq k+2 > k+1$, the k-distance coloring condition is satisfied between v_i and v_{n-s}, $1 \leq i \leq l'\alpha$, $0 \leq s < r'$; and $v_{l'\alpha+j}$ and v_{n-s}, $1 \leq j \leq n-r'$, $0 \leq s < r'$. □

Remark 6.2.2. *For $k < n$, from Theorem 6.1.2, $\chi_k(C_n) \geq k+1$ which is $\lceil \frac{r}{l} \rceil$, $0 \leq \lceil \frac{r}{l} \rceil \leq k$, less than the exact number.*

Example 6.2.3. In Figure 6.3, a 5-distance coloring, as in the proof of Theorem 6.2.1, for C_{17} is given.

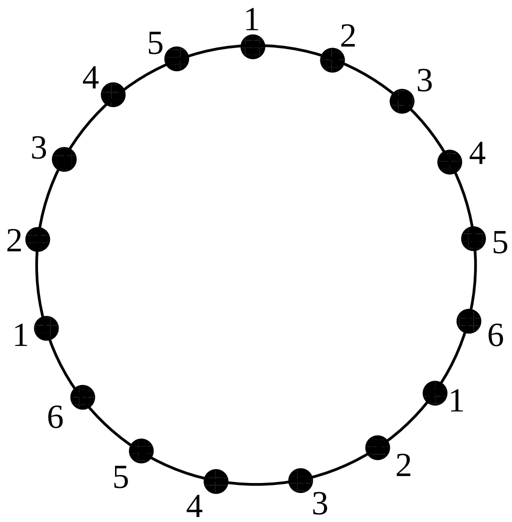

Figure 6.3 A 5-distance coloring of C_{17}

6.3 THE 2-DISTANCE CHROMATIC NUMBER OF CACTUS

Recall that, a cactus graph is a connected graph in which no two cycles share an edge. In this section, we determine the 2-distance chromatic number of cactus graph.

Theorem 6.3.1. *If G is a cactus graph with maximum degree $\Delta \geq 3$, then*

$$\chi_2(G) = \begin{cases} 5 & if\ G\ contains\ C_5\ and\ \Delta = 3, \\ \Delta + 1 & otherwise. \end{cases}$$

Proof: Let G be a graph with maximum degree $\Delta \geq 3$. Let $\alpha = 5$ if G contains C_5 and $\Delta = 3$, otherwise $\alpha = \Delta + 1$. Since $\chi_2(C_5) = 5$ and from Theorem 6.1.2, we have $\chi_2(G) \geq \alpha$. Now, we give a procedure to define a 2-distance coloring of G using α colors. Let C be any cycle in G. It is clear that $\chi_2(C) \leq \alpha$. We color the vertices of C using Theorem 6.2.1. Let u be a vertex on C with $deg(u) \geq 3$. As $deg(u) \leq \Delta$, u has at most $\Delta - 2$ neighbors which are not colored. Since $\alpha \geq \Delta + 1$, we have at least $\Delta - 2$ colors (excluding the colors given to u and its neighbors on C) to color the neighbors of u not on C. Suppose that u is on any other cycle C'. Let $C' : uu_3u_4\dots u_mu_1u$ and c_1, c_2 and c_3 be the colors assigned to u_1, u and u_3 respectively. We choose $\chi_2(C') - 3$ number of colors $c_4, c_5, c_6, \dots, c_{\chi_2(C')}$ from $\{1, 2, 3, \dots, \alpha\}$ other than c_1, c_2, c_3. We color the remaining vertices of C' using the colors $c_1, c_2, c_3, \ldots, c_{\chi_2(C')}$ as in Theorem 6.2.1 (color c_i refers to color i in Theorem 6.2.1). Now, we choose a colored vertex of G which has uncolored neighbors and continue as above until all the vertices of G are colored. It is easy to see that α is the maximum color and it is used to either a vertex in C_5 or to a maximum degree vertex or any one of its neighbors. □

Example 6.3.2. A cactus containing C_5 and having the maximum degree 3 along with the 2-distance coloring defined in the proof of Theorem 6.3.1 is given in Figure 6.4. First, the vertices of the cycle C are colored. The vertices v_1, v_2 and v_5 are the colored vertices which are at distance at most 2 from v_6. So, the least possible color available for v_6 is 3. After this, v_7, v_8, v_9 and v_{10} are colored, respectively. The cycle containing v_8 is C_5, so the vertices v_{11} and v_{12} are colored with a color different from that of v_8, v_9 and v_{10}. Similarly, the vertices $v_{13}, v_{14}, v_{15}, \ldots, v_{20}$ are colored. Since the cycle containing v_{18} is C_6 and $\chi_2(C_6)$ is 3, the remaining vertices of the cycle are colored using the colors of the vertices v_{18}, v_{19} and v_{20}. In Figure 6.5, a 2-distance coloring of a cactus with maximum degree 7 is given.

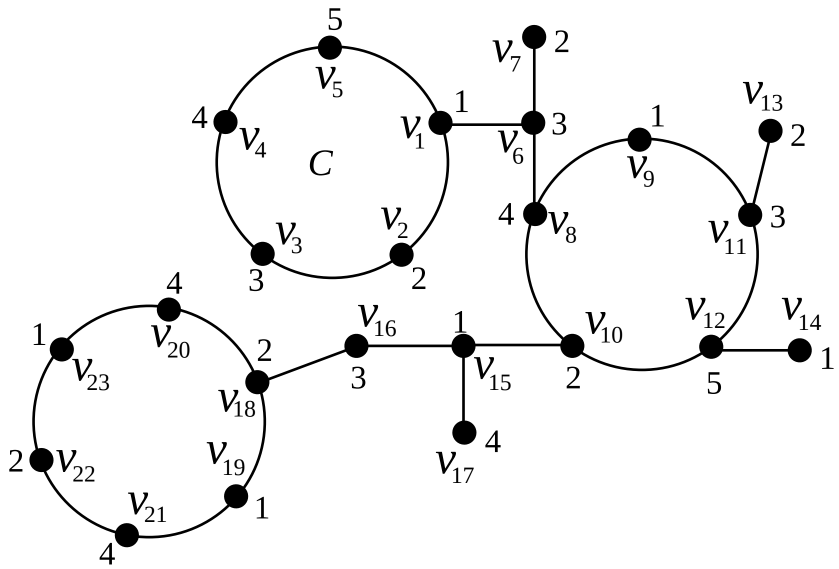

Figure 6.4 A 2-distance coloring, as in the proof of Theorem 6.3.1, of a cactus containing C_5 and having maximum degree 3

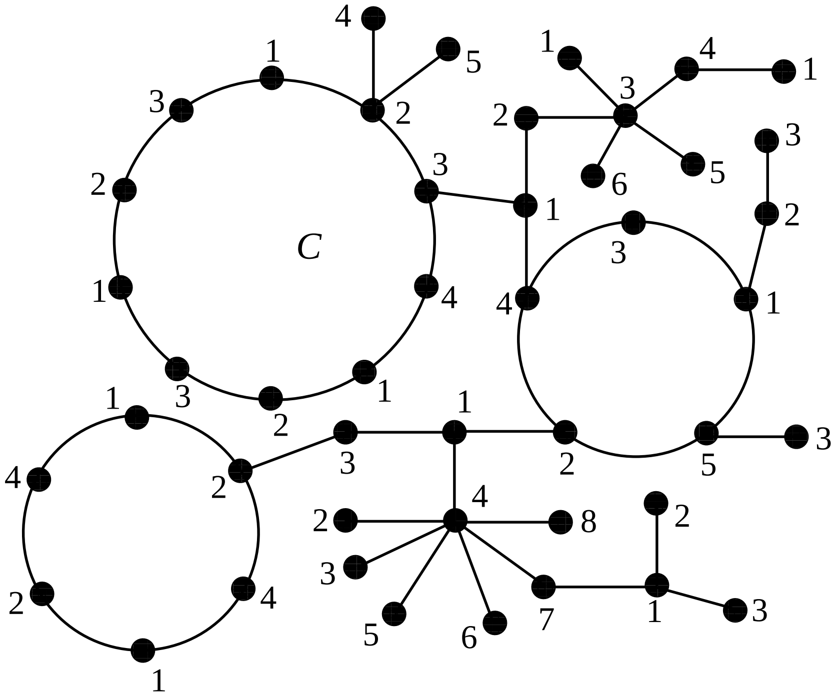

Figure 6.5 A 2-distance coloring, as in the proof of Theorem 6.3.1, of a cactus having maximum degree 7

6.4 SUMMARY

This chapter is dedicated to *k*-distance coloring of graphs. The *k*-distance chromatic number of trees, cycles and the 2-distance chromatic number of cactus graphs are determined in this chapter. It is clear that all the above graphs satisfy Conjecture 1.3.11.

CHAPTER 7

CONCLUSION AND FUTURE SCOPE

"*Every human activity, good or bad, except mathematics, must come to an end.*"
- Paul Erdős

The field of graph colorings has developed into one of the most popular areas of Graph Theory. The frequency assignment problem is the motivation for many of the graph coloring problems. Due to the relatively scarce radio spectrum and the rapid growth of wireless networks, the importance of the frequency assignment problem is growing significantly. Motivated by this, we have studied two graph coloring problems in this thesis, namely, radio k-coloring of graphs and k-distance coloring of graphs.

For any non-trivial class of graphs, the radio k-chromatic number is not known for arbitrary k, in fact, very less research is done when $k \leq diam(G) - 2$. One of the possible reasons could be, finding $rc_k(G)$ is difficult for smaller values of k, in general. As far as we know, $rc_k(G)$ is studied for $k < diam(G) - 3$, only for P_n. In Chapter 2, we have determined $rc_k(P_n)$ for $\frac{2n+1}{7} \leq k \leq diam(P_n) - 5$ if k is odd and for $\frac{2n-4}{5} \leq k \leq diam(P_n) - 6$ if k is even. From Theorem 2.2.5 and Theorem 2.3.5, for the infinite path P_∞, $rc_k(P_\infty) \geq \frac{k^2+k+4}{2}$ which improves the lower bound given by Das et al. (2017) by one, a step towards Conjecture 1.3.5.

Although, radio k-coloring of a graph G is defined for $1 \leq k \leq diam(G)$, some researchers have studied it for $k > diam(G)$, as it is useful to find the radio k-chromatic number of larger graphs containing G. In Chapter 3, for the trees in $\mathscr{G}$ and $\mathscr{G}'$, we have given upper and lower bounds for the radio k-chromatic number when $k \geq diam(T)$, which match when $diam(T)$ and k are of the same parity. Also, we have determined the radio d-chromatic number of the trees and graphs constructed from the trees in some subclasses of $\mathscr{G}$ and $\mathscr{G}'$. It is easy to see that paths P_n of even order are in $\mathscr{G}$ and hence $rc_k(P_n)$, n even, is determined for $k \geq n-1$ (when k and $n-1$ are of the same parity), which matches with the result of Liu and Zhu (2005) (for $k = n-1$) and with the result of Kchikech et al. (2007) (for $k \geq n$). Many authors have studied radio k-coloring for $k \in \{diam(G)-1, diam(G)\}$. Even for the simplest graph path P_n, the radio k-chromatic number is known only for $k \in \{1, 2, n-3, n-2, n-1\}$ and $k \geq n$. In Chapter 3, for each $k > 1$, we have determined the radio k-chromatic number of infinitely many trees whose diameter is much larger than k. We feel that the upper bounds given for the radio k-chromatic number of trees in $\mathscr{G}$ and $\mathscr{G}'$ are sharp and so one can try to improve the lower bounds to get the exact numbers. The problem of determining the radio k-chromatic number of a graph G for $k < diam(G)$ is comparatively hard problem. The way of construction of larger trees and graphs discussed in Chapter 3 is an idea to explore the radio k-chromatic number for $k < diam(G)$.

In Chapter 4, we have determined $rn(K_n \square C_m)$ when n even and m odd; any n and $m \equiv 6 \ (mod\ 8)$; n is odd and $m \equiv 5 \ (mod\ 8)$. In the remaining cases of n and m, to get an upper bound for $rn(K_n \square C_m)$ which matches with the lower bound obtained using Theorem 1.4.4, one needs an ordering $x_1, x_2, x_3, \ldots, x_{mn}$ of the vertices of $K_n \square C_m$ such that $d(x_i, x_{i+1}) + d(x_{i+1}, x_{i+2}) + d(x_i, x_{i+2}) = m + 6$ for all $i = 1, 2, 3, \ldots, mn-2$. From the proof of Lemma 4.1.2, it looks like getting such vertex ordering is difficult in general. In few of the remaining cases, it may be required to improve the lower bound also.

Radio k-coloring for corona $G \odot H$ of arbitrary graphs G and H is studied in Chapter 5. A best possible upper bound for $rc_k(G \odot H)$ is obtained. The upper bound is improved for the radio numbers of $P_{2p+1} \odot H$ and $Q_n \odot H$. Further, a lower bound for the radio number of $Q_n \odot H$ is obtained. The upper bound obtained for the radio number of $P_{2p+1} \odot H$ differs by 1 with the exact number. We feel that the upper bound is sharp. For $Q_n \odot H$, the upper and lower bounds obtained differ by at most $2^n - 2$. We feel that the lower bound obtained is sharp for n odd. For n even, we feel that the lower bound is close to the exact number. It will be interesting to classify the graphs whose radio k-chromatic numbers match with the upper bounds given in Theorem 5.1.1 and Theorem 5.1.2.

Chapter 6 is dedicated for k-distance coloring of graphs. In Chapter 6, the k-distance chromatic number of trees and cycles, and the 2-distance chromatic number of cactus graphs are determined. It is clear that all these graphs satisfy Conjecture 1.3.11. From Theorem 6.1.4, the 2-distance chromatic number of a tree with maximum degree Δ is $\Delta + 1$. If T is a tree obtained from a cactus graph G by deleting exactly one edge from each cycle without decreasing the maximum degree Δ, then $\chi_2(G)$ is at most 1 more than $\chi_2(T)$ (both differ by 1, only when G contains C_5 and $\Delta = 3$). The procedure given in the proof of Theorem 6.3.1 becomes difficult as k increases. Since any unicyclic graph can be obtained by adding an edge to a tree, one can try to find the k-distance chromatic number of unicyclic graph.

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Journal Publication:

1. Niranjan, P. K. and Srinivasa Rao, Kola. (2019). The *k*-distance chromatic number of trees and cycles. *AKCE International Journal of Graphs and Combinatorics,* 16(2):230 – 235. https://doi.org/10.1016/j.akcej.2017.11.007

2. Niranjan, P. K. and Srinivasa Rao, Kola. (2020). On the radio *k*-chromatic number of some classes of trees, *International Journal of Applied and Computational Mathematics,* 6(2):1–14. https://doi.org/10.1007/s40819-020-0778-9

3. Niranjan, P. K. and Srinivasa Rao, Kola. (2020). On the radio number for corona of paths and cycles. *AKCE International Journal of Graphs and Combinatorics* 17(1):269–275. https://doi.org/10.1016/j.akcej.2019.06.006

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1. Srinivasa Rao, Kola and Niranjan, P. K. (2018). The radio number for a class of Cartesian products of complete graphs and cycles. In proceedings of 7^{th} *International Engineering Symposium – IES2018, Kumamoto University, Japan,* E1-1-1–E1-1-5.

2. Niranjan, P. K. and Srinivasa Rao, Kola. (2020). The radio number for some classes of the Cartesian product of complete graphs and cycles. In proceedings of 2^{nd} *International Conference on Mathematical Modeling and Computational Methods in Sciences and Engineering–2020, Alagappa University Tamilnadu, India.* IOP Journal of Physics: Conference Series (Accepted).

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