EFFICIENT DOMINATION IN CARTESIAN PRODUCT OF GRAPHS AND ITS CRITICAL ASPECTS

Thesis

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

by

SUJATHA V SHET

DEPARTMENT OF MATHEMATICAL AND COMPUTATIONAL SCIENCES
NATIONAL INSTITUTE OF TECHNOLOGY KARNATAKA,
SURATHKAL, MANGALORE - 575 025.

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Dedicated to My beloved Family and Friends

DECLARATION

By the Ph.D. Research Scholar

I hereby declare that the Research Thesis entitled **EFFICIENT DOMINATION IN CARTESIAN PRODUCT OF GRAPHS AND ITS CRITICAL ASPECTS** which is being submitted to the National Institute of Technology Karnataka, Surathkal, in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Mathematical and Computational Sciences** is a *bonafide report of the research work carried out by me.* The material contained in this Research Thesis has not been submitted to any University or Institution for the award of any degree.

Sujatha V Shet
(Register No.: 145011MA14P03)
Department of Mathematical and Computational Sciences

Place: NITK, Surathkal.
Date: 29/07/2020

CERTIFICATE

This is to *certify* that the Research Thesis entitled **EFFICIENT DOMINATION IN CARTESIAN PRODUCT OF GRAPHS AND ITS CRITICAL ASPECTS** submitted by **Ms. SUJATHA V SHET**, (Register Number: 145011-MA14P03) as the record of the research work carried out by her is *accepted as the Research Thesis submission* in partial fulfillment of the requirements for the award of degree of **Doctor of Philosophy**.

Prof. SHYAM S KAMATH
Research Co-Guide

Dr. A SENTHIL THILAK
Research Guide

Chairman - DRPC
(Signature with Date and Seal)

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NOTATIONS

$V(G)$	$\rightarrow$	Vertex set of G
$E(G)$	$\rightarrow$	Edge set of G
n	$\rightarrow$	Order of graph G or $\lvert V(G)\rvert$
m	$\rightarrow$	Size of G or $\lvert E(G)\rvert$
$\lfloor x \rfloor$	$\rightarrow$	Largest integer at most x
$\lceil x \rceil$	$\rightarrow$	Smallest integer at least x
$N(u)$	$\rightarrow$	Open neighborhood of vertex u
$N[u]$	$\rightarrow$	Closed neighborhood of vertex u
$\delta(G)$	$\rightarrow$	Minimum degree of G
$\Delta(G)$	$\rightarrow$	Maximum degree of G
$d_G(u, v)$	$\rightarrow$	Distance between pair of vertices u and v in G
$\deg(v)$	$\rightarrow$	Degree of vertex v
$rad(G)$	$\rightarrow$	Radius of G
$diam(G)$	$\rightarrow$	Diameter of G
$ecc(v)$	$\rightarrow$	Eccentricity of vertex v
$\alpha(G)$	$\rightarrow$	Independence number of G
$I(S)$	$\rightarrow$	Influence of set S
$\rho(G)$	$\rightarrow$	Packing number of G
$\gamma(G)$	$\rightarrow$	Domination number of G
$\mathscr{E}$	$\rightarrow$	The class of all Efficiently dominatable graphs
$F(G)$	$\rightarrow$	Efficient Domination number of G
K_n	$\rightarrow$	Complete graph on n vertices
$K_{r,s}$	$\rightarrow$	Complete Bipartite graph with partition (V_1, V_2), where $\lvert V_1\rvert = r$ and $\lvert V_2\rvert = s$
P_n	$\rightarrow$	Path on n vertices
C_n	$\rightarrow$	Cycle on n vertices
$W(T)$	$\rightarrow$	Set of all weak Supports of tree T
$S(T)$	$\rightarrow$	Set of all strong supports of tree T
$L(T)$	$\rightarrow$	Set of all leaf nodes of tree T
$G - v$	$\rightarrow$	Induced subgraph of G obtained by deleting a vertex $v \in V(G)$
$G - e$	$\rightarrow$	Induced subgraph of G obtained by deleting an edge $e \in E(G)$
$G + e$	$\rightarrow$	Induced subgraph of G obtained by adding an edge $e \in E(\overline{G})$
$\mathscr{G}_{-v}$	$\rightarrow$	$\{G : G \in \mathscr{E}$ and $G - v \in \mathscr{E}$, for all $v \in V(G)\}$
$\mathscr{G}_{-e}$	$\rightarrow$	$\{G : G \in \mathscr{E}$ and $G - e \in \mathscr{E}$, for all $e \in E(G)\}$
$\mathscr{G}_{+e}$	$\rightarrow$	$\{G : G \in \mathscr{E}$ and $G + e \in \mathscr{E}$, for all $e \in E(\overline{G})\}$
$G \square H$	$\rightarrow$	Cartesian product of graphs G and H
$G^{(v)}$	$\rightarrow$	G-layer with respect to v in $G \square H$
$p_G(S)$	$\rightarrow$	Projection of set S onto G in $G \square H$

Acronyms

EDS	$\rightarrow$	Efficient Dominating Set
PWDED	$\rightarrow$	Pairwise Disjoint Efficient Dominating Set
WS	$\rightarrow$	Weak Support
SS	$\rightarrow$	Strong Support
UVR	$\rightarrow$	Unchanging Vertex Removal
CVR	$\rightarrow$	Changing Vertex Removal
UER	$\rightarrow$	Unchanging Edge Removal
CER	$\rightarrow$	Changing Edge Removal
UEA	$\rightarrow$	Unchanging Edge Addition
CEA	$\rightarrow$	Changing Edge Addition
$UVR_{\mathscr{E}}$	$\rightarrow$	$UVR \cap \mathscr{G}_{-v}$
$CVR_{\mathscr{E}}$	$\rightarrow$	$CVR \cap \mathscr{G}_{-v}$
$UER_{\mathscr{E}}$	$\rightarrow$	$UER \cap \mathscr{G}_{-e}$
$CER_{\mathscr{E}}$	$\rightarrow$	$CER \cap \mathscr{G}_{-e}$
$UEA_{\mathscr{E}}$	$\rightarrow$	$UEA \cap \mathscr{G}_{+e}$
$CEA_{\mathscr{E}}$	$\rightarrow$	$CEA \cap \mathscr{G}_{+e}$

ABSTRACT OF THE THESIS

In a graph $G = (V, E)$, every vertex $v \in V(G)$ dominates itself and its neighbors. A set $S \subseteq V(G)$ is a *dominating set* of G if each vertex in $V(G)$ is either in S or has a neighbor in S. The *domination number* of G, denoted by $\gamma(G)$, is the cardinality of a minimum dominating set of G. It is noted that if S is a dominating set, then the vertices in $V - S$ may have more than one neighbor in S. Imposing the additional constraint on a dominating set S that, each vertex in V must have exactly one neighbor in S (inclusive of vertices in S), leads to the notion of efficient domination in graphs.

A dominating set $S \subseteq V(G)$ is an *efficient dominating set* (EDS) of G, if each vertex in $V(G)$ is either in S or has exactly one neighbor in S. A graph G is *efficiently dominatable*, if it has an EDS. If S is an EDS of G, then S is also a minimum dominating set of G, but not conversely. Thus, all efficient dominating sets have the same cardinality, namely, $\gamma(G)$. Though an EDS of G has the same cardinality as its domination number, it is noted that for a given domination number, the properties of a graph which does not contain an EDS need not be true for an efficiently dominatable graph. This necessitates an exclusive study of the class of efficiently dominatable graphs. Though there is a significant amount of study in the literature related to efficient domination, both from graph theoretical as well as algorithmic perspective, to the best of our knowledge, it has not been much explored relative to the other domination parameters. Further, the concept of efficient domination also finds applications in diverse fields like coding theory, parallel computing, wireless ad hoc networks, etc. Motivated by the applications of efficient domination and the research gap identified in the literature, interest is shown in this thesis to study the concept of efficient domination for an arbitrary graph and for a particular type of graph product, namely cartesian product.

The contributions to this thesis are organized into three chapters, namely Chapter 3, 4 and 5. Chapter 3 deals with the study on Efficient domination in general/arbitrary graphs. Some basic results on efficient domination in general graphs including some improved bounds on domination number, efficient domination in trees and some special classes of graphs are discussed. Further, the structural properties of graphs possessing pairwise disjoint efficient dominating sets are studied along with an insight into the applications of such structures in ad hoc and sensor networks.

Chapter 4 focuses on the concept of criticality in the class of efficiently dominatable graphs, where the concept of criticality in general, deals with the study of the behaviour of a graph with respect to a graph parameter, upon the removal of a vertex or a set of vertices, removal or addition of one or more edges. The study in this chapter is restricted to the removal of a single vertex and removal or addition of a single edge, at a time. Based on the research gap identified in the literature, fascinated by the applications of the concept of criticality in the design of fault-tolerant networks and its significance in graph theory, the study is initiated with respect to efficient domination. A vertex whose removal from G alters $\gamma(G)$ is referred to as a *critical vertex*. Similarly, an edge, whose removal from G or whose addition between a pair of non-adjacent vertices in G alters $\gamma(G)$ is a *critical edge*. The collection of such vertices (or edges) is a vertex (or edge) critical set. In this chapter, an attempt is made to explore the properties of critical vertices, critical edges with respect to both removal and addition. The vertex critical sets, edge critical sets and six classes of graphs arising thereof are characterized. Finally, the relationship between all these classes is identified and discussed.

Finally, Chapter 5 deals with the study of efficient domination in the cartesian product of graphs. Here, the structural properties of the product in terms of its factors are discussed. The initial focus is on the product of two well-known graphs, followed by the product of an arbitrary graph G with a well-known graph. Further, the class of efficiently dominatable product graphs $G \square K_{1,p}$ and $G \square K_p$, for some positive integer p and an arbitrary graph G are characterized. The problem of deciding whether or not a graph is efficiently dominatable is $\mathcal{NP}$-complete and so also, for the the products mentioned above. So, an attempt is made to design exact exponential time algorithms, to determine whether the products are efficiently dominatable or not. The study is also extended to Hamming graphs.

Keywords: Efficient domination, Efficient domination number, 2-packing, Independent perfect domination, perfect 1-codes, perfect 1-domination, Efficiently dominatable trees, Changing efficient domination, Unchanging efficient domination, Cartesian product, Hamming graphs.

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Chapter 1

Introduction

1.1 Brief History

The study of graphs arose with various recreational problems, such as problem of Königsberg bridges and Knight's tour (Biggs et al., 1986). In 1735, the renowned Swiss Mathematician Leonhard Euler settled the famous Königsberg bridge problem, which has perplexed scholars for many years. His method of solution to the problem laid the foundation for an entirely new branch of Mathematics namely "Graph Theory". The origin of Graph Theory is well recorded in Biggs et al. (1986). This branch of mathematics has developed into a substantial body of knowledge with a variety of applications in diverse fields such as Physics, Chemistry, Economics, Psychology, Business, Sociology, Anthropology, Linguistics and Geography. Hence, Graph theory is considered to be one of the multi-faceted branches of Mathematics, rich in interesting research problems and applications. Further, a Graph is one of the useful tools to model and solve problems arising in Computer Science and its allied areas, especially those frequently experienced in networks. The design and analysis of interconnection networks is much inspired by the ongoing advancements in technologies. Computer scientists and Engineers from other fields commonly use graphs to model the topological structure of any interconnection network.

Among the various topics studied in Graph Theory, the concept of domination has its historical roots dating back to 1862, when the chess master C.F de Jaenisch studied the problem of determining the minimum number of queens that can be placed on a chessboard so that all squares are either attacked by a queen or are occupied by a queen. This is equivalent to the problem of dominating the squares of a chessboard. The evolution and the subsequent development of this fertile area of domination theory from the chess board problem is surveyed in Ore (1962); Haynes et al. (1998) and Berge (2001). The theory of domination finds application in diverse fields, of which facility location problems, coding theory, computer communication networks, biological networks and social networks are a few.

In the evolution of domination theory, one of the main reasons that captivated a wide research community is the multitude of variations of domination. Numerous types of domination have been defined and studied by imposing additional constraints on a dominating set. Each type of domination so obtained meets a specific purpose in real time applications. Bacsó and Tuza (1990) put forward the following problem: "Let **P** be a property satisfied by vertex subsets of a graph. Characterize all graphs having a dominating set satisfying the property **P**". By varying the property **P**, many different domination parameters have been introduced and studied. Generally, in the study of such domination parameters, interests are shown to characterize graphs having subsets possessing the respective properties, and in case, it is difficult to obtain such characterizations for a general graph, additional constraints are imposed to restrict the study to special classes of graphs; to obtain bounds or exact values of such parameters for various classes of graphs and so on. A detailed review on the motivation and applications of graph domination and comprehensive treatment of various domination parameters can be found in Cockayne and Hedetniemi (1977); Haynes et al. (1998) and Haynes (2017).

In line with that, this thesis deals with a particular variant of domination, namely "*Efficient domination*". An introduction to the notion of Efficient domination along with a brief discussion on its significance is given in Section 1.3. Following this, the motivation behind the choice of this research topic and the objectives of this thesis are discussed in Section 2.5.

Some of the basic terminologies and notations required for further discussion and understanding of this thesis are defined below, which are followed as in Bondy et al. (1976); Haynes et al. (1998) and West (2001), unless specified otherwise.

1.2 Preliminaries

A ***graph*** G is defined as an ordered triple consisting of a ***vertex set*** V (or $V(G)$ with reference to the graph under consideration), an ***edge set*** E (or $E(G)$) and a relation ψ (or ψ_G) called ***incidence relation*** that associates with each edge a pair of elements of $V(G)$ (not necessarily distinct) called its ***endpoints***. Each element of $V(G)$ is called a ***vertex*** (also called a node or a point) and each element of $E(G)$ is called an ***edge*** (or a line or a link).

Here, $V(G)$ may be finite or infinite and accordingly the graph is said to be a ***finite*** or an ***infinite*** graph. If the incidence relation ψ_G associates with each edge of G an ordered pair of vertices, then the graph G is said to be ***directed***. Otherwise, it is said to be ***undirected***.

The number of vertices or the cardinality of $V(G)$ is referred to as the ***order*** of G and is denoted by $|\boldsymbol{V(G)}|$. The number of edges or the cardinality of $E(G)$ is called the ***size*** of G, denoted by $|\boldsymbol{E(G)}|$.

Throughout this thesis, the symbols n and m are used to denote respectively the order and size of G, unless mentioned otherwise. A graph of order n and size m is referred to as an **(n, m)-graph**.

A ***loop*** is an edge whose endpoints are same and ***multiple or parallel edges*** are edges having the same pair of endpoints. A ***simple graph*** is a graph having no loops or multiple edges. In most of the applications, loops and parallel edges play relatively a less significant role. Hence, this study is restricted to simple graphs.

All graphs considered in this thesis are finite, simple and undirected, unless specified otherwise. In a simple graph, each edge can be uniquely identified by specifying its endpoints and hence ***throughout this thesis, ignoring***

the incidence relation in the definition of a graph, a graph G is denoted as an ordered pair (V, E), rather than representing as a triplet.

If $e \in E(G)$, where $e = uv$, then the vertices u and v are said to be ***adjacent*** and ***neighbors*** of each other; the edge e is said to be ***incident*** with u and v. Edges incident with the same vertex are said to be ***adjacent edges***.

The ***open neighborhood*** of a vertex u, denoted by $\boldsymbol{N(u)}$, is the set of vertices adjacent to u. That is, $N(u) = \{v \in V(G) : uv \in E(G)\}$. The ***closed neighborhood*** of u, denoted by $\boldsymbol{N[u]}$, is the set $N(u) \cup \{u\}$. For a set $S \subseteq V(G)$, the ***open neighborhood of S***, denoted by $\boldsymbol{N(S)}$, is $\bigcup_{u \in S} N(u)$ and the ***closed neighborhood $N[S]$ of S*** is $N(S) \cup S$.

The ***degree*** of a vertex v, denoted by $\mathbf{deg}(\boldsymbol{v})$, is the number of edges incident with v. That is, $\deg(v) = |N(v)|$. The minimum and maximum degrees of vertices in $V(G)$ are denoted by $\boldsymbol{\delta(G)}$ and $\boldsymbol{\Delta(G)}$ respectively. A graph G is said to be ***regular*** of degree r or ***r-regular***, if $\delta(G) = \Delta(G) = r$.

An ***odd vertex*** (or an ***even vertex***) is a vertex of odd (or even) degree. A ***pendant vertex*** is a vertex of degree one and a ***pendant edge*** is an edge incident with a pendant vertex. An ***isolated vertex*** or an ***isolate*** is a vertex of degree zero.

A ***walk*** $W = \{u_0, e_1, u_1, e_2, \ldots, u_{k-1}, e_k, u_k\}$ is a finite alternating sequence of vertices and edges such that the edge e_i has end points u_{i-1} and u_i, for each i $(1 \leq i \leq k)$. With the understanding that each edge occurring in this sequence can be identified using its preceding and succeeding vertices as endpoints, a walk is alternatively represented simply as a sequence of vertices (ignoring the edges) visited during the traversal. A ***u_0u_k-walk*** is a walk that begins and ends with vertices u_0 and u_k respectively. If $u_0 = u_k$, then W is said to be a ***closed walk***. Otherwise, it is said to be *open*. A ***trail*** is a walk with no repeated edges. A closed trail is a ***circuit***. A walk with $k + 1$ distinct vertices $u_0, u_1, \ldots, u_k$ is a ***path***. If $u_0 = u_k$ but $u_1, u_2, \ldots, u_{k-1}$ are distinct, then the trail is a ***cycle***. The ***length of a walk*** is the number of edges lying on the walk. Analogously, the length of a trail, a path and a cycle are defined. A path on n vertices (or of length $n-1$) is denoted by $\boldsymbol{P_n}$ and a cycle on n vertices (or of length n) by $\boldsymbol{C_n}$.

For a pair u, $v \in V(G)$, if there exists at least one uv-path in G, then the length of a shortest uv-path is referred to as the ***distance between u and v***, denoted by $\boldsymbol{d_G(u,v)}$ (or simply $\boldsymbol{d(u,v)}$, if no ambiguity). If no uv-path exists, then the distance between u and v is considered to be infinity, if $u \neq v$ and equal to zero, if $u = v$. The ***eccentricity*** of u in G, denoted by $\boldsymbol{ecc_G(u)}$ (or simply $\boldsymbol{ecc(u)}$), is the distance between u and a vertex farthest from u. That is, $ecc(u) = \max\{d(u,v) : v \in V(G)\}$. The minimum and maximum of eccentricities of all vertices in G are referred to as the ***radius*** ($\boldsymbol{rad(G)}$) and ***diameter*** ($\boldsymbol{diam(G)}$) of G respectively. That is, $rad(G) = \min\{ecc(v) : v \in V(G)\}$ and $diam(G) = \max\{ecc(v) : v \in V(G)\}$. A vertex v with $ecc(v) = rad(G)$ is a ***central vertex***. A path with its length equal to $diam(G)$ is called a ***diametral path*** in G.

A graph G is ***connected*** if there exists a uv-path for every pair of distinct vertices u, $v \in V(G)$. Otherwise, G is ***disconnected***. A graph H is a ***subgraph*** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph G is said to ***contain a copy of H***, if H is a subgraph of G. Thus, if H is a subgraph of G, then $uv \in E(H)$ implies that $uv \in E(G)$. If H satisfies the added property that for every pair of vertices u, $v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$, then H is an ***induced subgraph*** of G. The induced subgraph H with $S = V(H)$ is called the ***subgraph induced by S***, denoted by $\boldsymbol{< S >}$. A ***spanning subgraph*** of G is a subgraph of G with vertex set $V(G)$.

A subset S of $V(G)$ is ***minimal*** (or ***maximal***) in G with respect to a property P if no proper subset (or proper superset) of S possesses the property P in G. A set $S \subseteq V(G)$ is ***maximum*** with respect to property P in G if there exists no subset S' of $V(G)$ such that $|S'| > |S|$ and S' possesses the property P. Analogously, a ***minimum set*** is defined. A set which is maximum (minimum) with respect to a property P is also maximal (minimal).

A ***maximal connected subgraph*** of G is a subgraph that is connected and is not properly contained in any other connected subgraph of G. A ***component*** of G is a maximal connected subgraph of G. Clearly, G has exactly one component if and only if it is connected.

A ***complete graph*** is a simple graph whose vertices are pairwise adjacent and a complete graph on n vertices is denoted by $\boldsymbol{K_n}$. A graph with just one vertex is called as ***trivial*** and all other graphs are referred to as ***nontrivial***. Equivalently, a trivial graph is the complete graph K_1. A ***clique*** in a graph G is a maximal complete subgraph of G.

A subset S of $V(G)$ is an ***independent set*** of G if no two elements of S are adjacent in G. An independent set S is maximum in G if G has no independent set S' with $|S'| > |S|$ and an independent set S is maximal in G if G has no independent set S' properly containing S. The ***independence number*** of G, denoted by $\boldsymbol{\alpha(G)}$, is the cardinality of a maximum independent set of G. The minimum size of a maximal independent set in G is the ***lower independence number*** of G and is denoted by $\boldsymbol{i(G)}$.

For a set $S \subseteq V(G)$, $\boldsymbol{G - S}$ denotes the subgraph obtained by deleting the vertices in S (together with their incident edges). If $S = \{v\}$, then the corresponding graph $G - S$ is simply written as $\boldsymbol{G - v}$. Analogously, the graphs $\boldsymbol{G - E'}$, for $E' \subseteq E(G)$ and $\boldsymbol{G - e}$, for $e \in E(G)$ are defined.

A ***cut-edge*** (or ***cut-vertex***) of a graph is an edge (or vertex) whose deletion increases the number of components. A set $S \subseteq V(G)$ is a ***vertex cut*** of graph G if $G - S$ is disconnected. The minimum cardinality of S such that $G - S$ is either disconnected or trivial is the ***connectivity*** of G. A graph G is ***k-connected*** if its connectivity is at least k. Analogously, ***edge connectivity*** and ***k-edge-connectedness*** are defined.

An ***acyclic*** graph is a graph containing no cycles. A ***forest*** is an acyclic graph. A ***tree*** is a connected acyclic graph. A ***spanning tree*** is a spanning subgraph which is also a tree. A ***caterpillar*** is a tree in which the removal of all pendant vertices leaves a path and the resultant path is the ***spine*** of the caterpillar.

A ***rooted tree*** is a tree in which one vertex called the root, is distinguished from all the other vertices. In a rooted tree, a vertex v is said to be at ***level*** l_i if v is at a distance l_i from the root. Thus, the root is at level zero.

A graph G is ***bipartite*** if $V(G)$ is the union of two disjoint (possibly nonempty) independent sets, say V_1 and V_2 of G, where $(V_1,\ V_2)$ is called a ***bipartition*** of G and V_1 and V_2 are ***partite sets*** of G. In general, a graph G is ***k-partite*** if $V(G)$ can be partitioned into of k (possibly nonempty) independent sets, where $k \geq 2$.

For $k \geq 2$, a graph G is ***complete k-partite*** (or multipartite), if G is k-partite with partition $(V_1,\ V_2, \ldots, V_k)$ and $uv \in E(G)$ if and only if u and v belong to different partite sets. If $|V_i| = n_i$, for each i $(1 \leq i \leq k)$, then the complete k-partite graph is denoted by $K_{n_1, n_2, \ldots, n_k}$. Particularly, when $k = 2$, the graph is referred to as a ***complete bipartite graph***.

Two graphs G and H are ***isomorphic***, written as $\boldsymbol{G \cong H}$, if there exists a bijection $f : V(G) \rightarrow V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$.

The ***complement*** of a graph G, denoted by $\boldsymbol{\overline{G}}$, is the graph having vertex set $V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A graph G is ***self-complementary*** if $G \cong \overline{G}$.

The ***union*** of k graphs $G_1, G_2, \ldots, G_k$, denoted by $G_1 \cup G_2 \cup \cdots \cup G_k$, where $k \geq 2$ is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$. Similarly, the ***intersection*** of k graphs $G_1, G_2, \ldots, G_k$, denoted by $G_1 \cap G_2 \cap \cdots \cap G_k$, where $k \geq 2$ is the graph with vertex set $\bigcap_{i=1}^{k} V(G_i)$ and edge set $\bigcap_{i=1}^{k} E(G_i)$. Two graphs G_1 and G_2 are ***(vertex) disjoint*** if they have no vertex in common and ***edge disjoint*** if they have no edge in common. If G_1 and G_2 are disjoint, then $G_1 \cup G_2$ is also written as $\boldsymbol{G_1 + G_2}$.

$\boldsymbol{mG}$ is the graph formed by taking m copies of G. A graph G is ***H-free*** if G has no induced subgraph isomorphic to H. The ***k^{th} power of a graph G***, denoted by $\boldsymbol{G^k}$, has the same vertex set as G with two vertices adjacent in G^k if and only if they are at distance at most k in G. That is, $V(G^k) = V(G)$ and $E(G^k) = \{uv : d_G(u, v) \leq k\}$.

The ***cartesian product*** of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $\boldsymbol{G_1 \Box G_2}$, is the graph with vertex set $V_1 \times V_2$ and $((u_1, v_1), (u_2, v_2)) \in E(G_1 \Box G_2)$ if and only if either (i) $u_1 = u_2$ and $v_1v_2 \in E_2$ or (ii) $u_1u_2 \in E_1$ and $v_1 = v_2$ (Imrich and Klavžar, 2000).

In the literature, the notation "$\times$" is alternatively used in place of "$\Box$" in the definition of Cartesian product. However, throughout this thesis, the convention of using $\Box$ is followed as in Imrich and Klavžar (2000).

If G is an undirected graph without loops, where $V(G) = \{v_1, v_2, \ldots, v_n\}$, then the ***adjacency matrix*** of G, denoted by $\boldsymbol{A(G)}$, is the $n \times n$ matrix defined as $A(G) = (a_{ij})$, where a_{ij} is the number of edges with end points $\{v_i, v_j\}$. Clearly, $a_{ij} = a_{ji}$, for all i, j and hence the adjacency matrix of an undirected graph is symmetric. Further, for each i $(1 \leq i \leq n)$, $\deg(v_i)$ equals the sum of the entries in i^{th} row of $A(G)$.

The ***floor*** $\lfloor x \rfloor$ of x is the largest integer at most x and the ***ceiling*** $\lceil x \rceil$ of x is the smallest integer at least x.

For a given function $g(n)$, the notation $\boldsymbol{\mathcal{O}(g(n))}$ is used to denote the set of functions $\mathcal{O}(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that $0 \leq f(n) \leq cg(n)$, for all $n \geq n_0\}$. Similarly, for a given function $g(n)$, the notation $\boldsymbol{\Omega(g(n))}$ is used to denote the set of functions $\Omega(g(n)) = \{f(n) :$ there exist positive constants c and n_0 such that $0 \leq cg(n) \leq f(n)$, for all $n \geq n_0\}$. For a given function $g(n)$, a function $\boldsymbol{f(n) \in \mathcal{O}^*(g(n))}$, if there exists a polynomial $p(n)$ such that $f(n) \leq p(n).g(n)$, for all $n \geq n_0$.

Domination in Graphs

For a graph $G = (V, E)$, a set $S \subseteq V(G)$ is a ***dominating set*** of G if each vertex $v \in V(G)$ is either in S or has a neighbor in S. The cardinality of a minimum dominating set of G is the ***domination number*** of G, denoted by $\boldsymbol{\gamma(G)}$. In general, each vertex is said to dominate itself and all its neighbors. A dominating set S is a ***minimal dominating set*** if no proper subset of S is a dominating set.

A set $S \subseteq V(G)$ is a ***2-packing*** of G if $N[u] \cap N[v] = \emptyset$, for each pair u, v in S. The cardinality of a maximum 2-packing is the ***packing number*** of G, denoted by $\boldsymbol{\rho(G)}$. The minimum cardinality of a maximal 2-packing of G is called the ***lower packing number*** of G, denoted by $\boldsymbol{p_2(G)}$.

The ***influence*** of a set $S \subseteq V(G)$, denoted by $\boldsymbol{I(S)}$, is the number of vertices dominated by S. Since every vertex $v \in S$ dominates itself and $\deg(v)$ other vertices, $\boldsymbol{I(S) = \sum_{v \in S}(1 + \deg(v))}$, or equivalently, $\boldsymbol{I(S) = |N[S]|}$. In other words, $I(S)$ denotes the amount of domination done by S.

A dominating set $S \subseteq V(G)$ is a ***perfect dominating set*** if $|N(u) \cap S| = 1$, for all $u \in V(G) - S$. Every graph has at least the trivial perfect dominating set consisting of all vertices in $V(G)$.

1.3 Efficient Domination in Graphs

1.3.1 A Brief Overview

The concept of efficient domination in graphs has its origin back to early 1970's. In the literature, the concept has been studied using different terminologies, namely, perfect codes (Biggs, 1973; Kratochvíl, 1986), perfect 1-dominating sets (Livingston and Stout, 1990), independent perfect dominating sets (Fellows and Hoover, 1991) etc. The terminology "*efficient domination*" was introduced by Bange et al. (1978). A detailed review of the literature pertaining to the discussion of this thesis is given in Chapter 2. Throughout this thesis, the terminology namely, "*efficient domination" introduced by Bange et al. (1978) is adopted.*

In general, if a set S is a *dominating set* of a graph G, then each vertex in $V(G)$ is dominated at least once by S. That is, each vertex in $V - S$ has at least one neighbor in S and the vertices in S may or may not have neighbors in S. Now, suppose an additional constraint that each vertex in $V(G)$ is dominated exactly once by a dominating set, then such a dominating set is referred to as an *efficient dominating set*. Thus, a dominating set S is an efficient dominating set if S is independent and each vertex in $V - S$ has exactly one neighbor in S. It is formally defined as follows:

Definition 1.3.1. *(Haynes et al., 1998) A dominating set $S \subseteq V(G)$ is an **efficient dominating set** (EDS) of G if $|N[v] \cap S| = 1$, for all $v \in V(G)$.*

Equivalently, a dominating set S is an efficient dominating set if and only if S

is a 2-packing. Further, every efficient dominating set is an independent perfect dominating set.

For a vertex $v \in V(G)$ and a set $S \subseteq V(G)$, ***v is said to be efficiently dominated by S*** *if $|N[v] \cap S| = 1$.*

In general, unlike the case of a dominating set, a given graph may or may not possess an efficient dominating set. This leads to the following definition.

Definition 1.3.2. *(Haynes et al., 1998) A graph G is defined to be* ***efficiently dominatable*** *if it possesses an efficient dominating set.*

For example, the graph in Figure 1.1 is efficiently dominatable with S $= \{v_2, v_5\}$ as an EDS, whereas the graph in Figure 1.2 is not efficiently dominatable.

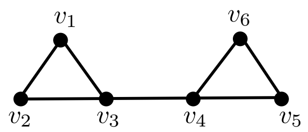


Figure 1.1: An efficiently dominatable graph

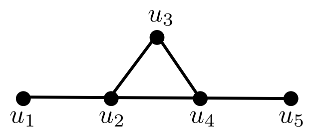


Figure 1.2: A graph which is not efficiently dominatable

An efficiently dominatable graph may have more than one EDS, but all EDSs have the same cardinality. For instance, the graph in Figure 1.1 has four efficient dominating sets, namely, $\{v_1, v_5\}$, $\{v_1, v_6\}$, $\{v_2, v_5\}$ and $\{v_2, v_6\}$. It can be observed that all the four sets have same cardinality. This fact was proved by Bange et al. (1988) which is stated as follows:

Theorem 1.3.1. *(Bange et al., 1988; Haynes et al., 1998) If G has an efficient dominating set, then the cardinality of any efficient dominating set equals the domination number of G. In particular, all efficient dominating sets of G have the same cardinality.*

However, it is noted that if a graph is not efficiently dominatable, then any two 2-packings with (same) maximum influence may have different cardinalities. *Therefore, in order to prove that a graph is efficiently dominatable, it is enough to show there exists a set (2-packing) which dominates all the vertices in the graph* *exactly once. On the other hand, in order that a graph is not efficiently dominatable, it is required to show that there exists no 2-packing with its influence equal to the order of the graph and it is required to search for a set (2-packing) which dominates the maximum number of vertices with the condition that each vertex is dominated exactly once. Precisely, in the study of efficient domination in graphs, the general focus is on finding the maximum number of vertices (efficiently) dominated by a 2-packing rather than the cardinality of a dominating set.* This leads to the following definition of efficient domination number of a graph.

Definition 1.3.3. *(Haynes et al., 1998) The maximum number of vertices dominated by a 2-packing of G is called the* ***efficient domination number*** *of G and is denoted by $F(G)$. That is, $F(G) = \max\{I(S) : S \text{ is a 2-packing}\}$.*

For every graph G, $1 \leq F(G) \leq n$ and G is *efficiently dominatable if and only if $F(G) = n$.* In other words, if a graph G is not efficiently dominatable, then $F(G) < n$.

It follows from the definition of an EDS that for any graph G of order n, a 2-packing of G with its influence equal to n is referred to as an EDS of G, provided one such exists. Whereas, for a convenient reference to a 2-packing with maximum influence (less than n) in graphs which are not efficiently dominatable, the following terminology is introduced in this thesis.

Definition 1.3.4. *Let G be a graph with $F(G) = k$ (possibly less than $|V(G)|$). Then a set $S \subseteq V(G)$ is an* ***F(G)-set*** *if S is a 2-packing and $I(S) = k$. That is, an $F(G)$-set is a 2-packing with maximum influence in G.*

It is understood that an $F(G)$-set with $F(G) = n$ is an EDS of G.

The following are some of the basic observations on efficient domination in graphs:

Observation 1.3.1.

1. *If $G \cong nK_1$, then $F(G) = n$ and $V(G)$ is the unique EDS of G.*

2. *$F(K_n) = n$ and for each $v \in V(K_n)$, $\{v\}$ is an EDS of K_n. In other words, K_n is efficiently dominatable, for all n.*

3. *P_n is efficiently dominatable for all n, $K_{p,q}$ is efficiently dominatable if and only if either $p = 1$ or $q = 1$.*

4. *C_n is efficiently dominatable if and only if $n \equiv 0\,(mod\ 3)$. Further,*

$$F(C_n) = \begin{cases} n-1 & \text{if } n \equiv 1\,(mod\ 3) \\ n-2 & \text{if } n \equiv 2\,(mod\ 3). \end{cases}$$

1.3.2 Significance of Efficient Domination

As discussed earlier, given a graph G it is always possible to find a dominating set D. Further, some vertices are dominated exactly once by D and some may be dominated more than once. So, the basic intention is to minimize the amount of excess domination done by a subset of $V(G)$. The idea of minimizing the amount of excess domination with the condition that each vertex is dominated exactly once led to the notion of *"efficiency"* or *"efficient domination"*. Alternatively, the idea of minimizing the amount of excess domination with the condition that every vertex is dominated at least once led to the notion of *"redundancy"*. The parameter namely, redundance of a graph G is a measure which determines how many times vertices in G are dominated by a subset of $V(G)$. That is, the ***redundance*** of G (also called the ***total redundance***), denoted by $\boldsymbol{R(G)}$ is defined as $R(G) = \min\{\sum_{v \in V(G)} |N[v] \cap S| : S \text{ is a dominating set}\}$. Equivalently, $R(G) = \min\{I(S) : S \text{ is a dominating set}\}$. Another measure, called ***"cardinality redundance"*** denoted by $\boldsymbol{CR(G)}$ is the minimum number of vertices dominated more than once by a dominating set (redundantly dominated).

It can be observed that the value of $F(G)$ is at most $|V(G)|$ and $R(G)$ is at least $|V(G)|$. And, both the parameters are equal to $|V(G)|$ if only if G is efficiently dominatable and in which case $CR(G) = 0$.

It is known that the influence of a set $S \subseteq V(G)$ measures the amount of domination done by S in G. Thus, based on the above discussion, the parameters namely, efficient domination number, redundance and cardinality redundance are alternatively referred to as *"influence parameters"* (Sinko and Slater, 2005, 2006). The study of influence parameters was introduced by Sinko and Slater (2005) for chessboard graphs, wherein the efficient domination number and the redundance number of such graphs were determined. Further, the other influence parameters like the closed neighborhood order domination number, the closed neighborhood order packing number were also studied together with their linear programming versions. Precisely, the parameters like domination number, independence number, packing number etc., are determined with a focus on the minimum or maximum cardinality of a subset S of $V(G)$, which is a dominating set or an independent set or a packing, as the case may be. But, in the case of influence parameters, one is interested in the *amount of domination* done by a subset S of $V(G)$, rather than the cardinality of S.

Though an EDS of G has the same cardinality as its domination number, it can be noted that for a given domination number, say k, all the properties of a graph which does not contain an EDS need not be true for an efficiently dominatable graph. Few trivial instances are: (i) the influence of a dominating set in an efficiently dominatable graph is exactly equal to n while that of a dominating set in a graph which is not efficiently dominatable is at least n. (ii) Further, if G is a graph of even order having no isolated vertices then $\gamma(G) = \frac{n}{2}$ if and only if the components of G are either C_4 or $H \circ K_1$, where H is connected (Haynes et al., 1998). But with the additional hypothesis that G is efficiently dominatable, every component of G must be $H \circ K_1$, where H is connected (as C_4 is not efficiently dominatable). There exist significant amount of properties which are true in the collection of efficiently dominatable graphs but not true in the complement and vice versa. This necessitates an exclusive study of efficiently dominatable graphs.

In the literature, there exists a significant amount of research dealing with the algorithmic aspects of the problem. However, most of these are restricted only to special classes of graphs like trees, perfect graphs and so on. The efficient dominating set problem is the problem of answering the question: *"Given a graph*

G, whether or not G is efficiently dominatable?". Equivalently, it is the task of answering the decision problem: *"Is $F(G) = n$, for a given graph G?"* This problem was proved to be $\mathcal{NP}$-complete on general graphs by Bange et al. (1988) and they also gave an $\mathcal{O}(|V(G)|)$ time algorithm for this problem on trees. In particular, the problem is $\mathcal{NP}$-complete on the cartesian product of graphs. Though this particular variant of domination has a long history, to the best of our knowledge, it has not been much explored relative to the other domination parameters.

Further, the concept of efficient domination has varied and interesting applications in coding theory, graph embedding, facility location, resource allocation in parallel processing systems and so on. For a given parallel computing architecture, it is often necessary to distribute a limited set of resources among the processors and it may be required to provide efficient file sharing mechanism. To effectively distribute the resources and guarantee ready access to every resource, one can initially represent their communication scenario as a graph by considering the set of processors as vertex set and joining any two processors capable of communicating with each other directly, by means of an edge. Then, determining a dominating set of processors in this underlying graph would suggest a good choice of locations for placement of resources. To avoid multiple sharing, an additional constraint is used; that is, a processor is permitted to access the resources available at only one location and this is accomplished by determining an efficient dominating set of the underlying graph. Various such considerations similar to the above assignment of designated resources to processors make an efficient dominating set the best choice of locations for resource allocation.

Similarly, considering a situation in computer networks where software packages like code libraries need to be placed at individual processor nodes. If every node is installed with the software package, then the total cost of the design becomes very high. Hence, finding a minimum dominating set for designating code libraries across the network is a definite solution to this problem. The more effective solution is the one which avoids overlaps in this allocation problem and this can be facilitated by finding a more restricted version of a minimum dominating set, namely an efficient dominating set.

In the same way, one of the main objectives in the design of communication protocols for wireless ad hoc and sensor networks is to provide an energy-efficient interference-free communication. This can be accomplished by establishing a non-overlapping cluster-based communication. The problem of designing non-overlapping clusters is equivalent to finding an efficient dominating set for the underlying network topology (with permissible dummy links to make the topology efficiently dominatable, in case it is not so) (Janakiraman and Thilak, 2012; Thilak, 2013). The significant properties possessed by an efficient dominating set namely, *domination, independence and* 2*-packing* makes it unique among all variants of domination and also suitable for the design of such protocols. Hence, the problem has applications in the design of efficient resource management protocols in distributed computing.

Summarizing the above discussion, the problem studied in this thesis is motivated by the applications of efficient domination in coding theory (Biggs, 1973; Hammond and Smith, 1975), resource allocation in distributed/parallel computing (Livingston and Stout, 1988, 1990; Van Wieren et al., 1993; Milanič, 2013), communication in sensor and ad hoc networks etc. (Yu and Chong, 2003, 2005; Janakiraman and Thilak, 2011; Thilak, 2013). Biggs (1973) studied perfect d-codes, wherein perfect domination is applied to coding theory. Related to the applications of interconnection networks in parallel computers, Livingston and Stout (1990) studied perfect d-dominating sets, which are exactly same as the perfect d-codes. The concept of efficient domination is precisely same as their perfect 1-domination. Further, the Cartesian product of graphs is one of the interesting structures in Graph theory. It is also one of the widely used multi-dimensional architectures in distributed computing systems and one of the commonly used topologies for ad hoc, sensor and vehicular networks. On these lines, the problem is of significant interest from both Graph theoretic as well as application perspective.

1.4 Organization of the Thesis

The contents of this thesis are organized as follows: Chapter 1 deals with the preliminaries required for the discussions carried out in this thesis followed by an introduction to the concept of efficient domination in graphs and its significance in terms of both theory and applications. In Chapter 2, a brief review of the literature related to efficient domination in graphs and its variants is presented. Next, the research gap identified from the literature and the objectives set for this research work are discussed.

The contributions in this research work are organized into three chapters: *Chapters 3, 4 and 5.* Chapter 3 deals with some basic results on efficient domination in general graphs, efficient domination in trees and efficient domination in some special graphs. Further, in this chapter, the structural properties of graphs possessing pairwise disjoint efficient dominating sets are discussed along with an insight into the applications of such structures in ad hoc and sensor networks.

In Chapter 4, the study of the concept of criticality is initiated with respect to efficient domination in graphs. Here, the notion of changing and unchanging efficient domination in graphs is studied with respect to vertex criticality (vertex removal) as well as edge criticality (edge removal and edge addition). The critical vertices, critical edges with respect to both removal and addition, vertex critical sets, edge critical sets and the six classes of graphs arising thereof are characterized. Finally, the relationship between all these classes is identified and discussed.

Chapter 5 deals with the study of efficient domination in the cartesian product of graphs. In this chapter, the structural properties of the product in terms of its factors are discussed. The initial focus is on the product of two well-known graphs, followed by product of an arbitrary graph G with a well-known graph. Further, the class of efficiently dominatable product graphs $G \square K_{1,p}$ and $G \square K_p$, for some positive integer p and an arbitrary graph G are characterized. In addition, as the efficient domination problem is known to be $\mathcal{NP}$-complete for an arbitrary graph and hence for the cartesian product of graphs, an attempt is made to design exact exponential time algorithms for finding an $\mathrm{F}(G \square K_{1,p})$-set and $\mathrm{F}(G \square K_p)$-set.

The study is also extended to Hamming graphs.

Finally, Chapter 6 deals with the summary and conclusion of this research work followed by the scope for future work.

Chapter 2

Literature Survey

The concept of efficient domination in graphs has its origin back to early 1970's. The notion of efficient domination has been studied in the literature using different terminologies, namely, perfect codes (Biggs, 1973; Kratochvíl, 1986), perfect 1-dominating sets (Livingston and Stout, 1990), independent perfect dominating sets (Fellows and Hoover, 1991) and efficient dominating sets. The terminology "*efficient domination*" was introduced by Bange et al. (1978). As this Thesis deals with results on general graphs and cartesian product of graphs both from graph theoretic and algorithmic perspective, the existing results on efficient domination in graphs is organized into three sections: *Efficient domination in graphs*, wherein the existing graph theoretical results related to general graphs and special classes of graphs is discussed, *Efficient domination in product graphs* and finally *Algorithmic aspects of efficient domination*, which is dedicated exclusively to those existing research works on efficient domination from an algorithmic perspective.

2.1 Efficient Domination in graphs

2.1.1 Prior work on Efficient Domination

Bange et al. (1978) have characterized the classes of trees with two disjoint minimum dominating sets, with two disjoint minimum independent dominating sets (any two vertices in the set must be at distance at least two) and trees with two disjoint minimum dominating sets where any two vertices in the obtained set must be at distance at least three. While giving such characterizations, they have defined dominating sets of the third category as efficient dominating sets, as such sets that have neither deficient nor excess domination. Later, Bange et al. (1988) have extended the work by characterizing efficiently dominatable trees of diameter at least three and proposed a procedure for computing $F(T)$ for an arbitrary tree T. As stated in Theorem 1.3.1, one of their significant results is that *if a graph G is efficiently dominatable, then all its efficient dominating sets have the same cardinality, namely, $\gamma(G)$*. Till then, the efficient dominating set problem was viewed as the problem of finding an efficient dominating set with minimum cardinality, if one such set exists. Later on, based on the above result, the researchers started reviewing the problem as that of simply finding an efficient dominating set in a graph (without concentrating on the cardinality) in a graph. As stated in Section 1.3.2, the efficient domination number is alternatively referred to as an influence parameter. There exists a significant amount of study related to the influence parameters like redundance, cardinality redundance, closed neighborhood order domination etc., as surveyed in (Haynes et al., 1998). Especially, it includes some significant fundamental results on efficient domination and its variants, both from graph theoretic as well as algorithmic perspective. The relationship between efficient domination number and other influence parameters is also discussed. Ten possible inequality chains connecting these parameters are identified and it is proved that for each chain of inequality, there exist infinitely many graphs satisfying the inequality. However, the convention of referring to such parameters as "influence parameters" was introduced by Sinko and Slater (2005). They have studied these parameters (including efficient domination number) for chessboard graphs.

A set $D \subseteq V(G)$ is a ***perfect d-code*** of a graph G if every vertex $u \in V$ is at most at a distance d from exactly one vertex in D. A perfect 1-code in a graph G is an independent subset of vertices $D \subseteq V(G)$ where every vertex of G is either an element of D or is adjacent to exactly one vertex in D. Based on this definition, Haynes et al. (1998) realized the notion of efficient domination as a generalization of perfect codes (also referred to as perfect 1-codes). Upon justifying the fact that a perfect code (or perfect 1-code) is same as an efficient dominating set, the following equivalent conditions have been proved.

Theorem 2.1.1. *(Haynes et al., 1998) The following statements are equivalent:*

(a) $S = \{u_1, u_2, ..., u_k\}$*is a perfect code for* G.

(b) $\{N[u_1], N[u_2], ..., N[u_k]\}$ *is a partition of* $V(G)$.

(c) S *is a packing and* $\sum_{u \in S}(1 + \deg(u)) = |V(G)|$.

So, based on the above fact, it is noted that the study of efficient dominating sets in graphs actually began in (Biggs, 1973), but using the terminology "perfect 1-codes". In this article, the authors have investigated the existence of perfect d-codes ($d \geq 1$) for the class of distance-transitive graphs, where a graph G with an associated distance function δ is ***distance-transitive*** if the following condition is satisfied: *Whenever* u, v, x, y *are vertices of* G *such that* $\delta(u, v) = \delta(x, y)$, *there exists an automorphism* h *of* G *such that* $h(u) = x$ *and* $h(v) = y$. The notion of perfect domination has been discussed by Livingston and Stout (1990), in which they define perfect d-dominating sets (equivalent to perfect d-codes). It can be observed that a perfect 1-dominating set or perfect 1-code is exactly the same as an efficient dominating set. In (Livingston and Stout, 1990), the authors have investigated the existence of perfect d-dominating sets in a wide variety of special classes of graphs like trees, hypercubes and hypercube related networks, tori, series-parallel graphs and so on. Interestingly, the authors have used different techniques namely, algorithmic, algebraic and combinatorial techniques to prove the existence of perfect d-dominating sets as appropriate for each (specific) class of graph under consideration. Weichsel (1994) has studied efficient domination in the name of perfect domination for hypercubes and have proved that a perfect dominating set (or EDS) of a hypercube induces a subgraph of the hypercube whose components are also hypercubes, but of lesser dimension.

Efficient domination has also been studied for other special classes of graphs, namely, Cocomparability graphs (Chang and Liu, 1993), Permutation graphs and

Trapezoid graphs (Liang et al., 1997), orientations of a graph (Bange et al., 1998), Sierpiński graphs (Klavžar et al., 2002), Cayley graphs (Dejter and Serra, 2003; Chelvam and Mutharasu, 2013; Caliskan et al., 2020), Labeled rooted oriented trees (Schwenk and Yue, 2005), Chessboard graphs (Sinko and Slater, 2005), Knight graphs (Sinko and Slater, 2006), Circulant graphs (Obradović et al., 2007; Kumar and MacGillivray, 2013; Deng, 2014; Deng et al., 2017), Vertex-transitive graphs (Huang and Xu, 2008), Generalized Petersen graphs (Ebrahimi et al., 2009), Bi-Cayley graphs (Chelvam and Mutharasu, 2010), Cubic Vertex-transitive graphs (Knor and Potočnik, 2012), Circular arc graphs (Lin et al., 2015), Cubic and Quartic Cayley graphs (Çalışkan et al., 2019), Mycielski's graphs (Anitha and Balamurugan, 2020).

Goddard et al. (2000) have studied the two measures namely, the efficient domination number (referred to as "efficiency" in the article) and the redundance (referred to as "total redundance" in the article) of a graph. Here, the authors have obtained upper and lower bounds on the efficient domination number and the redundance for general graphs and for trees. They have also obtained Nordhaus-Gaddum-type bounds for the efficient domination number of a graph G and its complement $\overline{G}$.

The paper by Brod and Skupien (2008) considers trees having the largest number of efficient dominating sets and characterizes them. They define a tree T on n vertices to be maximum if it has the largest number of efficient dominating sets among all n-vertex trees. They have characterized all such trees and have shown that the number of such n-vertex trees is bounded below by an increasing exponential function in n.

Thilak (2013) has studied the concept of efficient domination for general graphs. In particular, the author has obtained the necessary and/or sufficient conditions for a graph of diameter three and its complement to be efficiently dominatable. The relationship between this domination and other domination parameters like geodomination, k-perfect geodomination etc are also discussed. Among the various results discussed by the author, the following two results are used in further discussions of this Thesis.

Theorem 2.1.2. *(Thilak, 2013) If G is a connected efficiently dominatable graph with $rad(G) \geq 2$ and S is an EDS of G, then for each $u \in S$, there exists at least one vertex $v \in S$, such that $d(u, v) = 3$.*

Theorem 2.1.3. *(Thilak, 2013) If G is not efficiently dominatable, S is an $F(G)$-set and $S' = N[S]$, then for each $x \in V - S'$, there exists a vertex $u \in S$ such that $d(x, u) = 2$.*

With the perception that the difficulty in obtaining general characterizations for efficiently dominatable graphs is probably because of their subgraphs not inheriting the property of being efficiently dominatable, Milanič (2013) has introduced a new class of graphs, namely, hereditary efficiently dominatable graphs, defined as follows: *A graph G is said to be **hereditary efficiently dominatable** if every induced subgraph of G contains an efficient dominating set.* In this article, the hereditary property of a graph with respect to efficient domination is discussed. Presuming that the hereditary efficiently dominatable graphs must be contained in the class of (bull, fork, C_{3k+1}, C_{3k+2})-free graphs (refer to Figure 2.1) as the bull, fork and cycles of the form C_{3k+1} and C_{3k+2} are not efficiently dominatable, the author has initially proved a decomposition theorem for (bull, fork, cycle)-free graphs. Later, using this result, the author has proved that the class hereditary efficiently dominatable graphs equals the class of (bull, fork, C_{3k+1}, C_{3k+2})-free graphs. Further, it is shown that every hereditary efficiently dominatable graph can be constructed from paths and cycles C_n, where $n \equiv 0 \pmod 3$, with the help of a sequence of operations as detailed in Milanič (2013).

Barbosa and Slater (2016) have studied the class of ***super-efficient graphs*** (defined in the same way as hereditary efficiently dominatable graphs, introduced by Milanič (2013)). However, in this article, the focus is on a bigger class of graphs which includes the class of hereditary efficiently dominatable graphs as a subcollection. The authors have introduced and studied a new family of graphs, denoted by S_k, where S_k is the collection of all graphs G for which every induced subgraph $G - S$ with $0 \leq |S| \leq k < |V(G)|$ is efficiently dominatable. It is

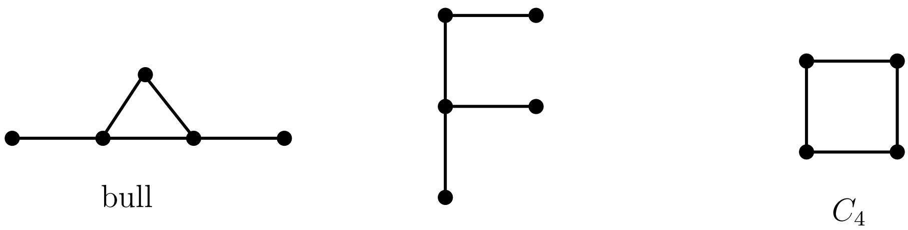


Figure 2.1: Graphs which are not efficiently dominatable (Milanič, 2013)

observed that S_0 is simply the collection of all efficiently dominatable graphs and $S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots$ and a graph G of order n is super-efficient (or hereditary efficiently dominatable) if and only if $G \in S_{n-1}$. They have defined the ***efficiency index*** of an efficiently dominatable graph G as the maximum value of k for which $G \in S_k$. Further, they have obtained characterizations for trees, torii, cylinders, grids and arbitrary graphs of diameter two to be super-efficient (or hereditary efficiently dominatable).

Cardoso et al. (2016) have studied efficient domination using eigen values. The authors have defined a set $S \subseteq V(G)$ to be a (k, τ)*-regular set* in G if every vertex in $V - S$ has exactly τ neighbors in S and S induces a k-regular subgraph in G. Thus, an EDS is nothing but a (0, 1)-regular set. It is discussed that the efficient domination problem can be viewed as a particular case of determining whether a graph possesses a $(0, \tau)$-regular set. They have given a simplex-like algorithm using the theory of star complements and some spectral results on (k, τ) -regular sets for finding a $(0, \tau)$-regular set in an arbitrary graph .

In general, a property P possessed by an efficiently dominatable graph G need not be possessed by G^2. Motivated by this fact, Karthick (2016) has posed and attempted the following problem: *Identify a family* $\mathcal{F}$ of graphs such that if G is efficiently dominatable and $\mathcal{F}$-free, then G^2 is also $\mathcal{F}$-free. The author has shown the existence of at least one particular class of graphs namely, $(P_6, banner)$-free graphs possessing the above property, where a *banner* is the graph obtained from a chordless cycle on four vertices by adding a vertex that has exactly one neighbor on the cycle. In this article, the author has proved that if G is efficiently dominatable and is $(P_6, banner)$-free, then G^2 is also $(P_6, banner)$-free.

2.1.2 Prior work on Variants of Efficient Domination

Apart from the above, several variants of efficient domination like efficient open domination, efficient edge domination, efficient multiple domination, weighted efficient domination etc., have also been studied from theoretical aspects as well as from algorithmic perspective.

In a graph G, every edge $e \in E(G)$ is said to dominate itself and all its adjacent edges. A set $E' \subseteq E(G)$ is an ***efficient edge dominating set (EEDS)*** of G if each edge in $E(G)$ is either in E' or dominated by exactly one edge in E'. The efficient edge dominating set problem (EED) asks for the existence of an EEDS in a given graph G. The *efficient edge domination* problem is proved to be $\mathcal{NP}$-complete on bipartite graphs and line graphs of bipartite graphs (Lu and Tang, 1998), planar bipartite graphs (Lu and Tang, 2002), p-regular graphs ($p \geq 3$) (Cardoso et al., 2008). In (Lu and Tang, 1998), a linear time algorithm is proposed to solve the weighted version of efficient edge domination problem on bipartite permutation graphs. The survey article by Brandstädt (2018) gives a brief review of existing research works related to efficient domination and efficient edge domination in graphs.

Given a graph G, a set $S \subseteq V(G)$ is an ***efficient open dominating set*** (also referred to as a perfect total dominating set) if for every $v \in V(G)$, $|N(v) \cap S| = 1$, that is, if the open neighborhoods $N(v)$, for $v \in S$ form a partition of $V(G)$. A graph G is called ***efficiently open dominatable*** if it possesses an efficient open dominating set. Gavlas and Schultz (2002) have studied the notion of efficient open domination for general graphs; discussed the existence of efficiently open dominatable graphs; characterized efficiently open dominatable trees and in addition, they have also defined and determined the efficient open domatic number for graphs under degree restriction, where the ***efficient open domatic number*** of an efficiently open dominatable graph G is the maximum number of disjoint efficient open dominating sets in G. Further, analogous to the result proved by Bange et al. (1988) for efficiently dominatable graphs, the authors have proved that, "If G has an efficient open dominating set S, then $|S| = \gamma_t(G)$, where $\gamma_t(G)$ denotes the total domination number of G. In particular, all efficient open dominating sets have the same cardinality". The authors have also shown that the efficient open dominating set problem is $\mathcal{NP}$-complete.

Rubalcaba and Slater (2007) have defined and discussed efficient versions of multiple domination, namely, k-tuple efficient domination, efficient double-domination and efficient (j, k)-domination.

The paper on *efficient edge domination* in regular graphs by Cardoso et al. (2008) relates maximum induced matchings and efficient edge dominating sets showing that efficient edge dominating sets are maximum induced matchings and that maximum induced matchings on regular graphs with efficient edge dominating sets are efficient edge dominating sets. A necessary condition for the existence of efficient edge dominating sets in terms of spectra of graphs is also established.

Efficient open domination in digraphs is discussed in (Knor, 2011). For a digraph G, a set $S \subseteq V(G)$ is called an *efficient open (total) dominating set* if the set of open out-neighborhoods $N^-(v) \in S$ form a partition of $V(G)$. If G is a digraph, then its reverse digraph, G^-, is obtained by reversing all the arcs of G. The author has shown that G is efficiently open dominatable if both G and its reverse digraph G^- have an efficient open dominating set. Further, properties of efficiently open dominated digraphs are also presented. A tournament is a digraph G such that, for every $u, v \in V(G)$, $u = v$, either $\overrightarrow{uv} \in E(G)$ or $\overrightarrow{vu} \in E(G)$. A special attention is also given to tournaments and directed tori (cartesian product of directed cycles).

In (Chelvam and Mutharasu, 2012), efficient open domination is studied for cayley graphs. The authors have characterized efficiently open dominatable bipartite cayley graphs and some classes of circulant Harary graphs. Further, they have derived a chain of efficient dominating sets and that of efficient open dominating sets in classes of circulant graphs.

Schaudt (2012) has proved that the efficient open domination problem is $\mathcal{NP}$-

complete for planar bipartite graphs of maximum degree 3 and is solvable in polynomial time with complexity $O(|V|^3)$ in T_3-free chordal graphs, where a ***T_3-free graph*** is a graph that does not contain as an induced subgraph a claw (or $K_{1,3}$), every edge of which is subdivided exactly twice. A graph is ***chordal*** if all its induced cycles have length 3. Let $n \geq 3$. An ***n-sun (or sun)*** is a chordal graph on $2n$ vertices whose vertex set can be partitioned into $W = \{w_1, \ldots, w_n\}$ and $U = \{u_1, \ldots, u_n\}$ such that W is independent and u_i is adjacent to w_j if and only if $i = j$ or $i = j + 1 \pmod{n}$, for all $1 \leq i, j \leq n$. It is also shown that the weighted version of efficient open domination problem on certain classes of graphs, like odd-sun-free chordal graphs, strongly chordal graphs and claw-free graphs ($O(|V|^3)$) is solvable in polynomial time. In the article by Kuziak et al. (2014), the efficiently open dominatable graphs among direct, lexicographic and strong products of graphs have been discussed in detail.

2.2 Efficient Domination and Graph Products

Many large networks can be efficiently modeled using graph products. While designing large scale networks, the product graphs serve as a base for easy and economical control of large scale systems. Hence, researchers have shown interest in studying various graph parameters for product graphs. On that line, there exist considerable amount of studies related to efficient domination in different graph products, as detailed below.

Cockayne et al. (1985) obtained bounds on the domination number of grid graphs. While deriving these bounds, they defined a particular type of dominating set, namely a $*$-dominating set, which is same as an efficient dominating set. The authors discussed by the method of construction that the infinite grid graphs $P_n \Box P_n$, for large n, must be efficiently dominatable. However, later Thilak (2013), has disproved this statement and shown that $P_n \Box P_n$ is efficiently dominatable if and only if $n = 4$ and computed the efficient domination number for all other products $P_n \Box P_n$, whenever n is finite. Kratochvíl (1986) has discussed the notion of perfect codes in cartesian product of graphs. Here, the author has focused on identifying those product graphs possessing 1-perfect codes. It is proved that for any graph G there exist infinitely many graphs H such that the product $G\Box H$ contains a 1-perfect code (or an efficient dominating set). Further, it is shown that if G is self-complementary, then there exists a 1-perfect code of cardinality $|V(G)|$ in G^2 and vice versa and for $k > 1$, the regular complete k-partite graphs having more than k vertices do not possess 1-perfect codes.

Dejter (2007) has studied perfect domination in the cartesian product of toroidal graphs $C_m\Box C_n$. The author has discussed about the existence of efficiently dominatable torus $C_p\Box C_q$, cylinders $P_n\Box C_n$ and grids $P_n\Box P_q$.

Mollard (2011) has studied perfect codes in cartesian products of graphs and discussed about the existence of perfect codes in these products. Given a n-regular graph, the author has defined a *code-colouring* as a vertex labeling c with the integers from $\{0, 1, \ldots, n\}$ such that for any vertex u, its neighbors $N(u)$ are coloured distinctly from the set of colours $\{0, 1, \ldots, n\} \setminus \{c(u)\}$. It is shown that for all $i \in \{0, 1, \ldots, n\}$, the set of vertices coloured i forms a perfect code. An extended code-colouring is defined as a labeling c of the vertices with integers from $\{0, 1, \ldots, n\}$ such that for any vertex u:

(i) The vertices in $N(u)$ coloured 0 are coloured with distinct colours from the set $\{1, \ldots, n\}$.

(ii) The vertices in $N(u)$ coloured with a colour from the set $\{1, \ldots, n\}$ are coloured 0.

It is shown that, if c is an extended code-colouring of an n-regular graph G, then G is bipartite. Also, the set of vertices coloured i, for all $i \in \{1, \ldots, n\}$, forms a perfect code. For any regular graph G of degree n, if there exists an extended code-colouring in $G\Box P_2$, then it is shown that G is bipartite and there exists a code-colouring in G. For any two regular graphs (finite or infinite) G and H each of degree n, if H is bipartite and if there exists a code-colouring in G and H, then there exists a code-colouring in $G\Box H\Box P_2$. Also, there exists a partition of perfect codes in $G\Box H\Box P_2$.

Chelvam and Mutharasu (2011) have discussed about the existence of an EDS in the cartesian product of two cycles and three cycles. They have also determined all possible efficient dominating sets in the cartesian product of n-cycles $\Box_{i=1}^{n} C_{k_i}$, where $k_i's$ are multiples of $2n + 1$ (prime numbers).

Thilak (2013) has studied efficient domination in the cartesian product of paths and cycles. It is proved that $P_n \Box P_2$ is efficiently dominatable if and only if n is odd. Further, as mentioned earlier, it is shown that $P_n \Box P_n$ is efficiently dominatable if and only if $n = 4$ and $C_n \Box K_2$ is efficiently dominatable if and only if $n \equiv 0 \pmod 4$. The exact values of the efficient domination number are obtained for the graphs $P_n \Box P_2$, where n is even, $P_n \Box P_3$, for all $n > 2$.

2.3 Algorithmic aspects of Efficient Domination

From an algorithmic perspective, the efficient dominating set problem is defined to be the problem of answering the following question: "*Given a graph G, whether or not G is efficiently dominatable?*". In other words, it is the task of answering the decision problem: "*Is $F(G) = n$, for an arbitrary graph G of order n?*"

Though there exists a significant amount of work concerning the algorithmic aspects of efficient domination, most of them are restricted to special classes of graphs. Bange et al. (1988) proved that the efficient domination problem is $\mathcal{NP}$-complete on general graphs and gave an $\mathcal{O}(|V(G)|)$ time algorithm for this problem on trees.

This problem is also proved to be $\mathcal{NP}$-complete even on special classes of graphs like, planar graphs of maximum degree at most three (Fellows and Hoover, 1991), bipartite graphs and chordal graphs (Smart and Slater, 1995; Chain-Chin and Lee, 1996), planar bipartite graphs and chordal bipartite graphs (Lu and Tang, 2002), planar bipartite graphs of maximum degree three with girth at least g, for every $g \geq 3$ (Brandstädt et al., 2013; Nevries, 2014), 3-regular graphs (Kratochvíl, 1994) and extended to p-regular graphs, for $p > 3$ (Cardoso et al., 2008), interval bigraphs, hypertrees and acyclic hypergraphs (Brandstädt et al., 2012), chordal unipolar graphs (Eschen and Wang, 2014). Apart from this, the weighted efficient domination problem is solved in polynomial time for special classes of graphs like split graphs ($\mathcal{O}(|V(G)| + |E(G)|)$) (Chang and Liu, 1993), series-parallel graphs ($\mathcal{O}(|V(G)| + |E(G)|)$) (Grinstead and Slater, 1994), interval graphs ($\mathcal{O}(|V(G)| + |E(G)|)$) (Chang and Liu, 1994), circular-arc graphs ($\mathcal{O}(|V(G)||E(G)| + |V(G)|^2)$) (Chang and Liu, 1994), cocomparability graphs ($\mathcal{O}(|V(G)||E(G)|)$) (Chang et al., 1995), block graphs ($\mathcal{O}(|V(G)| + |E(G)|)$) (Chain-Chin and Lee, 1996), permutation graphs ($\mathcal{O}(|V(G)| + |E(\overline{G})|)$) (Liang et al., 1997), trapezoid graphs ($\mathcal{O}(|V(G)| \log\log |V(G)| + |E(\overline{G})|)$) (Liang et al., 1997), bipartite permutation graph ($\mathcal{O}(|V(G)|)$), distance-hereditary graph ($\mathcal{O}(|V(G)|)$) (Lu and Tang, 2002), AT-free graphs ($\mathcal{O}(\min\{|V(G)||E(G)| + |V(G)|^2, |V(G)|^{\omega}\})$, where $\omega < 2.3727$) (Brandstädt et al., 2015), (P_6, banner)-free graphs ($\mathcal{O}(|V(G)|^3)$) (Karthick, 2016).

One of the recent articles by Brandstädt (2018) gives a survey of the research progress in the Efficient domination problem from an algorithmic viewpoint. Further, a dichotomy of the complexity of efficient domination is discussed for H-free graphs, where H is a disjoint union of chordless paths P_k, for any k. H is a linear forest if H is claw-free and C_k-free, for every $k \geq 3$. Efficient domination problem is $\mathcal{NP}$-complete for chordal graphs, bipartite graphs and claw-free graphs (Brandstädt, 2018). Efficient domination problem is $\mathcal{NP}$-complete for (C_k, claw)-free graphs (Brandstädt, 2018). For linear forests H, Efficient domination problem is $\mathcal{NP}$-complete for $2P_3$-free graphs (Brandstädt, 2018). Efficient domination problem is solvable in linear time for $2P_2$-free graphs ($\mathcal{O}(|V(G)| + |E(G)|)$) (Brandstädt et al., 2013), P_5-free graphs ($\mathcal{O}(|V(G)||E(G)|)$) and ($P_4 + P_2$)-free graphs ($\mathcal{O}(|V(G)||E(G)|)$) (Nevries, 2014). If efficient domination problem is polynomial for H-free graphs, then it is polynomial for ($H + kP_2$)-free graphs, for every fixed k (Brandstädt and Giakoumakis, 2014). Efficient domination problem is solved in polynomial time for P_6-free graphs (Lokshtanov et al., 2017; Brandstädt et al., 2017).

2.4 Research gap

Efficient domination stands unique among other variants of domination because of its three properties: Domination, Independence and 2-packing. Efficient domination has its wide applications in communication networks, mobile ad hoc networks, coding theory, fault tolerance analysis, wireless sensor networks.

Though this particular type of domination has a long history, to the best of our knowledge, it has not been much explored like the other domination parameters from a graph theoretic perspective. Most of the research papers in the literature deal with the algorithmic aspects of the problem either on arbitrary special graphs or on some special classes of graphs like perfect graphs and so on. Related to general graphs the results are very limited. Regarding product graphs, the existing results are focused involving well known graphs like paths, cycles, etc. Only a few results deals with arbitrary graphs and products like cross product, lexicographic product, etc. To summarize, unlike other domination invariants, the concept of efficient domination has not been studied much for general graphs and the study on the properties of efficiently dominatable graphs and graphs which are not efficiently dominatable have not been explored completely.

This research gap has led to our motivation on this problem and necessitates an independent study of efficiently dominatable graphs.

2.5 Objectives of the Thesis

With the above motivation and research gap identified, the following were set as the objectives for this research work.

Objective 1: To study the concept of efficient domination in general graphs

- To obtain improved bounds on the domination number of efficiently dominatable graphs, bounds on efficient domination number in terms of degree, order and size.

- To explore the structural properties of graphs which are efficiently dominatable and those which are not efficiently dominatable.

- To study the notion of efficient domination in trees.

Objective 2: To study the critical aspects of efficient domination in graphs.

In general, the removal of a vertex from a given graph may increase or decrease or leave unaltered the domination number of the graph. Similar effects are observed upon removal of an edge as well as addition of an edge. The study related to this analysis is referred to as the study of criticality aspects with respect to domination. While the concept of criticality is well explored with respect to domination and its variants, to the best of our knowledge, the concept has not been studied with respect to efficient domination, except for the study of super-efficient graphs by Barbosa and Slater (2016). Therefore, motivated by the study of changing and unchanging properties with respect to the classical/ordinary domination surveyed in (Haynes et al., 1998) and with respect to its other variants in (Edwards, 2006; Hou and Edwards, 2008; Ebrahimi and Ebadi, 2011; Samodivkin, 2016), the study is initiated on criticality aspects for efficiently dominatable graphs. On these lines, the following were set as the sub-objectives.

- To study changing and unchanging efficient domination with respect to vertex criticality (Vertex removal).

- To study changing and unchanging efficient domination with respect to edge criticality (Edge removal and Edge addition).

- To classify and relate all the critical sets and classes generated due to vertex removal, edge removal and edge addition.

Objective 3: To study efficient domination in Cartesian product of graphs.

With the intention of exploring the structural properties of efficiently dominatable cartesian product of two or more arbitrary graphs and those which are not efficiently dominatable, in this Thesis, the study is initiated on Cartesian product of two graphs, one of whose factors is an arbitrary graph and the other factor is a well-known graph like P_n, C_n etc. Based on this, the following were set as the sub-objectives.

- To study efficient domination in the cartesian product of two well-known graphs, namely, P_n, C_n, K_n and $K_{1,n}$.
- To study efficient domination in the cartesian product of two graphs, where one of the factors is an arbitrary graph and other is a well-known graph.
- To study efficient domination in the cartesian product of two arbitrary graphs.
- To extend the study for cartesian product of graphs with more than two factors.
- To design exact-exponential time algorithms to determine whether or not the Cartesian product of two graphs is efficiently dominatable. (Here, one of the factors is restricted to be K_n or $K_{1,n}$).

Chapter 3

Efficient Domination in Graphs

Based on the research gap identified from the literature and the objectives set for this thesis, in this chapter, an attempt is made to obtain some basic results on efficient domination in general graphs, graphs with restricted conditions and trees. One of the critical issues in the design of network topology is to obtain a fault tolerant structure so as to facilitate an uninterrupted efficient communication. In the literature, significant contribution has been made to the design of fault-tolerant structures by adopting various graph theoretic techniques. In line with that, some fault-tolerant graph structures are proposed that are suitable for efficient communication in wireless sensor networks, based on the notion of efficient domination.

3.1 Efficient Domination in general graphs

As stated in Theorem 1.3.1, if a graph G is efficiently dominatable, then any EDS of G has its cardinality equal to $\gamma(G)$. So, with the intent to examine if the property of being efficiently dominatable has any influence on the interval for the domination number of a graph, the initial focus is on bounds for the domination number of efficiently dominatable graphs. Section 3.1.1 deals with some improved bounds on γ for efficiently dominatable graphs. Section 3.1.2 deals with a discussion on the existence of efficiently dominatable graphs having domination number k, for any positive integer k and includes a procedure to construct such graphs.

In Section 3.1.3, some basic necessary conditions are determined for a graph of diameter three to be efficiently dominatable. In general, an efficiently dominatable graph may possess either a unique EDS or more than one EDS. When it has more than one EDS, the sets may or may not intersect. Based on this fact, the structural properties of those graphs possessing pairwise disjoint efficient dominating (PWDED) sets are studied with a brief discussion on the applications of such structures; a characterization for such graphs is also obtained in Section 3.1.4.

Notation 3.1.1. *For a convenient reference, throughout this thesis, the notation* $\mathscr{E}$ *is used to denote the collection of all efficiently dominatable graphs.*

Proposition 3.1.1. *If* G *is a graph of order* n*, where* n *is even and* $G \in \mathscr{E}$*, then a vertex of degree* $n-2$ *does not belong to any EDS of* G*.*

Proof. Let $G \in \mathscr{E}$ and $|V(G)| = n$, where n is even. Let u be an arbitrary vertex of degree $n-2$ in G. Then, there exists a vertex $v \in V(G)$ such that $d_G(u,v) = 2$. Suppose that S is an EDS of G containing u, then u dominates exactly $(n-1)$ vertices (including itself) and no other vertex can be included in S. Hence, v is left undominated by S, contradicting that S is an EDS of G. Therefore, $u \notin S$. □

3.1.1 Bounds on Domination number of Efficiently Dominatable graphs

Various bounds on $\gamma(G)$ have been obtained in terms of degree, order and size of G (refer to (Haynes et al., 1998)). By revisiting those bounds for efficiently dominatable graphs, the results are discussed below, some of which are immediate consequences from the known bounds while few others are improved by restricting the graphs under consideration to be efficiently dominatable.

Bounds on γ in terms of order of a graph

Theorem 3.1.2. *(Haynes et al., 1998) If* G *is a graph of even order* n *with no isolated vertices, then* $\gamma(G) = \frac{n}{2}$ *if and only if the components of* G *are either* C_4 *or* $H \circ K_1$*, for any connected graph* H*.*

But, it is observed that C_4 is not efficiently dominatable. Hence, Theorem 3.1.2 leads to the following immediate characterization for an efficiently dominatable graph of even order and whose domination number is half the order.

Theorem 3.1.3. *If G is an efficiently dominatable graph of even order n with no isolated vertices, then $\gamma(G) = \frac{n}{2}$ if and only if each component of G is isomorphic to $H \circ K_1$, for some connected graph H.*

Let $\mathcal{A}$ denote the collection of graphs given in Figure 3.1. Then, the following bound exists for a graph with minimum degree at least 2.

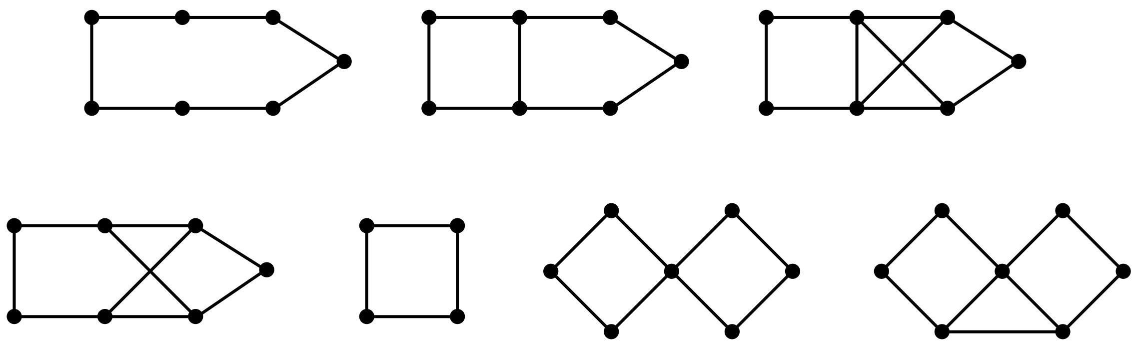


Figure 3.1: Graphs belonging to the family $\mathcal{A}$
(Haynes et al., 1998)

Theorem 3.1.4. *(Haynes et al., 1998) If G is a connected graph of order n with $\delta(G) \geq 2$ and $G \notin \mathcal{A}$, then $\gamma(G) \leq \frac{2n}{5}$.*

It can be observed that none of the graphs in the family $\mathcal{A}$ is efficiently dominatable. Hence, the following result is obtained as an immediate consequence of Theorem 3.1.4, when restricted to the class $\mathscr{E}$.

Theorem 3.1.5. *If G is an efficiently dominatable connected graph of order n with $\delta(G) \geq 2$, then $\gamma(G) \leq \frac{2n}{5}$.*

Lemma 3.1.6. *Let G be an efficiently dominatable connected graph of order n with $\delta(G) \geq 2$. Then, $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if $G \cong K_3$.*

Proof. If $G \cong K_3$, then clearly, $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Conversely, let $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Suppose that n is even. Then $n = 2k$, for some k and $\gamma(G) = k$. As G is a connected graph such that $G \in \mathscr{E}$ and $\delta(G) \geq 2$, it follows from Theorem 3.1.5 that $\gamma(G) \leq \frac{2n}{5}$. That is, $k \leq \frac{4k}{5}$, which is absurd. Hence, n must be odd.

Further, as $\left\lfloor \frac{n}{2} \right\rfloor \leq \frac{2n}{5}$, n must be equal to either 3 or 5. The graphs depicted in Figure 3.2 are the only possible graphs of order 3 or 5 with $\delta(G) \geq 2$ and $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Of these, K_3 is the only graph which is efficiently dominatable. Hence, the result follows. □

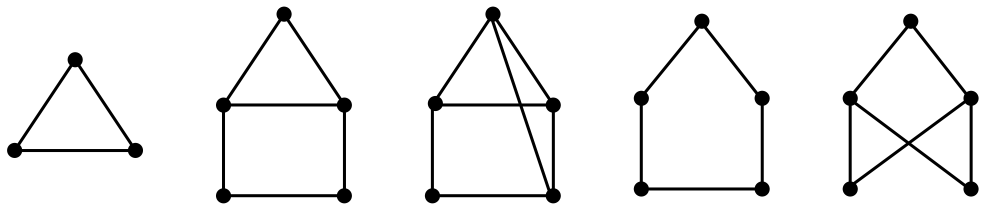


Figure 3.2: Graphs of order 3 or 5 with $\delta(G) \geq 2$ and $\gamma(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Notation 3.1.2. *For any graph H, let $S(H)$ denote the set of all connected graphs obtained from $H \circ K_1$ by adding a new vertex, say u, such that u is made adjacent to exactly one pendant vertex of $H \circ K_1$ and one or more vertices of H. (Refer to Figure 3.3)*

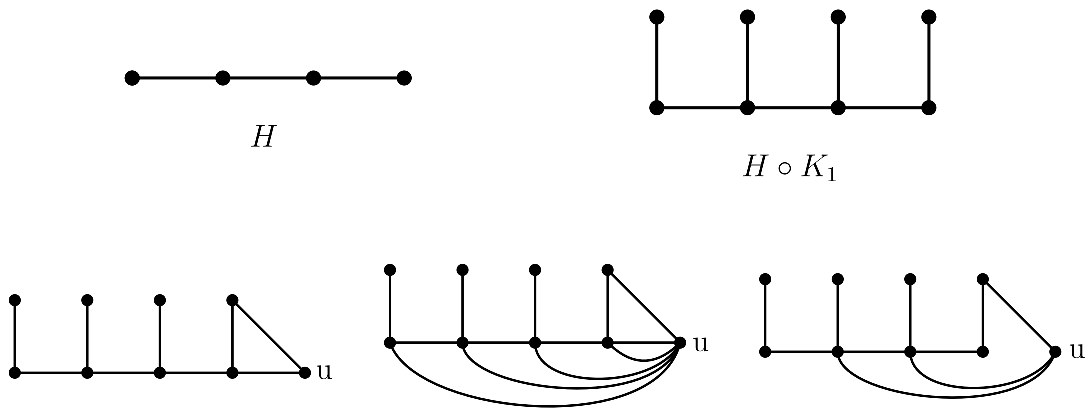


Figure 3.3: Some graphs in $S(H)$

Lemma 3.1.7. *Let G be an efficiently dominatable connected graph of order n with $\delta(G) = 1$. Then, $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if the following conditions hold:*

(i) Whenever n is even, $G \cong H \circ K_1$, for some connected graph H.

(ii) Whenever n is odd, either $G \cong P_3$ or G must have exactly $\left\lfloor \frac{n-2}{2} \right\rfloor$ pendant vertices, one vertex of degree two and the remaining vertices of degree at least two (Precisely, $G \in S(H)$, for some connected graph H).

Proof. If either condition (i) or condition (ii) holds according as n is odd or even, then clearly $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$.
Conversely, let $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$. Suppose that n is even, then condition (i) follows from Theorem 3.1.3. Whereas, if n is odd, then $n = 2k + 1$ for some k. Suppose $n = 3$, then as $\delta(G) = 1$, $G \cong P_3$. On the other hand, if $n > 3$, let S be an EDS of G, where $S = \{v_1, v_2, \ldots, v_k\}$. Then, $|V - S| = n - \left\lfloor \frac{n}{2} \right\rfloor = n - k = k + 1$. Let $V - S = \{u_1, u_2, \ldots, u_k, u_{k+1}\}$. Then, as S is an EDS of G, each u_i must have exactly one neighbor in S. Equivalently, as $|V - S| = |S| + 1$, every vertex in S must have exactly one neighbor in $V - S$, except for one vertex which has two neighbors in $V - S$. That is, each vertex in S is a pendant vertex except for one vertex which is of degree two. Without loss of generality, let $\deg(v_i) = 1$, for each i, where $1 \leq i \leq k - 1$ and $\deg(v_k) = 2$; let v_i be adjacent to u_i, for each i $(1 \leq i \leq k - 1)$ and v_k be adjacent to the two vertices u_k and u_{k+1}. Then, as S is independent and G is connected, the subgraph induced by $V - S$ must also be connected. Therefore, by defining $H = < V - S >$, the graph G belongs to $S(H)$. Hence, the result follows. □

Theorem 3.1.8. *Let G be an efficiently dominatable connected graph of order n. Then, $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if one of the following conditions hold.*

(i) $G \cong K_3$

(ii) $G \cong P_3$

(iii) $G \cong H \circ K_1$, for some connected graph H.

(iv) $G \in S(H)$, for some connected graph H.

Proof. Clearly, if any of the conditions (i) to (iv) hold, then $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$.
Conversely, let $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$. If n is even, then by Theorem 3.1.3, condition (iii) holds. On the other hand, suppose that n is odd. Here, if $\delta(G) = 1$, then it follows

from Lemma 3.1.7 that either condition (ii) or (iv) must hold and when $\delta(G) \geq 2$, it follows from Lemma 3.1.6 that condition (i) holds. $\square$

Bounds on γ in terms of size of a graph

Next, some bounds on γ are discussed for efficiently dominatable graphs, in terms of size. A known bound on γ in terms of the size 'm' of a graph G is as stated below.

Theorem 3.1.9. *(Haynes et al., 1998) For any graph G with $\gamma(G) \geq 2$,* $m \leq \left\lfloor \frac{1}{2}(n-\gamma(G))(n-\gamma(G)+2) \right\rfloor$.

Revisiting this result for efficiently dominatable graphs, it is observed that the bound is improved by a factor of $\frac{n-\gamma}{2}$. Further, the result holds even for all graphs $G \in \mathscr{E}$ with $\gamma(G) = 1$, as discussed below.

Theorem 3.1.10. *Let G be a simple, connected (n, m) graph such that $G \in \mathscr{E}$. Then,* $m \leq \dfrac{(n-\gamma(G))(n-\gamma(G)+1)}{2}$.

Proof. Let $\gamma(G) = k$ and S be an EDS of G. Then, $|S| = k$ and $|V - S| = n - k$. Further, every vertex in $V - S$ has a unique neighbor in S. Therefore, as G is connected, exactly $(n-k)$ edges connect S with $V - S$. As S is independent, $< S >$ has zero edges. And, $< V - S >$ has at most $\dfrac{(n-k)(n-k-1)}{2}$ edges. Thus, the maximum number of edges in G is $(n-k) + 0 + \dfrac{(n-k)(n-k-1)}{2}$. That is, $m \leq \dfrac{(n-k)(n-k+1)}{2}$. $\square$

Corollary 3.1.10.1. *For every connected (n, m) graph G, if $G \in \mathscr{E}$ then* $\gamma(G) \leq \dfrac{2n+1-\sqrt{8m+1}}{2}$.

Proof. Let G be a connected (n, m)-graph such that $G \in \mathscr{E}$ and let $\gamma(G) = k$. Then, it follows from Theorem 3.1.10 that $2m \leq (n-k)^2 + (n-k)$. On completing the square, $\left(n-k+\frac{1}{2}\right)^2 \geq 2m+\frac{1}{4}$. Here, as $k \leq \frac{n}{2}$, $\left(n-k+\frac{1}{2}\right) \geq 0$. Therefore, $k \leq \dfrac{2n+1-\sqrt{8m+1}}{2}$. $\square$

Bounds on γ in terms of minimum and maximum degree of a graph

In this section, some bounds are obtained for the domination number of an efficiently dominatable graph, in terms of the minimum and maximum degree of the graph. The following result gives a lower and an upper bound on γ for an arbitrary graph, in terms of the maximum degree.

Theorem 3.1.11. *(Haynes et al., 1998) For any graph G, $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq n - \Delta(G)$.*

Remark 3.1.1. *It can be observed that the lower bound in Theorem 3.1.11 is sharp, that is, $\gamma(G) = \frac{n}{1+\Delta(G)}$ if and only if G is efficiently dominatable and in particular, if S is any EDS of G, then $deg(v) = \Delta(G)$, for all $v \in S$. In other words, the lower bound for γ in Theorem 3.1.11 is best possible for efficiently dominatable graphs.*

Proposition 3.1.12. *If $G \in \mathscr{E}$, then $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \leq \gamma(G) \leq \left\lfloor \frac{n}{1+\delta(G)} \right\rfloor$.*

Proof. Let $G \in \mathscr{E}$ and S be an EDS of G. Then, $|S| = \gamma(G)$ and $I(S) = n$. Further, for each $v \in V(G)$, $\delta(G) \leq \deg(v) \leq \Delta(G)$. Thus, $|S|(1 + \delta(G)) \leq I(S) \leq |S|(1 + \Delta(G))$. That is, $\gamma(G)(1 + \delta(G)) \leq n \leq \gamma(G)(1 + \Delta(G))$. Hence, the result follows. □

Remark 3.1.2. *It can be observed from Remark 3.1.1 and Proposition 3.1.12 that if G is a regular graph with $\gamma(G) = \frac{n}{1+\Delta(G)}$, then G must be efficiently dominatable. However, if G is an efficiently dominatable graph with $\gamma(G) = \frac{n}{1+\Delta(G)}$, then G need not be regular. (refer to Figure 3.4)*

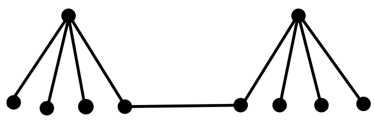


Figure 3.4: An efficiently dominatable graph with $\gamma(G) = \frac{n}{1+\Delta(G)}$, but not regular

Theorem 3.1.13. *For any connected graph G with $\gamma(G) \geq 2$ and $\gamma(G) = n - \Delta(G)$, $G \in \mathscr{E}$ if and only if $rad(< V - S >) = 1$, where S is a γ-set of G.*

Proof. Let $\gamma(G) \geq 2$ and S be a γ-set of G.

Let $G \in \mathscr{E}$ and S be its EDS. For $v \in V(G)$, let $\deg(v) = \Delta(G)$.

Claim: $v \in V - S$

Suppose that $\gamma(G) = k$. Then, $\Delta(G) = n-k = \deg(v)$. If $v \in S$, then v is adjacent to none of the $k-1$ vertices in S and its neighbors and hence $\deg(v) < n-k$, a contradiction. Thus, $v \in V - S$.

Since $v \in V - S$ and $\deg(v) = n - k$, v is adjacent to all the vertices in $V - S$ and thus $rad(< V - S >) = 1$.

Conversely, let $rad(< V - S >) = 1$. Let $w \in V - S$ be adjacent to all the other vertices in $V - S$. Then, for any pair $u, v \in S$, $d(u, v) = d(u, w) + d(w, v) = 3$ or 4, accordingly when w is adjacent or nonadjacent to one of the neighbors of u or v. Thus, S is an EDS of G and hence $G \in \mathscr{E}$. □

Corollary 3.1.13.1. *Let T be a tree with $\gamma(T) = n - \Delta(T)$. Then, $T \in \mathscr{E}$ if and only if $T \cong K_{1,n-1}$.*

3.1.2 Existence of Efficiently Dominatable graphs with domination number k, for any integer $k > 0$

Given any positive integer k, the existence is proved for efficiently dominatable graphs having domination number k and a method is proposed to construct such graphs.

Theorem 3.1.14. *Given any pair of integers n and k, where $n \geq 2$ and $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, there exists an efficiently dominatable graph G of order n with $\gamma(G) = k$.*

Proof. Let G' be an arbitrary connected graph of order k and $V(G') = \{v_1, v_2, \ldots, v_k\}$. Now, construct a graph G from G' as follows: Add k new vertices $u_1, u_2, \ldots, u_k$ such that for each $i \in \{1, 2, \ldots, k\}$, $u_iv_i \in E(G)$ and $\deg(u_i) = 1$. Clearly, $|V(G)| = 2k$ and $|E(G)| \geq 2k - 1$. If n is even and $k = \frac{n}{2}$, then $G \in \mathscr{E}$ with $\{u_1, u_2, \ldots, u_k\}$ as its EDS and $\gamma(G) = \frac{n}{2}$. Else, add $n - 2k$ new vertices $w_1, w_2, \ldots, w_{n-2k}$ to G. For each $i \in \{1, 2, \ldots, n - k\}$, join w_i to u_j, for some j, such that $1 \leq j \leq k$, subject to the condition that $\deg(w_i) \geq 1$. Here, each w_i is made adjacent to exactly one u_j, while each u_j may be adjacent to more than one w_i. Then, the set $\{u_1, u_2, \ldots, u_k\}$ will be an EDS of the resultant graph G and $\gamma(G) = k$. □

3.1.3 Graphs of diameter three

Let us consider graphs G of order n having diameter three. It can be observed that $n \geq 4$ and the eccentricities of all the vertices of G are either 2 or 3.

Proposition 3.1.15. *Let $G \in \mathscr{E}$ and $diam(G) = 3$. Then the following results hold:*

(i) $\triangle(G) \leq n - 2$ *and* $\gamma(G) \geq 2$.

(ii) $\gamma(G) = \dfrac{n}{2}$ *if and only if* $< V - S >$ *is complete.*

(iii) All the vertices in any EDS of G is of eccentricity three.

(iv) Any EDS of G contains all the pendant vertices of G, if exists.

(v) For $n \geq 4$, G is cyclic.

Proof. Let S be an EDS of G.
(i) If $\triangle(G) = n - 1$, then $rad(G) = 1$ and $diam(G) \leq 2$. Thus, $\triangle(G) \leq n - 2$. Since $diam(G) = 3$, at least two vertices are needed to efficiently dominate G. Therefore, $\gamma(G) \geq 2$.
(ii) Let $S = \{u_1, u_2, \ldots, u_k\}$. Since $\gamma(G) = \dfrac{n}{2}$, $\deg(u_i) = 1$, for every $u_i \in S$. Suppose that, there exist two nonadjacent vertices, say u, $v \in V - S$, such that $u \in N(u_1)$ and $v \in N(u_2)$, for $u_1, u_2 \in S$. Then, $d(u_1, u_2) > 3$, contradicting that $diam(G) = 3$. Thus, all the vertices in $V - S$ are adjacent to each other. That is, $< V - S >$ is complete.
(iii) Let $v \in S$ and suppose that $ecc(v) \neq 3$. Then, $ecc(v) = 2$. Since S is a 2-packing, all the vertices at a distance 2 from v cannot be in S and hence are left undominated efficiently. Thus, $v \notin S$, a contradiction. Thus, for all $v \in S$, $ecc(v) = 3$ holds.

(iv) *Claim:* There can be at most one pendant vertex adjacent to any vertex of G.

Suppose there exist at least two pendant vertices adjacent to a vertex, say u, of G. Then, $u \in S$ and also $ecc(u) = 2$, which is not possible. Thus, G can have at most one pendant vertex adjacent to any vertex.

Since $diam(G) = 3$, all the pendant vertices, if it exist, will have eccentricity three. All the vertices adjacent to these pendant vertices will have eccentricity two and hence cannot belong to any EDS of G. Since $G \in \mathscr{E}$, all the pendant vertices must be included in any EDS of G.

(v) For $n \geq 4$, if G is acyclic, then Theorem 3.2.15 implies that $G \notin \mathscr{E}$. Thus, G is cyclic. □

3.1.4 Graphs having at least two pairwise disjoint efficient dominating sets and Applications

Let $G \in \mathscr{E}$ with $\gamma(G) = k$. For $l \geq 2$, let $S_1, S_2, \ldots, S_l$ be l PWDED sets of G. Then, $|S_1| = |S_2| = \cdots = |S_l| = k$. Let $S^* = V(G) - (S_1 \cup S_2 \cup \cdots \cup S_l)$. Then, $|S^*| = n - kl = n^*$ (say). The set S^* may or may not be empty. Let $|E(S_i, S_j)|$ represents the number of edges between the sets S_i and S_j. Then G is isomorphic to the structure shown in Figure 3.5. As S_i $(1 \leq i \leq l)$ is an EDS, for every $u \in V - S_i$, $|N(u) \cap S_i| = 1$. Every vertex in S^* is adjacent to a unique vertex from each S_i. Based on the structure of G and discussion above, the following properties are observed in G.

Proposition 3.1.16.

(i) *For each* $v \in V(S^*)$, $|N(v) \cap S_i| = 1$, *for* $i \in \{1, 2, \ldots, l\}$.

(ii) *For each* $v \in V(S_i)$, $|N(v) \cap S_j| = 1$, *for each* $i \neq j$ *and* $1 \leq i, j \leq l$

(iii) *If* G *contains at least* l $(l \geq 2)$ *PWDED sets, the* $\gamma(G) \leq \dfrac{p}{l}$.

(iv) *If* G *has at least* l *PWDED sets, then* $\dfrac{kl(l-1)}{2} + ln^* \leq |E(G)| \leq \dfrac{kl(l-1)}{2} + ln^* + |E < S^* > |$.

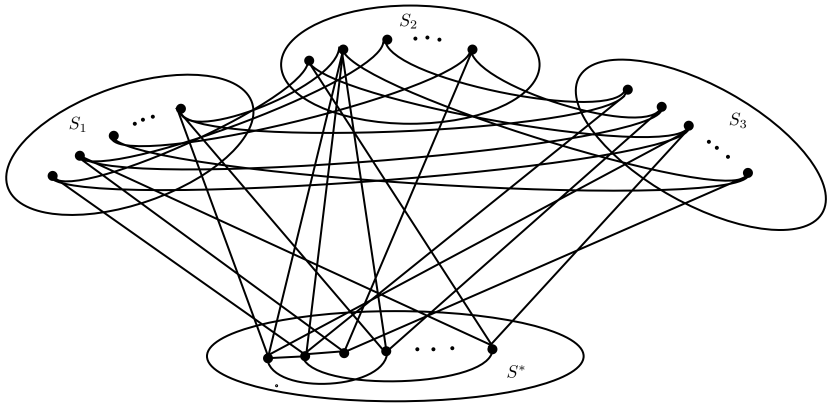


Figure 3.5: A graph with three pairwise disjoint efficient dominating sets

(v) For any $v \in V(G)$, $l \le \deg_{<S^*>}(v) \le n^* + l - 1$ and $l - 1 \le \deg_{<S_i>}(v) \le n^* + l - 1$, for each i, $i \in \{1, 2, \ldots l\}$.

Proof. Properties (i) to (iii) and (v) follow from the discussion above.

Proof of (iv): For each $i \in \{1, 2, \ldots l\}$, since each vertex in each S_i has a unique neighbor in S_j, for $i \ne j$, $|E(S_i, S_j)| = k$ and hence $\sum_{1 \le i \ne j \le l} |E(S_i, S_j)| = \frac{kl(l-1)}{2}$. Also, as every vertex in S^* has a unique neighbor in S_i, for $i \in \{1, 2, \ldots l\}$, $|E(S_i, S^*)| = n^*$. Thus, $\sum_{1 \le i \ne j \le l} |E(S_i, S^*)| = ln^*$. Since, $E(G) = E(S_i, S_j) + E(S_i, S^*) + E < S^* >$, it follows that, $\frac{kl(l-1)}{2} + ln^* \le |E(G)| \le \frac{kl(l-1)}{2} + ln^* + |E < S^* > |$, where $0 \le |E < S^* > | \le \frac{n^*(n-1)}{2}$. □

Proposition 3.1.17. *Let $G \in \mathscr{E}$. If $\Delta(G) \le l$, then there can be at most $(l + 1)$ PWDED sets of G. If G has at least l PWDED sets, then $\delta(G) \ge l - 1$.*

Proof. As $G \in \mathscr{E}$, for all $u \in V(G)$, either $u \in S$ or one of its neighbors $N(u)$ belongs to S. If there exist k pairwise disjoint efficient dominating sets of G, then k distinct vertices of $N[u]$ will belong to k different efficient dominating sets of G. Thus, if $\Delta(G) \le l$, then a maximum of $(l + 1)$ such efficient dominating sets are possible in G. And if G has at least l pairwise disjoint efficient dominating sets, then $\delta(G) \ge l - 1$. □

Proposition 3.1.18. *If G is connected and $G \in \mathscr{E}$, then G has three pair wise disjoint efficient dominating sets if and only if for all pairs $u, v \in V(G)$, there exists an EDS of G not containing both u and v.*

Proof. Suppose that G contains at least three pairwise disjoint EDSs. Now, for any $u \in V(G)$, u and its neighbors belong to distinct EDSs. Hence, for all pairs $u, v \in V(G)$ (adjacent or nonadjacent), there exists at least one EDS not containing both u and v.

Conversely, suppose that for each vertex pairs $u, v \in V(G)$, there exists an EDS not containing both u and v. Then, as G is connected, it must have at least three EDS. Suppose that G has exactly three EDS, say S_1, S_2 and S_3. Clearly, $S_1 \cap S_2 \cap S_3 = \emptyset$.

Claim: $S_i \cap S_j = \emptyset$, for $i \neq j$.

Suppose that $S_1 \cap S_2 \neq \emptyset$. If $u \in V(G)$ where $u \in S_1 \cap S_2$, then $u \notin S_3$. As $u \notin S_3$, a neighbor of u, say v, must be in S_3. But then, the hypothesis does not holds for the pair u, v. Hence, the result follows. □

Remark 3.1.3. *If G has l PWDED sets $S_1, S_2, \ldots, S_l$, then $S_i \subseteq V - S_j$, for $i \neq j$ and $1 \leq i, j \leq l$.*

Theorem 3.1.19. *For $r \geq 1$, G is an r-regular graph containing $(r+1)$ pairwise disjoint efficient dominating sets if and only if $V(G)$ can be partitioned into $(r+1)$ independent sets S_i (for $i = 1$ to $r+1$), each of cardinality $\frac{|V(G)|}{r+1}$, such that each vertex $u \in S_i$ has a unique neighbor in S_j, for every $i \neq j$.*

Proof. Let G be an r-regular efficiently dominating graph and $|V(G)| = n$. Let $S_1, S_2, \ldots, S_{r+1}$ be $r+1$ PWDED sets of G. Since G is r-regular, $\gamma(G) = \frac{n}{r+1}$. Thus, $|S_1| = \frac{n}{r+1} = |S_2| = \cdots = |S_{r+1}|$. Also, for any $u \in V(G)$, since $\deg(u) = r$, the $r+1$ distinct vertices of $N[u]$ will belong to $r+1$ distinct efficient dominating sets of G. Thus, $\bigcup_{i=1}^{r+1} S_i = V(G)$. In other words, S_i's (for $1 \leq i \leq r+1$) form a partition of $V(G)$. For any i, $1 \leq i \leq r+1$, since S_i is an EDS, any vertex $u \in V - S_i$ is adjacent to a unique vertex of S_i. Hence, the result follows.

Conversely, Let $V(G)$ be partitioned into $(r+1)$ sets, say S_1, S_2, $\ldots$, S_{r+1}, where each S_i is independent and $|S_i| = \frac{n}{r+1}$. Also, assume that each vertex $u \in S_i$ has a unique neighbor in S_j, for every $i \neq j$. Thus, $\deg_{<S_i>} u = r$. Hence, G is r-regular. Since S_i's are independent, for each $u \in S_i$, $|N[u] \cap S_i| = 1$. Also by our assumption, for each $u \in V - S_i$, $|N(u) \cap S_i| = 1$. Thus, for each $u \in V(G)$,

$|N[u] \cap S_i| = 1$, for $i = \{1, 2, \ldots, r+1\}$. That is, S_i's are efficient dominating sets of G. Hence, the result follows. □

Remark 3.1.4. *It follows from Theorem 3.1.19 that the set S^* in Figure 3.5 is empty and hence if G is an r-regular graph having $(r+1)$ PWDED sets, then G is isomorphic to the structure shown in Figure 3.6.*

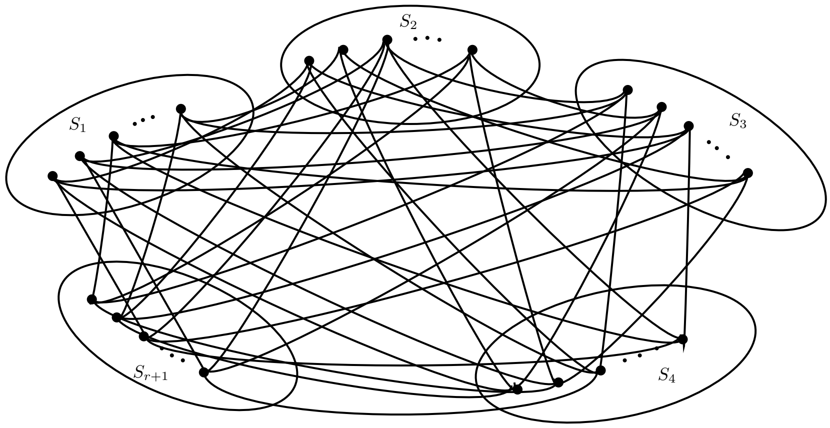


Figure 3.6: A graph with $r+1$ pairwise disjoint efficient dominating sets

An Application to Wireless Ad hoc and Sensor Networks

Treating different functional units of a mobile (or static) device in a network as a single unit, termed collectively as a node, a network topology can be modelled as follows: In the network, the set of all nodes are considered as the vertex set and vertices are joined by an edge if the respective network nodes are in the range of transmission. The graph so obtained is termed as the "*underlying graph*" or "*network graph*". This general graph abstraction can be extended further to construct graph models that satisfy specific network characteristics. Incorporating additional constraints on transmission range or other similar communication criteria, the graph so obtained can be either directed or undirected.

Unlike wired/static network communication, in an ad hoc network environment, link failures, new link establishments, node failures and new node arrivals are frequent occurrences. Therefore, in topology design, it is essential to ensure that network communication is not affected much by node and link failures. In other words, the network must be fault-tolerant.

In general, the problem of fault-tolerance can be addressed at two levels. One is at the level of network topology design as discussed above. The other one is at the level of designing fault-tolerant virtual backbone (generally termed as Clusters). The graph structures which are discussed in Section 3.1.3 focus on the latter approach and help in designing fault-tolerant virtual backbone, also termed as clusters for ad hoc and sensor networks. *Clustering represents partitioning the network into subnetworks, referred to as clusters, of varied or equal sizes.* This results in a virtual organization of the ad hoc network and is basically the problem of graph partitioning. Clustering can be of two categories: Head-based and Non-head-based. In head-based clustering, communication within the entire network is facilitated with the help of cluster heads which are special nodes identified within the network based on some strategy. Clustering is accomplished by determining a set which dominates the underlying graph. Every vertex in the dominating set together with its (1-hop) neighbors will form a cluster.

It is assumed that all wireless nodes are within a uniform transmission range so that the underlying network graph is undirected.

Design Strategy of Networks supporting Fault-tolerant Communication in Sensor Networks

Let us consider the structure as in Figure 3.6 given in Section 3.1.3. The underlying graph consists of $r+1$ pairwise disjoint efficient dominating sets. Considering this structure as the underlying topology of a network of a set of static wireless nodes, its properties are analyzed. The structure proposed here (a) facilitates interference-free communication, (b) Possesses a built-in non-overlapping clustering architecture (c) Possesses an optimal cluster partition and (d) supports fault-tolerant communication. A brief outline is given in the discussion to follow.

Clustering: The proposed structure satisfies the desired characteristics to be a "well formed" cluster architecture. Every node is in exactly one cluster and maintains full coverage. Each cluster possesses a distinctive node called cluster head (CH) and the structure is in such a way that the set of all CH's form an

EDS of the underlying graph. It can also be observed that the set of CH's of this network forms an independent set and they are at a distance at least three from other CH's. In the process of cluster-based routing using the proposed cluster architecture, the clusters are well-separated. At that same time, the CH's are neither too close nor too far from each other. Moreover, for each CH, it is always possible to find a CH exactly at distance three (Thilak, 2013) and the two nodes between these CH's are said to form a gateway. The induced subgraph of the CH's together with these gateway nodes forms a dominating set which is also connected. The degree of every node is r and hence this network is $r - 1$ connected.

A structure similar to Figure 3.6 possesses the following properties:

- The structure is efficiently dominatable and hence has at least one EDS, thereby supports the process of clustering using EDS.

- The structure has $r+1$ pairwise disjoint efficient dominating sets, facilitating a proper load balanced communication among all network nodes with the help of a suitable activity scheduling. This makes the structure more suitable for sensor networks.

- Each set S_i $(1 \leq i \leq r + 1)$ will induce a non-overlapping cluster, so as to facilitate interference-free communication.

- For each S_i $(1 \leq i \leq r + 1)$, every vertex $u \in S_i$ has a unique neighbor in S_j for all $i \neq j$. Therefore, in case of failure of node u, the role of node u can be handled by any one of its neighbors in one of the sets S_j $(i \neq j)$, thereby supporting fault-tolerant communication.

- To support an efficient channel assignment, for each i, where $1 \leq i \leq r + 1$, the set S_i is independent. Assuming that if two vertices are adjacent in the graph, those two wireless components cannot use the same channel/spectrum for communication simultaneously due to possible wireless interference, as each S_i is independent, for all the vertices in S_i, the same spectrum is associated from an available list of spectra. The concept of list colouring in graphs will facilitate such a spectral assignment for the vertices in S_i. At the same time, a vertex in S_i will not be assigned the same spectrum as its neighbors in S_j, for any $i \neq j$. Further, because of the differences in their geographic locations, it is preferred to assign different sets of spectra for different vertices and this can be facilitated with the help of list colouring. The list colouring being a proper vertex colouring, guarantees that no two vertices adjacent to each other are allocated the same spectrum. Thus, the channel assignment problem can be effectively managed.

- Finally, as the network structure is in such a way that each set S_i is an EDS, any of these sets can be used to facilitate cluster-based routing in ad hoc networks.

3.2 Efficient domination in Trees

This section deals with the properties of vertices in trees T, where $T \in \mathscr{E}$ and $T \notin \mathscr{E}$; the necessary conditions for a tree to be efficiently dominatable (or not efficiently dominatable). In Section 3.1.4, trees T are considered, for which $S(T) = \emptyset$. It is observed that if $S(T) = \emptyset$, then the distance between any two leaf nodes is at least three. When $T \in \mathscr{E}$, the bounds or exact values for $\gamma(T)$ are obtained. It is shown that trees with $\gamma(T) = \frac{n}{2}$ are efficiently dominatable and has $S(T) = \emptyset$. Section 3.2.2 identifies some special classes of efficiently dominatable trees. Efficiently dominatable trees of bounded diameter at most five are classified, based on the number of paths and strong supports adjacent to the central vertex (vertices) and spiders containing efficient dominating sets are characterized.

Definition 3.2.1. *A pendant vertex in any tree T is referred to as a leaf node. The unique neighbor of a leaf node is referred to as its support vertex. A support vertex with exactly one adjacent leaf node is called a weak support (WS) and a support with at least two adjacent leaf nodes is called a strong support (SS). A vertex which is neither a leaf node nor a support (WS/SS) is called an internal vertex of T.*

Notation 3.2.1. *In the discussions to follow, T represents a tree; n denotes the* *order of T; $S(T)$ and $W(T)$ represent the set of all strong supports and weak supports in T, respectively and $L(T)$ denotes the set of all leaf nodes in T, unless specified otherwise.*

Observation 3.2.1. *The path $P_n \in \mathscr{E}$, for all n and $\gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Let $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. If $n \equiv 0 \pmod 3$, then $S = \{u_2, u_5, \ldots, u_{n-1}\}$ forms an EDS of P_n and it can be observed that it is the only EDS of P_n. If $n \equiv 1 \pmod 3$, $S = \{u_1, u_4, \ldots, u_n\}$ is the unique EDS of P_n. When $n \equiv 2 \pmod 3$, $S = \{u_1, u_4, \ldots, u_{n-1}\}$ and $S = \{u_2, u_5, \ldots, u_n\}$ are the only two efficient dominating sets of P_n. Also, it can be observed that an EDS of P_n contains a leaf node if and only if either $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$.*

3.2.1 Results on arbitrary Trees

The following theorem proves the existence of a tree T on n vertices, where $T \in \mathscr{E}$ and $\gamma(T) = k$ and also defines a procedure to generate such a tree T.

Theorem 3.2.1. *Given a pair of integers n and k, where $n \geq 2$ and $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$, there exists an efficiently dominatable tree T on n vertices with $\gamma(T) = k$.*

Proof. Let T' be an arbitrary tree on k vertices and $V(T') = \{v_1, v_2, \ldots, v_k\}$. Construct a tree T from T' as follows. Add k new vertices $u_1, u_2, \ldots, u_k$ such that for each i, for $i = \{1, \ldots, k\}$, $u_i v_i \in E(T)$. Clearly, T is a tree with $|V(T)| = 2k$ and $|E(T)| = 2k - 1$. If n is even and $k = \frac{n}{2}$, then $T \in \mathscr{E}$ and $\{u_1, u_2, \ldots, u_k\}$ forms an EDS of T. Otherwise, add $n - 2k$ new vertices $w_1, w_2, \ldots, w_{n-2k}$ to T. For each $j = \{1, \ldots, n - 2k\}$, w_j is made adjacent to u_i, for some i, $1 \leq i \leq k$, with the condition that $\deg(w_j) = 1$ and each u_i may be adjacent to more than one w_j. Then, the set $\{u_1, u_2, \ldots, u_k\}$ will be an EDS of the resultant tree T and $\gamma(T) = k$. □

The corollaries given below follow immediately from Theorem 3.2.1.

Corollary 3.2.1.1. *If $T \in \mathscr{E}$, then $\gamma(T) = 1$ if and only if $T \cong K_{1,n}$.*

Using Theorem 3.1.3, a characterization is given for efficiently dominatable trees whose domination number is half their order and is stated below.

Theorem 3.2.2. *Let T be a tree of even order and $\gamma(T) = \frac{n}{2}$. Then, $T \in \mathscr{E}$ if and only if $T \cong T' \circ K_1$, for some tree T' of order $\frac{n}{2}$.*

Corollary 3.2.2.1. *If $T \in \mathscr{E}$ is of even order, then $\gamma(T) = \frac{n}{2}$ if and only if $L(T)$ is the unique EDS of T.*

Corollary 3.2.2.2. *If $T \in \mathscr{E}$, then $\gamma(T) = \left\lfloor \frac{n}{2} \right\rfloor$ if and only if n is odd; $|S \cap L(T)| = \left(\frac{n}{2} - 1\right)$ and $|S \cap \overline{L(T)}| = 1$, for any EDS S of T.*

Based on the definition of an EDS and support vertices, the following trivial conditions necessary for a tree $T \in \mathscr{E}$ are observed.

Proposition 3.2.3. *If $T \in \mathscr{E}$ and S is an EDS of T, then the following conditions hold.*

(i) $S(T) \subseteq S$.

(ii) *For each vertex $w \in W(T)$, either $w \in S$ or the leaf node adjacent to w is in S.*

(iii) *No internal vertex adjacent to a weak support is in S.*

(iv) *If $\{w_1, w_2\} \subseteq W(T)$ and $w_1w_2 \in E(T)$, then neither $w_1 \in S$ nor $w_2 \in S$.*

Proof. Let $T \in \mathscr{E}$ and S be any EDS of T.

(i) Let s be any strong support of T. Then for each vertex pairs $x, y \in L(T) \cap N(s)$, $d(x, y) = 2$. Therefore, $|L(T) \cap N(s) \cap S| = \emptyset$ and hence, $s \in S$.

Conditions (ii) and (iii) follow immediately from the fact that S must be a 2-packing.

(iv) Let w_1 and w_2 be two adjacent weak supports of T and v_1, v_2 be their adjacent leaf nodes respectively. As $T \in \mathscr{E}$, for each i, where $1 \leq i \leq 2$, either $v_i \in S$ or $w_i \in S$. Therefore, the following cases arise: (a) $v_1, v_2 \in S$, (b) $w_1, v_2 \in S$, (c) $v_1, w_2 \in S$ and (d) $w_1, w_2 \in S$. But, both w_1 and w_2 cannot simultaneously be in S. Now suppose $w_1 \in S$, then as $d(w_1, v_2) = 2$, v_2 will be left undominated. Similarly, if $w_2 \in S$, then v_1 will be left undominated. Thus, the only possibility is that both v_1 and v_2 must be in S. In other words, S does not contain any two adjacent weak supports of T. □

Remark 3.2.1. *If $T \in \mathscr{E}$, then it follows from Proposition 3.2.3-(i) that for every vertex pairs $u, v \in S(T)$, $d(u, v) \geq 3$.*

Proposition 3.2.4. *For any tree T, if there exists a strong support adjacent to a weak support, then $T \notin \mathscr{E}$.*

Proof. Suppose that $u \in S(T)$ and $w \in W(T)$ such that $uw \in E(T)$. Let $v \in N(w) \cap L(T)$. Then, as $u \in S(T)$, by Proposition 3.2.3-(i), $u \in S$ and also u dominates w. But then, as $d(u, v) = 2$, neither $v \in S$ nor $v \in N(x)$, for any $x \in S$, contradicting that $T \in \mathscr{E}$. □

Theorem 3.2.5. *(Thilak, 2013) If $G \notin \mathscr{E}$, S is an $F(G)$-set and $S' = N[S]$, then for each $x \in V - S'$, there exists $u \in S$ such that $d(x, u) = 2$.*

The following theorem states the necessary conditions for a vertex to be left undominated by an F(T)-set, whenever $T \notin \mathscr{E}$.

Theorem 3.2.6. *Let $T \notin \mathscr{E}$ and S' be an $F(T)$-set. If $u \in V(T) - N[S']$, then the following conditions hold.*

(i) u is neither a weak support nor a strong support.

(ii) If u is an internal vertex adjacent to a weak support, say w, then $\deg(w) \geq 3$.

(iii) u cannot be adjacent to a strong support.

Proof. Since $T \notin \mathscr{E}$, $1 \leq F(T) \leq n - 1$. Suppose that $u \notin N[S']$.
(i) Let u be a weak support in T and v be the leaf node adjacent to u. Then as $u \notin N[S']$, v is left undominated efficiently. Since $u \notin S'$, Theorem 3.2.5 follows that there exists a vertex $w \in S'$ where $d(u, w) = 2$. Then, $d(v, w) = 3$ and so v can be included in S', contradicting that v is left undominated by S'. On the other hand, if u is a SS and if u is left undominated, then by a similar argument as above, all the leaf nodes adjacent to u are also left undominated. Thus, u is neither a weak support nor a strong support.
(ii) Let w be a weak support and u be an internal vertex adjacent to w. Suppose $\deg(w) = 2$ and v is the leaf node adjacent to w, then the vertex $w \in S'$ will efficiently dominate u, v and w, contradicting that $u \notin N[S']$. Hence, $\deg(w) \geq 3$.

(iii) Suppose that u is adjacent to a strong support, say s. Then, as $u \notin N[S']$ it follows that $s \notin S'$. But, as S' is an $F(T)$-set, s must be in $N(S')$. Let $s \in N(v)$, where $v \in S'$. If v is a leaf node, then the other leaf nodes adjacent to s will be at distance two from v and hence are left undominated efficiently. On the other hand, if v is an internal vertex, then all the leaf nodes adjacent to s must be left undominated efficiently. In either case, the set $S'' = (S' - v) \cup \{s\}$ will be such that $I(S'') > I(S')$, contradicting that S' is an $F(T)$-set. Therefore, u is not adjacent to any SS of T. □

3.2.2 Trees with no strong support

Every tree $T \ncong K_{1,n}$ has at least two support vertices. However, there exist trees in which all the support vertices are weak supports. For example, all the trees listed in Table 3.1 do not have any strong support. Here, the trees T are considered, for which $S(T) = \emptyset$.

Proposition 3.2.7. *Let $T \in \mathscr{E}$ and $S(T) = \emptyset$. Then, the following conditions holds.*

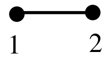
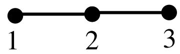
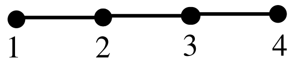
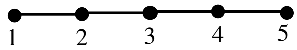
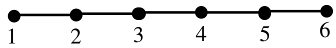
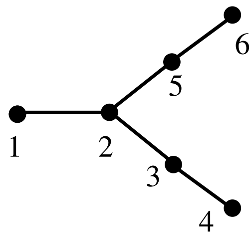
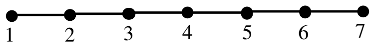
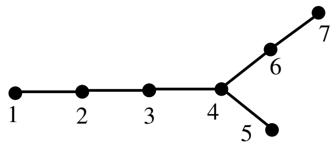
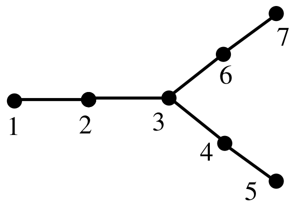
(i) For every vertex pairs $x, y \in L(T)$, $d(x, y) \geq 3$.

(ii) The set of all leaf nodes which are mutually at a distance three is a subset of every EDS of T.

Proof. (i) For any two vertices $x, y \in L(T)$, since $S(T) = \emptyset$, x and y do not have common neighbors and hence $d(x, y) \geq 3$.
(ii) Let S be an EDS of T. Since $T \in \mathscr{E}$, either the leaf node or the vertex adjacent to it must be in S. Let u and v be any two leaf nodes such that $d(u, v) = 3$. If $u' \in N(u)$ and $v' \in N(v)$, then $d(u', v) = 2$, $d(u, v') = 2$ and u' is adjacent to v'. Hence, neither u' nor v' will belong to S. Thus, both u and v must be in S. □

Theorem 3.2.8. *(Lemańska, 2004) If $n \geq 3$ and $|L(T)| = l$, then $\gamma(T) \geq \dfrac{n - l + 2}{3}$.*

Table 3.1: Efficiently dominatable trees of order n ($n \leq 7$) with no strong support

$n = \|V(T)\|$	**Trees T with no SS**	**γ-set of T**	$\gamma(T)$**,** $F(T)$
$n = 1$		{1}	1, 1
$n = 2$		{1}, {2}	1, 2
$n = 3$		{2}	1, 3
$n = 4$		{1, 4}	2, 4
$n = 5$		{1, 4}, {2, 5}	2, 5
$n = 6$		{2, 5}	2, 6
		{1, 4, 6}	3, 6
$n = 7$		{1, 4, 7}	3, 7
		{2, 7, 5}	3, 7
		{1, 6, 5}, {2, 7, 5} {1, 4, 7}	3, 7

Theorem 3.2.9. *For any tree T with $S(T) = \emptyset$ and $|W(T)| = p$, the following holds.*

(i) If $p = 2$, then $T \in \mathscr{E}$.

(ii) If $T \in \mathscr{E}$, then $\gamma(T) \geq p$.

(iii) $\left\lceil \dfrac{n+2}{4} \right\rceil \leq \gamma(T) \leq \left\lfloor \dfrac{n}{2} \right\rfloor$.

Proof. (i) Let $w_1, w_2 \in W(T)$ and v_1 and v_2 be the leaf nodes adjacent to w_1 and w_2 respectively. Since T has exactly two weak supports, it has exactly two pendant vertices and $\deg(u) = 2$, for all $u \in V(T)$ and $u \neq v_1$, $u \neq v_2$. Therefore, T is a path on n vertices and hence $T \in \mathscr{E}$.
(ii) Let $w_1, w_2, \ldots, w_m \in W(T)$. For each $i = \{1, 2, \ldots, m\}$, let v_i be the leaf node adjacent to w_i. If $T \in \mathscr{E}$, then either $v_i \in S$ or $w_i \in S$. Hence, at least m vertices must be in any EDS of T and hence $\gamma(T) \geq m$.
(iii) Since $S(T) = \emptyset$, $|L(T)| = l = p = |W(T)|$. Using Theorem 3.2.8 and the result (ii) above, $\gamma(T) \geq p \geq n + 2 - 3\gamma(T)$ and hence the lower bound follows. The upper bound follows. □

Among all trees of order n, for $n \leq 7$, those trees without strong support are depicted in Table 3.1 and it can also be observed that for each of these trees T, $F(T) = |V(T)|$ and hence all are efficiently dominatable. But for $n > 7$, it is observed that the trees of order n having no strong support may or may not be efficiently dominatable. Particularly, in Proposition 3.2.10, the value of $F(T)$ is determined for all such trees of order n, $1 \leq n \leq 10$.

Proposition 3.2.10. *For any tree T with $S(T) = \emptyset$, the following is true.*

(i) For all n, $n \leq 7$, $T \in \mathscr{E}$.

(ii) For $8 \leq n \leq 10$, if $T \notin \mathscr{E}$, then $F(T) = n - 1$.

3.2.3 Some Classes of Efficiently Dominatable Trees

Every tree has at least two leaf nodes. If there are more than two leaf nodes, then for any pair of distinct leaf nodes x and y, $d(x, y) \equiv c \pmod 3$, where $c \in \{0, 1, 2\}$. For example, consider a star $K_{1,n}$. Then, $d(u, v) \equiv 2 \pmod 3$, for every two distinct leaf nodes u and v.

For any tree T, let $|V(T)| = n$ and $|L(T)| = l$. Let $\mathscr{L}$ denote the family of trees in which for every pair of distinct leaf nodes x and y, $d(x, y) \equiv c \pmod 3$, where c is constant. For every pair of distinct leaf nodes x and y, let $\mathscr{L}_0 = \{$Trees $T : d(x, y) \equiv 0 \pmod 3\}$, $\mathscr{L}_1 = \{$Trees $T : d(x, y) \equiv 1 \pmod 3\}$, $\mathscr{L}_2 = \{$Trees

$T : d(x, y) \equiv 2 \pmod 3\}$. Then, $\mathscr{L} = \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2$.
Figures 3.7, 3.8 and 3.9 illustrates trees $T \in \mathscr{L}_0$, $T \in \mathscr{L}_1$ and $T \in \mathscr{L}_2$. The encircled vertices form an EDS of T.

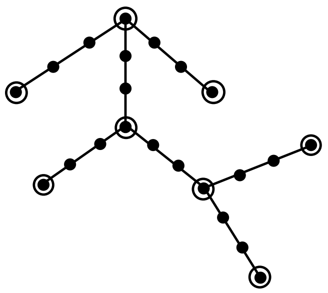


Figure 3.7: Efficiently dominatable tree in $\mathscr{L}_0$

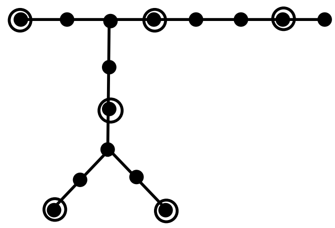


Figure 3.8: Efficiently dominatable tree in $\mathscr{L}_1$

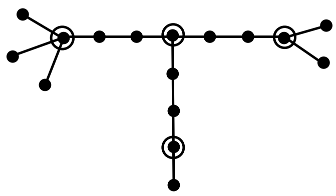


Figure 3.9: Efficiently dominatable tree in $\mathscr{L}_2$

Lemańska (2004) gives a characterization for trees T, $T \in \mathscr{L}_2$, which is stated in the result below.

Lemma 3.2.11. *(Lemańska, 2004) Let $T \in \mathscr{L}_2$ and D be its minimum dominating set having no leaf nodes. Then, $d(u, v) \equiv 0$ (mod 3), for every vertex pairs u, $v \in D$. In addition, $\gamma(T) = \dfrac{n - l + 2}{3}$.*

Theorem 3.2.12. *Let T be a tree and $T \in \mathscr{L}$. Then, $T \in \mathscr{E}$.*

Proof. Let $T \in \mathscr{L} = \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2$.
Case(i): $T \in \mathscr{L}_2$
It follows from Lemma 3.2.11 that, if $T \in \mathscr{L}_2$, then any dominating set D containing all the weak supports forms an efficient dominating set. Thus, $T \in \mathscr{E}$ and $l = n + 2 - 3\gamma(T)$.
Case(ii): $T \in \mathscr{L}_0$
Let $P = \{v_0, v_1, \ldots, v_l\}$ be an arbitrary diametral path. Then, $l \equiv 0 \pmod 3$.

Claim: $\deg(v_1) = 2 = \deg(v_2) = \deg(v_{l-2}) = \deg(v_{l-1})$

Suppose that $\deg(v_1) \geq 3$. Let $P' = \{v_0, v_1, v_1^1, v_1^2, \ldots, v_1^k\}$ be a diametral path through v_1, different from P. Then, $k \equiv 2 \pmod 3$. Hence, $d(v_1^k, v_l) = d(v_1^k, v_1) + d(v_1, v_l) = k + l - 1 \equiv 2 + 0 - 1 = 1 \pmod 3$, a contradiction. Thus, $\deg(v_1) = 2$. By a similar argument it can be shown that, $\deg(v_2) = 2 = \deg(v_{l-2}) = \deg(v_{l-1})$.

Consider the tree $T^* \cong T - N[L(T)]$. Then, $T^* \in \mathscr{L}_2$. If D^* is the dominating set of T^*, then $S = D^* \cup L(T)$ is an EDS of T and hence $T \in \mathscr{E}$. Let n^* and l^* respectively denote the order and number of leaf nodes of tree T^*. Then by Lemma 3.2.11, $n^* = l^* - 2 + 3|D^*|$ and $n = n^* + 2l$, where $l = l^*$. Since $\gamma(T) = |D^*| + l$, it follows that $n = 3\gamma(T) - 2$.

Case(iii): $T \in \mathscr{L}_1$

Let $P = \{v_0, v_1, \ldots, v_l\}$ be a diametral path. Then, $l \equiv 1 \pmod 3$.

Claim: $\deg(v_1) = 2 = \deg(v_{l-1})$

Suppose that $\deg(v_1) \geq 3$. Let $P' = \{v_0, v_1, v_1^1, v_1^2, \ldots, v_1^k\}$ be a diametral path other than P. Then, $k \equiv 0 \pmod 3$. Hence, $d(v_1^k, v_l) = d(v_1^k, v_1) + d(v_1, v_l) = k + l - 1 \equiv 0 + 1 - 1 = 0 \pmod 3$, a contradiction. By a similar approach it can be shown that $\deg(v_{l-1}) = 2$.

Consider the tree $T' \cong T - L(T)$. Then, $T' \in \mathscr{L}_2$. Hence $T \in \mathscr{E}$ and $\gamma(T) = l + |D'| - 1$, where D' is the dominating set of T'. Let S be an EDS of T. Since, either $v_0 \in S$ or $v_1 \in S$, it is easy to observe that $D' \not\subseteq S$. Let n' and l' respectively denote the order and the number of leaf nodes of T'. Then, $n = n' + l'$ and $l' = l$. Since, $l' = n' - 3|D'| + 2$, it follows that $n = 3\gamma(T) - l + 1$.

Thus combining all the cases, if $T \in \mathscr{L} = \mathscr{L}_0 \cup \mathscr{L}_1 \cup \mathscr{L}_2$, then $T \in \mathscr{E}$. □

Efficiently Dominatable Spiders

Definition 3.2.2. *(Haynes et al., 1998) A wounded spider is the graph obtained by subdividing at most $(n-1)$ edges of the star $K_{1,n}$, for $n \geq 1$. A healthy spider is the graph obtained by subdividing all the n edges of the star $K_{1,n}$, for $n \geq 1$.*

Theorem 3.2.13. *Let k represent the number of subdivided edges in a star $K_{1,n}$ to obtain a spider T, where $1 \leq k \leq n$. Then, $T \in \mathscr{E}$ if and only if either $k = n$*

or $k = n-1$ *and* $\gamma(T) = n$. *When* $T \notin \mathscr{E}$,

$$F(T) = \begin{cases} n+1; & \text{if } k < \dfrac{n}{2} \\ 2(k+1); & \text{if } \dfrac{n}{2} \leq k < n-1 \end{cases}$$

Proof. Here $|V(T)| = n + 1 + k$. Let v_0 be the central vertex (vertex of degree $n-1$) of $K_{1,n}$.

Case(i): $k = n$

Then, T is a healthy spider and $|V(T)| = 2n + 1$. Let w be a weak support in T. Then, w efficiently dominates v_0 and the leaf node adjacent to it. The set consisting of w together with the remaining $(n-1)$ leaf nodes efficiently dominates $V(T)$ and hence $T \in \mathscr{E}$ and $\gamma(T) = n$.

Case(ii): $k = n - 1$

Here $|V(T)| = 2n$. All the n leaf nodes will form an EDS of T. Thus, $T \in \mathscr{E}$ and $\gamma(T) = n$.

Case(iii): $k < n - 1$

Then, v_0 is a strong support. Suppose that S is an EDS of T. Then, $v_0 \in S$. Since $ecc(v_0) = 2$, this is not possible. Thus, $T \notin \mathscr{E}$. If $0 \leq k < \frac{n}{2}$, then $\{v_0\}$ will be an $F(T)$-set and $F(T) = n+1$. When $\frac{n}{2} \leq k < n-1$, the k leaf nodes will efficiently dominate $2k$ vertices. To dominate the vertex v_0, choose one of the leaf nodes adjacent to v_0. It is observed that these $k+1$ leaf nodes efficiently dominate the maximum number of vertices and thus, $F(T) = 2k + 2 = 2(k+1)$. □

The Figure 3.10 illustrates efficiently dominatable spiders. The encircled vertices forms an EDS.

Figure 3.10: Efficiently dominatable Spiders

Efficiently Dominatable Trees of bounded diameter

Consider a tree T on n vertices and diameter d, where $d \geq 1$. Then, T has either one centre or a pair of adjacent centres, according as d is even or odd, respectively. Let c_i $(1 \leq i \leq 2)$ denote the central vertices of T (with the understanding that $c_1 = c_2$, if d is even). The notations are as described below.

Notation 3.2.2.

- $s_i \rightarrow$ *The number of strong supports adjacent to* c_i*.*
- $l_i \rightarrow$ *The number of leaf nodes adjacent to* c_i*.*
- $k_i \rightarrow$ *The number of paths of length two appended to* c_i*.*

a) *Trees of Diameter three*

If T is a tree of diameter three, then $l_i \geq 1$, $s_i = 0$ and $k_i = 0$. (Refer to Figure 3.11)

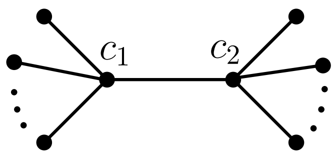


Figure 3.11: Structure of a tree of diameter three

Lemma 3.2.14. *Any tree T of order n, for $n \leq 4$ and $diam(T) \leq 3$ is efficiently dominatable.*

Proof. Let T be a tree of order n and $n \leq 4$. If $diam(T) \leq 2$, then $T \cong K_1$ or $T \cong P_2$ or $T \cong P_3$. When $diam(T) = 3$, $T \cong P_4$. Hence in all these cases $T \in \mathscr{E}$. □

Theorem 3.2.15. *Any tree T whose diameter is three is efficiently dominatable if and only if $T \cong P_4$. Otherwise, $F(T) = \Delta(T) + 1$, where $\Delta(T)$ is the maximum degree of T.*

Proof. As $diam(T) = 3$, all the leaf nodes will be of eccentricity three. Also, there are exactly two vertices of eccentricity two, namely, the central vertices c_1 and c_2, which are adjacent to each other. Since $n > 4$, these two vertices must be support vertices and hence, must be included in any EDS of T. But this is not possible since c_1 and c_2 are adjacent and hence $T \notin \mathscr{E}$. Also, either c_1 or c_2 or both have the maximum degree (Refer to Figure 3.11). Thus, $F(T) = \Delta(T) + 1$. □

Figure 3.12 illustrates the only tree $T \in \mathscr{E}$ whose $diam(T) = 3$. The encircled vertices form an EDS.

Figure 3.12: Efficiently dominatable tree of diameter three

b) *Trees of Diameter four*

Let $diam(T) = 4$. Then, $n \geq 5$ and $\gamma(T) \geq 2$ and the structure of T will be as shown in Figure 3.13. Thus, $l_1 \geq 0$, $s_1 \geq 0$ and $k_1 \geq 0$.

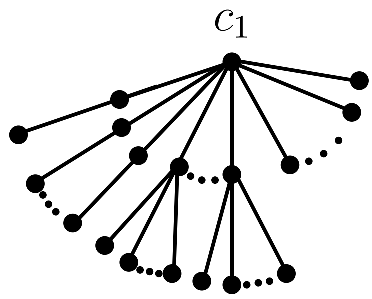


Figure 3.13: Structure of a tree of diameter four

Theorem 3.2.16. *Consider a tree T of diameter four. For $k_1 \geq 0$, $T \in \mathscr{E}$ if and only if T satisfies one of the conditions given below:*

(i) $l_1 = 1$ *and* $s_1 = 0$.

(ii) $l_1 = 0$ *and* $s_1 = 1$.

(iii) $l_1 = 0$ *and* $s_1 = 0$.

Proof. Let $k_1 \geq 1$. If $l_1 = 1$ and $s_1 = 0$, then any EDS of T includes all the leaf nodes and $T \in \mathscr{E}$. In this case, $\gamma(T) = k_1 + 1$. Let $l_1 = 0$ and $s_1 = 1$. Let u be the strong support adjacent to c_1. Then, any EDS of T includes the strong support u and all the leaf nodes except $N(u)$. In this case, $\gamma(T) = k_1 + 1$. Suppose $l_1 = 0$ and $s_1 = 0$, then any EDS of T includes any of the $k_1 - 1$ leaf nodes and one support vertex adjacent to the leaf node. In this case, $\gamma(T) = k_1$.
Conversely, let $T \in \mathscr{E}$. If $s_1 > 1$, then there will be at least two strong supports at a distance two from each other and hence $T \notin \mathscr{E}$, a contradiction. Thus, $s_1 \leq 1$. If $l_1 > 1$, then the central vertex c_1 becomes a strong support. Since $T \in \mathscr{E}$, $s_1 = 0$ and $k_1 = 0$. Let $l \leq 1$, $s_1 \leq 1$ and $k_1 \geq 1$. Since $T \in \mathscr{E}$, it follows from Theorem 3.2.3 that either $l_1 = 1$ and $s_1 = 0$ or $l_1 = 0$ and $s_1 = 1$ or $l_1 = 0$ and $s_1 = 0$. □

Note: It can be observed that if T satisfies either condition (i) or condition (ii) in Theorem 3.2.16, then $\gamma(T) = k_1 + 1$ and if it satisfies condition (iii), then $\gamma(T) = k_1$.
Any tree T whose $diam(T) = 4$ is efficiently dominatable if and only if it is isomorphic to one of the trees shown in Figure 3.14. The encircled vertices form an EDS.

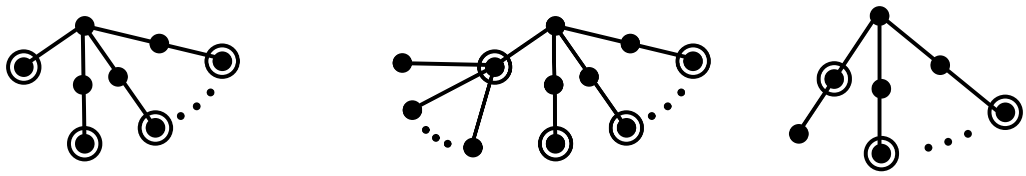


Figure 3.14: Efficiently dominatable trees of diameter four

c) *Trees of Diameter five*

Let $diam(T) = 5$. Then, $n \geq 6$ and $\gamma(T) \geq 2$. Also, for $i = \{1, 2\}$, $k_i \geq 1$, $s_i \geq 0$ and $l_i \geq 0$. (Refer to Figure 3.15)
By a similar reasoning as in Theorem 3.2.16, the following theorem can be discussed.

Theorem 3.2.17. *Consider a tree T of diameter five. For $i \in \{1, 2\}$, if $k_i \geq 0$ for each i, then $T \in \mathscr{E}$ if and only if T satisfies one of conditions given below:*

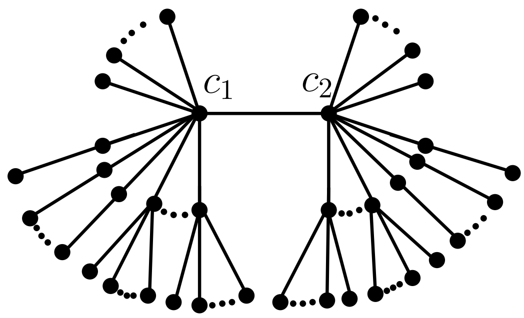


Figure 3.15: Structure of a tree of diameter five

(i) $l_i = 1$ *and* $s_i = 0$.

(ii) $l_i = 0$ *and* $s_i = 1$.

(iii) $l_i = 0$ *and* $s_i = 0$.

Note: It can be observed in Theorem 3.2.17 that, if T satisfies condition (i), then $\gamma(T) = l_1 + l_2 + k_1 + k_2$, if it satisfies condition (ii), then $\gamma(T) = s_1 + s_2 + k_1 + k_2$ and if it satisfies condition (iii), then $\gamma(T) = k_1 + k_2$.

Any tree T whose $diam(T) = 5$ is efficiently dominatable if and only if it belongs to one of the trees in Figure 3.16. The encircled vertices forms an EDS.

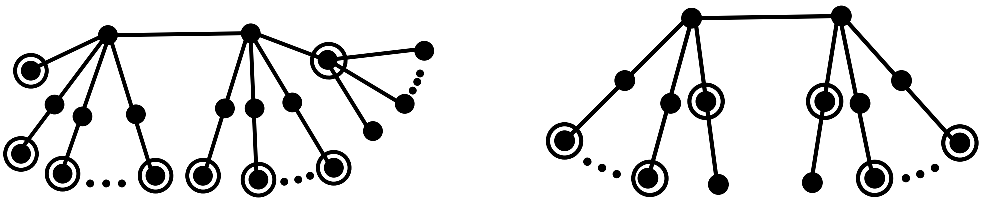


Figure 3.16: Efficiently dominatable trees of diameter five

3.3 Efficient Domination in some special graphs

3.3.1 Efficient Domination in Ciliates

This section deals with a special class of graphs, namely, the Ciliates, which was introduced by Fajtlowicz (1988).

Definition 3.3.1. *(Dankelmann et al., 1998; Fajtlowicz, 1988) For* p, $q \in \mathbb{N}$, *the Ciliate* $C_{p,q}$ $(p \geq 3)$ *is the graph obtained from* p *disjoint copies of the path of length* *q by linking together one end-vertex of each path in a cycle C_p. Equivalently, the Ciliate $C_{p,q}$ $(p \geq 3)$ is the graph obtained by appending a path of length q to each vertex on the cycle C_p. The Ciliate $C_{4,2}$ is shown in Figure 3.17.*

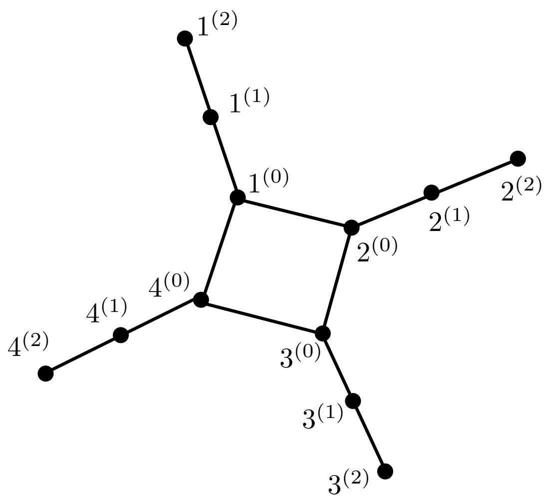


Figure 3.17: Ciliate $C_{4,2}$

Let the vertices of $C_{p,q}$ be labelled as follows. The vertices lying on the cycle C_p are labelled as $1^{(0)}$, $2^{(0)}$, $\ldots$, $p^{(0)}$ in the clockwise direction. Let P_q^i denote the path of length q, appended to the vertex $i^{(0)}$ on the cycle C_p and its vertex be labelled as $i^{(0)}$, $i^{(1)}$, $\ldots$, $i^{(q)}$, as shown in Figure 3.17. Clearly, $|V(C_{p,q})| = p + pq = p(1+q) = |E(C_{p,q})|$. An EDS of $C_{p,q}$ or an F$(C_{p,q})$-set can be generated by extending an EDS of C_p (or P_q^i) or an F(C_p)-set (or F(P_q^i)-set) respectively. With these notations, the following theorems are proved.

Theorem 3.3.1. *For $q \equiv 1, 2 \ (mod\ 3)$, $C_{p,q} \in \mathscr{E}$ and $\gamma(C_{p,q}) = p\left\lceil \frac{q}{3} \right\rceil$.*

Proof. Let $p \geq 3$. For $q \equiv 1$ (mod 3), the set $S = \bigcup_{i=1}^{p} S_i$, where $S_i = \{i^{(1)}, i^{(4)}, i^{(7)}, \ldots, i^{(q)}\}$ is an EDS of $C_{p,q}$. Thus, $\gamma(C_{p,q}) = p\left\lceil \frac{q}{3} \right\rceil$. When $q \equiv 2$ (mod 3), the set $S = \bigcup_{i=1}^{p} S_i$, where $S_i = \{i^{(1)}, i^{(4)}, i^{(7)}, \ldots, i^{(q-1)}\}$ is an EDS of $C_{p,q}$. In this case, $\gamma(C_{p,q}) = p\left(\frac{q}{3}\right)$. □

Theorem 3.3.2. *For $q \equiv 0 \ (mod\ 3)$, $C_{p,q} \in \mathscr{E}$ if and only if $p \equiv 0 \ (mod\ 3)$. Otherwise,* $F(C_{p,q}) = \begin{cases} pq + p - 1 & \text{if } p \equiv 1 \ (mod\ 3) \\ pq + p - 2 & \text{if } p \equiv 2 \ (mod\ 3) \end{cases}$

Proof. Let $q \equiv 0 \pmod 3$. An EDS of $C_{p,q}$ cannot be generated by using EDSs of P_q^i $(1 \leq i \leq q)$. Therefore, choose an EDS for C_p or an F(C_p)-set to generate an EDS for $C_{p,q}$ or an F($C_{p,q}$)-set. For any $p(\geq 3)$, the following three cases arise.

Case(i): $p \equiv 0 \pmod 3$

Since $C_p \in \mathscr{E}$ if and only if $p \equiv 0 \pmod 3$, it follows that $C_{p,q} \in \mathscr{E}$ if and only if $p \equiv 0 \pmod 3$.

Let $S' = \{1^0, 4^0, \ldots, (p-2)^0\}$ be an EDS of C_p. For $1 \leq i \leq p$, the set $S = \{i^{(0)}, i^{(3)}, i^{(6)}, \ldots, i^{(q)} : i \in S'\} \cup \{i^{(2)}, i^{(5)}, i^{(8)}, \ \ldots \ , i^{(q-1)} : i \notin S'\}$ forms an EDS of $C_{p,q}$. Also, as C_p has three pairwise disjoint efficient dominating sets, $C_{p,q}$ also has three pairwise disjoint efficient dominating sets. In particular, $\gamma(C_{p,q}) = \dfrac{p}{3} + p\left(\dfrac{q}{3}\right) = \dfrac{pq+p}{3}$.

Case(ii): $p \equiv 1 \pmod 3$

In this case, $C_p \notin \mathscr{E}$ and $F(C_p) = p - 1$. There is one vertex left undominated efficiently on the cycle C_p. Let $S' = \{1^0, 4^0, \ldots, (p-3)^0\}$ be an $F(C_p)$-set. Then, the set $S = \{i^{(0)}, i^{(3)}, i^{(6)}, \ldots, i^{(q)} : i \in S'\} \cup \{i^{(2)}, i^{(5)}, i^{(8)}, \ldots, i^{(q-1)} : i \notin S'\}$ forms an $F(C_{p,q})$-set which efficiently dominates all, except one vertex on the cycle C_p. Thus, $F(C_{p,q}) = pq + p - 1$.

Case(iii): $p \equiv 2 \pmod 3$

In this case, $C_p \in \mathscr{E}$ and $F(C_p) = p - 2$. There are two vertices left undominated efficiently on the cycle C_p. With the same $F(C_{p,q})$-set as discussed in Case(ii) above, it is observed that S efficiently dominates all, except the two vertices on the cycle C_p. Thus, $F(C_{p,q}) = pq + p - 2$. □

3.3.2 Efficient Domination in Join, One-point union and Corona of graphs

In this section, the concept of efficient domination is discussed for composite graphs/graph operations such as join, one-point union and corona of graphs.

Definition 3.3.2. *The join of simple graphs G and H, denoted by $G \vee H$, is the graph obtained from the disjoint union $G + H$ and adding the edges $\{xy : x \in V(G), y \in V(H)\}$.*

If graphs G and H are of order p and q respectively, then $V(G \vee H) = p + q$. Figure 3.18a represents the join of G and H. The encircled vertices form an EDS of $G \vee H$.

Theorem 3.3.3. *Let G and H be graphs of order p and q respectively. $G \vee H \in \mathscr{E}$ if and only if $\gamma(G) = 1$ or/and $\gamma(H) = 1$. In particular, $\gamma(G \vee H) = 1$.*

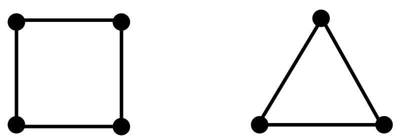
Proof. By definition, $diam(G \vee H) = 2$. Thus, $G \vee H \in \mathscr{E}$ if and only if $rad(G \vee H) = 1$. But, $rad(G \vee H) = 1$ if and only if either $rad(G) = 1$ or $rad(H) = 1$ or both holds. Thus, it follows that $G \vee H \in \mathscr{E}$ if and only if $\gamma(G) = 1$ or/and $\gamma(H) = 1$. □

Definition 3.3.3. *One-point union, $G^{(p)}$ of p copies of G is obtained by identifying the roots of p copies of G.*

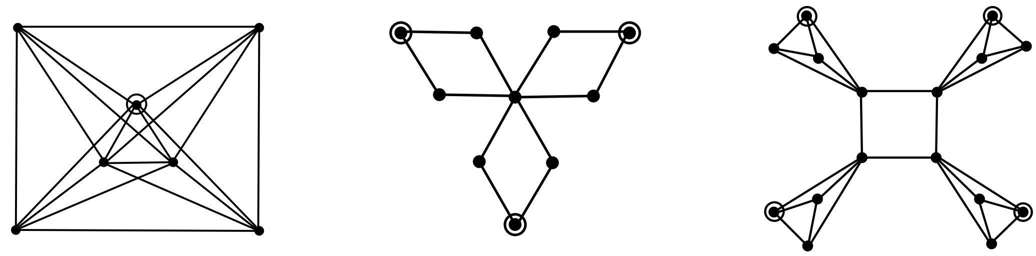
Since $|V(G)| = n$, $|V(G^{(p)})| = p(n-1) + 1$. Figure 3.18b represents one-point union $H^{(3)}$. The encircled vertices form an EDS of $H^{(3)}$.

Theorem 3.3.4. *$G^{(p)} \in \mathscr{E}$ if and only if $G \in \mathscr{E}$. In particular, $\gamma(G^{(p)}) = p(\gamma(G) - 1) + 1$.*

Proof. Let $G \in \mathscr{E}$ and $S = \{u_1, u_2, \ldots, u_k\}$ be its EDS. Let $G^{(p)}$ be obtained by identifying p copies of G at the vertex v, where $v \in S$. Without loss of generality, let $v = u_1$. Let $S' \subseteq V(G^{(p)})$ contain u_1 and the remaining $(k-1)$ vertices of S from each of the p copies of G. Then, $S' = \{u_1, (u_2, \ldots, u_k), (u_2, \ldots, u_k), \ldots, (u_2, \ldots, u_k)$ (p times)$\}$ is an EDS of $G^{(p)}$. Thus, $G^{(p)} \in \mathscr{E}$ and $\gamma(G^{(p)}) = p(k-1) + 1$.
Conversely, suppose that $G \notin \mathscr{E}$ and S' be an $F(G)$-set. Let $F(G) = l$, $l < n$. Then, $G^{(p)}$ can be obtained by identifying p copies of G at the vertex v, where $v \in V - S'$. Then, $F(G^{(p)}) = pl < (n-1)p + 1 = V(G^{(p)})$. Thus, $G^{(p)} \notin \mathscr{E}$. □



G H



(a) $G \vee H$ (b) $H^{(3)}$ (c) $G \circ H$

Figure 3.18: Illustration for the operations join, one-point union and corona

Definition 3.3.4. *Let G be a graph of order n. The corona of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G and n copies of H, and then joining the i^{th} vertex of G to every vertex in the i^{th} copy of H.*

If $|V(H)| = p$, then $|V(G \circ H)| = n(p+1)$. For every $v \in V(G)$, the subgraph H^v of the corona $G \circ H$ is the copy of H whose vertices are attached one by one to the vertex v.

Figure 3.18c represents the corona graph of G and H. The encircled vertices form an EDS of $G \circ H$.

Theorem 3.3.5. *Let G and H be connected graphs of order p and q respectively. $G \circ H \in \mathscr{E}$ if and only if $\gamma(H) = 1$. In particular, $\gamma(G \circ H) = p$.*

Proof. Let $G \circ H \in \mathscr{E}$ and S be its EDS. Let $v \in S$.

Claim: $v \in V(H)$

Suppose that $v \in V(G)$. Then v will efficiently dominate $N_G[v]$ in G and all the vertices in the corresponding subgraph H^v. S will be an EDS of $G \circ H$ if and only if G contains only isolated vertices. This is not possible since G is connected. Thus, $v \notin V(G)$.

Thus, $v \in V(H)$. Then, v dominates $N_H[v]$ in H and the corresponding vertex in G. In this case, v will dominate all the vertices of H if and only if $rad(H) = 1$, that is, if and only if $\gamma(H) = 1$. Thus, any EDS S of $G \circ H$ contains exactly one vertex from each copy of H. Since there are p copies of H, S contains exactly p vertices and $\gamma(G \circ H) = p$. □

Conclusion

In this chapter, a few results on efficient domination in arbitrary graphs are presented. Some bounds in terms of order, degree and size on domination number of efficiently dominatable graphs are discussed. The properties of graphs possessing pairwise disjoint efficient dominating sets are identified and a structure is proposed which supports fault-tolerant communications in ad hoc and sensor networks. Some significant results on efficient domination in trees are obtained. Categorizing the vertices in a tree as support (strong and weak), the properties of vertices in efficiently dominatable trees and trees which are not efficiently dominatable are studied. Efficiently dominatable trees of diameter up to five are characterized. Characterization are obtained for the join of two graphs, one-point union of graph and corona of graphs to be efficient dominatable.

Chapter 4

Changing and Unchanging Efficient Domination in graphs

A critical constraint in the topological design of a network is to ensure uninterrupted network communication in the event of an unexpected occurrence of faulty components like nodes or links. The influence of a faulty node (or a link) on network communication can be analyzed by examining the influence of removal of vertices (or edges) from the underlying graph. Further, to provide a cost-effective communication, most of the applications require a subset of the network nodes to be designated with special roles such as servers or heads and are preferably as smaller subsets as possible. Such a subset can be identified by finding a minimum dominating set of the underlying graph. Further, to provide a non-overlapping/interference-free communication, it is required to fix an additional constraint that each network node must have a unique neighbor in the subset. This is accomplished by identifying an EDS in the underlying network. The concept of criticality in graph theory deals with the study of the behaviour of a graph with reference to a parameter, upon removing a vertex or a set of vertices, removing or adding an edge or a set of edges. Hence, due its significance from theoretical as well as application perspectives, a special interest is shown in the study of critical concept at least for the past three decades.

In general, the removal of a vertex or the removal/addition of an edge in a graph G may increase or decrease or leave unaltered the value of $\gamma(G)$. That is, if a vertex $v \in V(G)$ is removed from G, then $\gamma(G-v)$ may be greater than or less than or equal to $\gamma(G)$. A vertex $v \in V(G)$ such that $\gamma(G-v) \neq \gamma(G)$ is referred to as a ***critical vertex***. Similarly, an edge $e \in E(G)$ (or $e \in E(\overline{G})$) such that $\gamma(G-e) \neq \gamma(G)$ (or $\gamma(G+e) \neq \gamma(G)$) is referred to as a ***critical edge***. Based on this, the vertices and edges in a graph are categorized into nine sets: V^0, V^+, V^-, ER^0, ER^+, ER^-, EA^0, EA^+ and EA^-; which in turn results in a categorization of the entire collection of graphs into six classes: UVR, CVR, UER, CER, UEA and CEA, defined as in Section 4.1.

Though the properties of critical vertices have been well explored in the literature with respect to domination and other variants of domination, to the best of our knowledge, the concept of criticality has not been much explored with respect to efficient domination, except for the studies by Milanič (2013) and Barbosa and Slater (2016). Furthermore, the properties possessed by a critical vertex (or a critical edge) in a graph which is not efficiently dominatable need not be the same for such a vertex (or an edge) in an efficiently dominatable graph. For an instance, it is known that for a vertex $v \in V(G)$, $\gamma(G-v) < \gamma(G)$ if and only if $pn[v,S] = \{v\}$, where $pn[v,S] = \{u : N[u] \cap S = \{v\}\}$ (refer to (Haynes et al., 1998)). But, if G is efficiently dominatable, it follows from the definition of an EDS that for an arbitrary EDS of G, say S, if $v \in S$ then $pn[v,S] = N[v]$ and consequently, the properties possessed by $v \in V(G)$ such that $\gamma(G-v) < \gamma(G)$ differ in an efficiently dominatable graph (as explored in Section 4.2). The existence of such properties necessitates to revisit the study on critical concept with respect to efficient domination. Thus, motivated by the significance of the concept of criticality and based on the research gap identified in the literature, in this chapter, the study of the concept of criticality is initiated with respect to efficient domination.

By extending this study with respect to efficient domination, the following classes are analogously introduced: $UVR_{\mathscr{E}}$ and $CVR_{\mathscr{E}}$ with respect to vertex removal; $UER_{\mathscr{E}}$ and $CER_{\mathscr{E}}$ with respect to edge removal; $UEA_{\mathscr{E}}$ and $CEA_{\mathscr{E}}$ with respect to edge addition in Sections 4.2, 4.3 and 4.4 respectively. Here, the subscript $\mathscr{E}$ is used to indicate that the respective classes are restricted to the class ($\mathscr{E}$) of efficiently dominatable graphs. The main objective of this chapter is to explore those structures (referred to as fault-tolerant structures) which are efficiently dominatable and continue to remain efficiently dominatable even after the removal of a vertex or removal of an edge or addition of an edge. On that line, initially, the properties of critical vertices, critical edges with respect to both removal and addition, vertex critical sets, edge critical sets with respect to both removal and addition are discussed. Later, the structural properties of the above six classes of graphs arising thereof are studied and these classes are characterized. Finally, the relationship between all these classes are identified and represented in terms of a Venn diagram. At the end of this chapter, some of the significant properties discussed for the class of efficiently dominatable graphs in this chapter are compared against the respective properties for the class of arbitrary graphs (that is, graphs which may or may not be efficiently dominatable) and presented in Tables 4.1, 4.2 and 4.3.

4.1 Preliminaries

Throughout this chapter, the following acronyms are used as in (Haynes et al., 1998): *(C stands for changing; U for unchanging; V stands for vertex; E for edge; R for removal and A for addition)*. With this convention, the following abbreviations are in general used to denote the six classes of graphs which arise due to the removal of a vertex or removal/addition of an edge, defined as below:

Let $\overline{G}$ denote the complement of a graph G. Then, any graph G belongs to one or more classes defined below, based on the conditions stated for each class.

(a) **UVR** (**U**nchanging **V**ertex **R**emoval) if $\gamma(G-v) = \gamma(G)$, for all $v \in V(G)$

(b) **CVR** (**C**hanging **V**ertex **R**emoval) if $\gamma(G-v) \neq \gamma(G)$, for all $v \in V(G)$

(c) **UER** (**U**nchanging **E**dge **R**emoval) if $\gamma(G-e) = \gamma(G)$, for all $e \in E(G)$

(d) **CER** (**C**hanging **E**dge **R**emoval) if $\gamma(G-e) \neq \gamma(G)$, for all $e \in E(G)$

(e) **UEA** (**U**nchanging **E**dge **A**ddition) if $\gamma(G+e) = \gamma(G)$, for all $e \in E(\overline{G})$

(f) **CEA** (**C**hanging **E**dge **A**ddition) if $\gamma(G+e) \neq \gamma(G)$, for all $e \in E(\overline{G})$

Similarly, the vertices of G, edges of G and the edges of $\overline{G}$ are categorized as follows:

(a) $V^0 = \{v \in V(G) : \gamma(G-v) = \gamma(G)\}$

(b) $V^+ = \{v \in V(G) : \gamma(G-v) > \gamma(G)\}$

(c) $V^- = \{v \in V(G) : \gamma(G-v) < \gamma(G)\}$

(d) $ER^0 = \{e \in E(G) : \gamma(G-e) = \gamma(G)\}$

(e) $ER^+ = \{e \in E(G) : \gamma(G-e) > \gamma(G)\}$

(f) $ER^- = \{e \in E(G) : \gamma(G-e) < \gamma(G)\}$

(g) $EA^0 = \{e \in E(\overline{G}) : \gamma(G+e) = \gamma(G)\}$

(h) $EA^+ = \{e \in E(\overline{G}) : \gamma(G+e) > \gamma(G)\}$

(i) $EA^- = \{e \in E(\overline{G}) : \gamma(G+e) < \gamma(G)\}$

In general, a given graph G may or may not be efficiently dominatable. In the same way, for any $u \in V(G)$, $G-u$ may or may not be efficiently dominatable. And, for any $e \in E(G)$, $G-e$ (or $G+e$, for any $e \in E(\overline{G})$) may or may not be efficiently dominatable. Based on these facts, *an element p (may be a vertex or an edge) is said to* ***preserve the efficient domination property*** *if and only if* $G \pm p \in \mathscr{E}$ and based on this we categorize the entire collection $\mathscr{G}$ of all (finite) graphs into four sets as below:

(i) $\mathscr{G}_1 = \{G : G \notin \mathscr{E}\}$

(ii) $\mathscr{G}_2 = \{G : G \in \mathscr{E}$ and $G-v \notin \mathscr{E}$, for all $v \in V(G)\}$

(iii) $\mathscr{G}_3 = \{G : G \in \mathscr{E}$ and $G-v \in \mathscr{E}$, for some $v \in V(G)\}$

(iv) $\mathscr{G}_4 = \{G : G \in \mathscr{E}$ and $G-v \in \mathscr{E}$, for all $v \in V(G)\}$

A similar categorization can be done with respect to edge removal and edge addition. For a convenient reference and comparison, throughout this thesis, the set $\mathscr{G}_4$ is alternatively referred to as $\mathscr{G}_{-v}$. **That is, $\mathscr{G}_{-v}$ $(= \mathscr{G}_4)$ refers to the collection of all efficiently dominatable graphs G such that every vertex in G preserves the efficient domination property. Analogously, the set $\mathscr{G}_{-e}$ is defined as $\{G : G \in \mathscr{E}$ and $G - e \in \mathscr{E}$, for all $e \in E(G)\}$ and $\mathscr{G}_{+e} = \{G : G \in \mathscr{E}$ and $G + e \in \mathscr{E}$, for all $e \in E(\overline{G})\}$.** As this chapter deals with the concept of criticality with respect to efficient domination, the graphs considered in this chapter are restricted to the class $\mathscr{G} - \mathscr{G}_1$ (or $\mathscr{G}_2 \cup \mathscr{G}_3 \cup \mathscr{G}_4$).

But, it will be shown in the sections to follow that the class $\mathscr{G}_2$ does not exist. Further, it follows from the definition that $\mathscr{G}_4 \subsetneqq \mathscr{G}_3$. Hence, the overall focus is reduced to those graphs belonging to the class $\mathscr{G}_3$. As defined in Chapter 2, a graph G is hereditary efficiently dominatable (also called super-efficient graph) if every induced subgraph of G contains an efficient dominating set. The studies carried out by Milanič (2013) and Barbosa and Slater (2016) on hereditary efficiently dominatable graphs (or super-efficient graphs) justify the existence of the class $\mathscr{G}_3$. That is, every hereditary efficiently dominatable graph belongs to the class $\mathscr{G}_3$, but not conversely.

Since the graphs considered in this chapter are restricted to the class $\mathscr{G}_3$, it is noted that all graphs G considered throughout this chapter are efficiently dominatable, unless mentioned otherwise. Further, for any $u \in V(G)$ (or $e \in E(G)$), $G - u$ (or $G - e$) may or may not be connected. An EDS of an efficiently dominatable graph is the union of EDS of its components (taken one for each component).

4.2 Vertex removal

Let $v \in V(G)$. Then, the vertex v is defined to be

(a) γ-critical if $\gamma(G - v) \neq \gamma(G)$

(b) γ^+-critical if $\gamma(G - v) > \gamma(G)$

(c) γ^--critical if $\gamma(G - v) < \gamma(G)$

Thus, *a vertex is said to be* **$\boldsymbol{\gamma}$-critical** *if it is either γ^+-critical or γ^--critical.* With respect to vertex removal, restricting the two known classes UVR and CVR to the class $\mathscr{E}$ of efficiently dominatable graphs, the two classes $UVR_{\mathscr{E}}$ and $CVR_{\mathscr{E}}$ are defined as follows:

(a) $UVR_{\mathscr{E}} = UVR \cap \mathscr{G}_{-v}$

(b) $CVR_{\mathscr{E}} = CVR \cap \mathscr{G}_{-v}$

Remark 4.2.1. *(Haynes et al., 1998)*
For an arbitrary graph G, it is observed that

(a) removal of a vertex can increase $\gamma(G)$ by more than one.
For example, $\gamma(K_{1,n}) = 1$, $\gamma(K_{1,n} - u) = n\ (> 1)$, where u is the central vertex.

(b) removal of a vertex can decrease $\gamma(G)$ by at most one.
Hence, for any vertex $v \in V(G)$, v is γ^--critical if and only if $\gamma(G - v) = \gamma(G) - 1$.

(c) any isolated vertex in G is γ^--critical.

Hence, the above properties are also true for all graphs in the class $\mathscr{G}_{-v}$.

4.2.1 Results on some well-known graphs

This section is devoted to the discussion on the concept of criticality with respect to efficient domination for some well known graphs.

Proposition 4.2.1. *For $n \geq 2$, $K_{1,n} \notin UVR_{\mathscr{E}} \cup CVR_{\mathscr{E}}$.*

Proof. Let $V(K_{1,n}) = \{u_0, u_1, \ldots, u_n\}$, where u_0 is the central vertex and $n \geq 2$. Clearly, $\{u_0\}$ is an EDS of both $K_{1,n}$ and $K_{1,n} - u_j$, for each j $(1 \leq j \leq n)$. Hence, $\gamma(K_{1,n} - u_j) = \gamma(K_{1,n})$, for each j $(1 \leq j \leq n)$. But, for the graph $K_{1,n} - u_0$, $\gamma(K_{1,n} - u_0) = n$ with the set $\{u_1, \ldots, u_n\}$ as its EDS. Hence, $\gamma(K_{1,n} - u_0) > \gamma(K_{1,n})$ resulting in the conclusion that $K_{1,n} \notin UVR_{\mathscr{E}}$ and $K_{1,n} \notin CVR_{\mathscr{E}}$. □

Proposition 4.2.2. *For $n \geq 2$, $K_n \in UVR_{\mathscr{E}}$.*

Proof. Let $V(K_n) = \{u_1, \ldots, u_n\}$. Then, for $1 \leq i \leq n$, the set $\{u_i\}$ is an EDS of K_n. It can be observed that for a fixed i and $i \neq j$, $(i, j \in \{1, 2, \ldots, n\})$, the set $\{u_j\}$ is an EDS of $K_n - u_i$. Thus, $\gamma(K_n - u) = \gamma(K_n)$, for all $u \in V(K_n)$ and hence, $K_n \in UVR_{\mathscr{E}}$. □

It is known that $C_n \in \mathscr{E}$ if and only $n \equiv 0 \pmod 3$. The result given below shows that every efficiently dominatable cycle belongs to the class $UVR_{\mathscr{E}}$.

Proposition 4.2.3. *For $n \geq 3$, $C_n \in UVR_{\mathscr{E}}$ if and only if $n \equiv 0$ (mod 3).*

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $n \equiv 0 \pmod 3$. Then, $C_n \in \mathscr{E}$ and $\gamma(C_n) = \frac{n}{3}$. For any $u_i \in V(C_n)$, $C_n - u_i \cong P_{n-1}$. It follows that $n - 1 \equiv 2 \pmod 3$ and $\gamma(C_n - u_i) = \gamma(P_{n-1}) = \lceil \frac{n-1}{3} \rceil = \frac{n}{3}$. Thus, $\gamma(C_n - u) = \gamma(C_n)$, for all $u \in V(C_n)$ and $C_n \in UVR_{\mathscr{E}}$.

Conversely, let $C_n \in UVR_{\mathscr{E}}$. If $n \not\equiv 0 \pmod 3$, then $C_n \notin \mathscr{E}$ and hence $C_n \notin UVR_{\mathscr{E}}$, which is a contradiction. Thus, $n \equiv 0 \pmod 3$. □

It is known that P_n is efficiently dominatable for all $n \geq 1$. Propositions 4.2.4 and 4.2.5 given below deal with the conditions under which P_n belongs to either $UVR_{\mathscr{E}}$ or $CVR_{\mathscr{E}}$ or neither.

Proposition 4.2.4. *For $n \geq 1$, $P_n \in UVR_{\mathscr{E}}$ if and only if $n \equiv 2$ (mod 3).*

Proof. Let $n \equiv 2 \pmod 3$ and $V(P_n) = \{u_1, u_2, \ldots, u_n\}$. Then, the sets $S_1 = \{u_1, u_4, \ldots, u_{n-1}\}$ and $S_2 = \{u_2, u_5, \ldots, u_n\}$ are two disjoint EDSs of P_n. Let $u_i \in V(P_n)$. Then, one of the following three cases arise: *(i) $u_i \in S_1$ (or) (ii) $u_i \in S_2$ (or) (iii) u_i is in neither S_1 nor S_2.*

If $u_i \in S_1$ (or $u_i \in S_2$), then S_2 (or S_1) will be an EDS of $P_n - u_i$ and $\gamma(P_n - u_i) = \gamma(P_n)$. If $u_i \notin S_1$ and $u_i \notin S_2$, then both S_1 and S_2 are two disjoint EDSs of $P_n - u_i$ and hence, $\gamma(P_n - u_i) = \gamma(P_n)$. Since u_i is arbitrary, it follows that $P_n \in UVR_{\mathscr{E}}$.

Conversely, let $P_n \in UVR_{\mathscr{E}}$ and $n \not\equiv 2 \pmod 3$. Then, one of the following two cases arise: $n \equiv 0$ *(mod 3) or* $n \equiv 1$ *(mod 3)*

Case (i): $n \equiv 0 \pmod 3$

In this case, $\gamma(P_n) = \frac{n}{3}$ with the set $S = \{u_2, u_5, \ldots, u_{n-1}\}$ as its unique EDS. For any $u_i \notin S$, S still remains as an EDS of $P_n - u_i$ and hence $\gamma(P_n - u_i) = \gamma(P_n)$. On the other hand, let $u_i \in S$. Since $P_n - u_i \cong P_{i-1} \cup P_{n-i}$, where $i \equiv 2 \pmod 3$, it follows that $\gamma(P_n - u_i) = \gamma(P_{i-1}) + \gamma(P_{n-i}) = \lceil \frac{i-1}{3} \rceil + \lceil \frac{n-i}{3} \rceil = \frac{(i-1)+2}{3} + \frac{(n-i)+2}{3} = \frac{n}{3} + 1$. Therefore, $\gamma(P_n - u_i) > \gamma(P_n)$.

That is, for any $u_i \in V(P_n)$, if $u_i \notin S$, then $\gamma(P_n - u_i) = \gamma(P_n)$ and if $u_i \in S$, then $\gamma(P_n - u_i) > \gamma(P_n)$. Hence, $P_n \notin UVR_{\mathscr{E}}$, which is a contradiction.

Case (ii): $n \equiv 1 \pmod 3$

In this case, the set $S = \{u_1, u_4, \ldots, u_n\}$ is the unique EDS of P_n and hence, $\gamma(P_n) = \lceil \frac{n}{3} \rceil = \frac{n+2}{3}$.

Clearly, for any $u_i \notin S$, S is an EDS of both P_n and $P_n - u_i$ and hence, $\gamma(P_n - u_i) = \gamma(P_n)$. Now, let $u_i \in S$, where $1 \leq i \leq n$. Since, $\gamma(P_n - u_1) = \gamma(P_n - u_n) = \gamma(P_{n-1}) = \lceil \frac{n-1}{3} \rceil = \frac{n}{3}$, it follows that $\gamma(P_n - u_i) < \gamma(P_n)$, when $i = 1$ or $i = n$. For any i $(1 < i < n)$, $P_n - u_i \cong P_{i-1} \cup P_{n-i}$, where $i \equiv 1 \pmod 3$. Therefore, if $u_i \in S$ where $1 < i < n$, then $\gamma(P_n - u_i) = \gamma(P_{i-1}) + \gamma(P_{n-i}) = \lceil \frac{i-1}{3} \rceil + \lceil \frac{n-i}{3} \rceil = \frac{i-1}{3} + \frac{n-i}{3} = \frac{n-1}{3}$. Therefore, $\gamma(P_n - u_i) < \gamma(P_n)$, for every $u_i \in S$ and hence, $P_n \notin UVR_{\mathscr{E}}$, which is a contradiction.

So, it can be concluded from the above discussions that if $P_n \in UVR_{\mathscr{E}}$, then $n \equiv 0 \pmod 3$. □

Further, the arguments stated in proving the converse part of Proposition 4.2.4 also lead to the following proposition.

Proposition 4.2.5. *If $n \geq 1$ and $n \not\equiv 2$ (mod 3), then $P_n \notin UVR_{\mathscr{E}} \cup CVR_{\mathscr{E}}$.*

Remark 4.2.2. *In connection with Proposition 4.2.5, the following conditions are also noted.*

If $n \geq 1$ and $n \not\equiv 2$ (mod 3), then two cases arise:

Case (i): $n \equiv 0$ (mod 3)

If $u \in V(P_n)$ such that $ecc(u) \equiv 1$ (mod 3), then $\gamma(P_n - u) = \gamma(P_n) + 1$ and for every other vertex u, $\gamma(P_n - u) = \gamma(P_n)$. That is,

$$\gamma(P_n - u) > \gamma(P_n); \text{ if } ecc(u) \equiv 1 \ (mod\ 3)$$
$$= \gamma(P_n); \text{ otherwise}$$

Case (ii): $n \equiv 1 \ (mod\ 3)$

In this case, if $u \in V(P_n)$ such that $ecc(u) \equiv 0 \ (mod\ 3)$, then $\gamma(P_n - u) = \gamma(P_n) - 1$ and for every other vertex u, $\gamma(P_n - u) = \gamma(P_n)$. That is,

$$\gamma(P_n - u) < \gamma(P_n); \text{ if } ecc(u) \equiv 0 \ (mod\ 3)$$
$$= \gamma(P_n); \text{ otherwise}$$

Thus, it follows from cases (i) and (ii) that $P_n \notin UVR_{\mathscr{E}}$ and $P_n \notin CVR_{\mathscr{E}}$. Hence, the result follows.

Remark 4.2.3. *A common observation made in the discussions of Propositions 4.2.1 to 4.2.5 is that if G denotes any of the well known graphs discussed above and $u \in V(G)$ such that there exists at least one EDS of G which does not contain u, then $\gamma(G) = \gamma(G - u)$. This property is in general true for an arbitrary graph (as evident from the result to be proved in Theorem 4.2.8).*

4.2.2 Properties of Critical vertices

This section deals with some significant properties of the critical vertices of an efficiently dominatable graph. The equivalent conditions for a vertex to be γ-critical or otherwise in an efficiently dominatable graph are discussed. Further, the structural properties of graphs belonging to the classes $UVR_{\mathscr{E}}$ and $CVR_{\mathscr{E}}$ are studied and based on these results, the two classes are characterized.

Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G - u \in \mathscr{E}$. That is, $G \in \mathscr{G}_3$. Suppose that S is an EDS of G. For any vertex $u \notin S$, as S itself is an EDS of $G - u$, $\gamma(G - u) = \gamma(G)$ and hence, $u \in V^0$. On the other hand, if $u \in S$, then u may or may not be γ-critical in G. Since S is an EDS of G containing u, upon removing u from G, it can be observed that $S - \{u\}$ cannot be an EDS of $G - u$. However, $S - \{u\}$ efficiently dominates all the vertices in $G - u$ except $N_G(u)$. So, the following natural question arises - *"Is it possible to append one or more vertices from $N_G(u)$ to $S - \{u\}$ so as to efficiently dominate $N_G(u)$ and hence* *to efficiently dominate $V(G-u)$ or replace one or more vertices in $S-\{u\}$ by their neighbors suitably so that the resultant set is an EDS of $G-u$?"* If yes, then it may be easier to compare $\gamma(G)$ and $\gamma(G-u)$, which in turn helps in easily categorizing whether or not the vertex u is γ-critical. Focusing in this direction, some of the properties of critical vertices and critical sets are discussed for an efficiently dominatable graph in the results to follow. Based on these results, a general construction is proposed to generate an EDS of $G-u$ by starting with an arbitrary EDS of G containing u and this procedure leads to a simpler way to compare $\gamma(G)$ and $\gamma(G-u)$.

Proposition 4.2.6. *Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $deg(u) \geq 1$ and $G-u \in \mathscr{E}$. If S_u is an arbitrary EDS of $G-u$, then $S_u \cup \{u\}$ will not be an EDS of G.*

Proof. Let S_u be an arbitrary EDS of $G-u$. Then, in $G-u$, S_u efficiently dominates $N(u)$ and hence, there exists at least one vertex $x \in S_u$ for which $d(x, u) \leq 2$. Therefore, $S_u \cup \{u\}$ will not be a 2-packing in G and hence cannot be an EDS of G. □

Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G-u \in \mathscr{E}$. If S is an arbitrary EDS of G such that $u \in S$, then it follows from the definition of an EDS that for each $x \in N_G(u)$, $N_G[x] \cap S = \{u\}$. Hence, $S-\{u\}$ cannot be an EDS of $G-u$. However, as discussed earlier, $S-\{u\}$ efficiently dominates all vertices in $V(G-u)$ except $N_G(u)$. Further, for each $x \in N_G(u)$ and for each $y \in S-\{u\}$, $d_{G-u}(x,y) \geq 2$. Therefore, based on these facts, it may be possible to generate an EDS of $G-u$ in one of the following ways:

(i) If there exist vertices $x \in N_G(u)$ such that $d_{G-u}(x,y) \geq 3$, for all $y \in S-\{u\}$, then appending $S-\{u\}$ with one or more such vertices from $N_G(u)$ which can also dominate $N_G(u)$, may result in an EDS of $G-u$.

(ii) If there exist vertices in $S-\{u\}$ which are at distance two from some or all vertices in $N_G(u)$, then deleting such vertices from $S-\{u\}$ and replacing each such deleted vertex by exactly one of its suitable neighbors in $V(G-u)-[(S-\{u\}) \cup N_G(u)]$ may result in an EDS of $G-u$.

Suppose the set, say S', so generated does not dominate some vertices in $V(G-u)$ (precisely, in $V(G-u) \cap N_G(u)$) then further addition of one or more suitable vertices from $N_{G-u}[S'] - N_G(u)$ may result in an EDS of $G-u$.

The Lemma 4.2.7 discussed below, guarantees the possibility of generating an EDS of $G-u$ by the above methods and in fact, it proves that every EDS of $G-u$ should have been generated in one of the above ways. Further, this lemma helps in comparing $\gamma(G)$ and $\gamma(G-u)$ and thereby, helps in characterizing the critical vertices in an efficiently dominatable graph.

Lemma 4.2.7. *Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G-u \in \mathscr{E}$. If S is an arbitrary EDS of G such that $u \in S$ and S_u is an arbitrary EDS of $G-u$, then there exists a partition of S_u into subsets, say, A and B of $V(G-u)$, such that exactly one of the following conditions hold:*

(i) $A = S - \{u\}$ *and* $B \subseteq N_G(u)$.

(ii) $B = \emptyset$ *and there exists a one-to-one correspondence between the sets* $S - \{u\}$ *and* A.

(iii) $B \neq \emptyset$ *and there exists a one-to-one correspondence between the sets* $S - \{u\}$ *and* A.

Proof. Let S be an arbitrary EDS of G such that $u \in S$. Then, as discussed earlier, $S - \{u\}$ cannot be an EDS of $G-u$. Let S_u be an arbitrary EDS of $G-u$. Then, as $S - \{u\} \subseteq V(G-u)$, the following two cases arise: (i) $S - \{u\} \subsetneqq S_u$ and (ii) $S - \{u\} \nsubseteqq S_u$. That is, neither $S_u \supset S - \{u\}$ nor $S_u \subset S - \{u\}$.
Case (i): $S - \{u\} \subsetneqq S_u$
Then, as S_u is an EDS of $G-u$, $S - \{u\}$ dominates only the vertices in $V(G-u) - N_G(u)$ and hence, there exists at least one vertex, say x, such that $x \in S_u \cap N_G(u)$. Define, $A = S - \{u\}$ and $B = S_u \cap N_G(u)$. Then, clearly $B \subseteq N_G(u)$, $A \cap B = \emptyset$ and $A \cup B = S_u$. Hence, condition (i) holds.
Case (ii): $S - \{u\} \nsubseteqq S_u$ (Or equivalently, neither $S_u \supset S - \{u\}$ nor $S_u \subset S - \{u\}$)
In this case, the following facts are noted:

(a) There exist vertices $x, y \in V(G-u)$ such that $x \in S - \{u\}$, but $x \notin S_u$ and similarly, $y \in S_u$, but $y \notin S - \{u\}$.

(b) For each $x \in (S - \{u\}) - S_u$, $|N_{G-u}(x) \cap S_u| = 1$ and hence, for each $x \in (S - \{u\}) - S_u$, there exists a unique $y_x \in S_u$ such that $xy_x \in E(G-u)$.

(c) S_u may or may not intersect with $N_G(u)$.

Define, $A = S_u - N_G(u)$ and $B = N_G(u) \cap S_u$. Clearly, $A \cap B = \emptyset$ and $A \cup B = S_u$. Further, it follows from the above stated property (c) that either $B = \emptyset$ or $B \neq \emptyset$. In order to show that there exists a one-to-one correspondence between A and $S - \{u\}$, define a mapping $f :\ S - \{u\} \to A$ such that for each $x \in S - \{u\}$,

$$f(x) = \begin{cases} x; & \text{if } x \in A \\ y_x; & \text{otherwise, where } y_x \text{ is the unique neighbor of } x \text{ in } S_u \end{cases}$$

Then, as $S - \{u\}$ is a 2-packing of $G - u$, it is clear that for any two distinct vertices x_1, x_2 in $S - \{u\}$, $f(x_1) \neq f(x_2)$ and for each $y_x \in A$, there exists a unique $x \in S - \{u\}$ such that either $y_x = x$ or $xy_x \in E(G-u)$. Therefore, f is an isomorphism between the sets $S - \{u\}$ and A, irrespective of whether $B = \emptyset$ or $B \neq \emptyset$. Hence, conditions (ii) and (iii) hold. □

Theorem 4.2.8. *Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G - u \in \mathscr{E}$. Then u is γ-critical if and only if u is in every EDS of G.*

Proof. Let u be γ-critical in G. Suppose there exists an EDS of G, say S, such that $u \notin S$, then S is an EDS of $G - u$ also and hence, $\gamma(G-u) = \gamma(G)$, contradicting that u is γ-critical in G. Hence, u must be in every EDS of G.

Conversely, suppose that u is in every EDS of G. Let S and S_u be arbitrary efficient dominating sets of G and $G - u$, respectively. Then, $u \in S$ and it follows from Lemma 4.2.7 that there exist disjoint subsets A and B of $V(G-u)$ such that $A \cup B = S_u$ satisfying exactly one of the three conditions stated in Lemma 4.2.7. Thus, $|S_u| = |A| + |B| = |S - \{u\}| + |B| = \gamma(G) - 1 + |B|$. Further, as S_u is an arbitrary EDS of $G - u$, either $N_G(u) \cap S_u = \emptyset$ or $N_G(u) \cap S_u \neq \emptyset$.

Case (i): $N_G(u) \cap S_u = \emptyset$.

Then, as $B \subseteq S_u$, $B = \emptyset$ and hence, $\gamma(G-u) = |S_u| = \gamma(G) - 1$. That is,

$\gamma(G-u) < \gamma(G)$. Therefore, u is γ-critical in G.

Case (ii): $N_G(u) \cap S_u \neq \emptyset$

Then, either $|N_G(u) \cap S_u| = 1$ or $|N_G(u) \cap S_u| > 1$. If $|N_G(u) \cap S_u| = 1$, then S_u efficiently dominates all the vertices in $V(G)$ and hence, is an EDS of G, as well. This implies that G has an EDS not containing u, which is a contradiction to our hypothesis.

On the other hand, if $|N_G(u) \cap S_u| > 1$, then $|B| > 1$ and hence, $\gamma(G-u) = |S_u| = \gamma(G) - 1 + |B| > \gamma(G) - 1 + 1 = \gamma(G)$. That is, $\gamma(G-u) > \gamma(G)$, which implies that $u \in V^+$. Hence, the result follows. □

It can be observed from the discussion in Theorem 4.2.8 that, if u satisfies the hypothesis of Theorem 4.2.8 and is γ-critical, then for every EDS S_u of $G-u$, either $|N_G(u) \cap S_u| = 0$ or $|N_G(u) \cap S_u| > 1$ and vice-versa. This leads to three equivalent conditions for a vertex to be γ-critical in an efficiently dominatable graph, as stated in Corollary 4.2.8.1 and also leads to Corollary 4.2.8.2

Corollary 4.2.8.1. *Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G - u \in \mathscr{E}$. Then the following conditions are equivalent.*

(i) u is γ-critical.

(ii) u is in every EDS of G.

(iii) $|N_G(u) \cap S_u| \neq 1$, for every EDS S_u of $G-u$.

Proof.

It follows from Theorem 4.2.8 that condition (i) is equivalent to condition (ii).

Next, to prove condition (ii) is equivalent to condition (iii), suppose that u is in every EDS of G. If there exists an EDS of $G-u$, say S_u, such that $|N_G(u) \cap S_u| = 1$, then it follows by the same argument as in Theorem 4.2.8 that S_u is an EDS of G. This contradicts the hypothesis that u is in every EDS of G. Therefore, $|N_G(u) \cap S_u| = 1$.

Conversely, let $|N_G(u) \cap S_u| \neq 1$, for every EDS S_u of $G-u$. Then, either $|N_G(u) \cap S_u| = 0$ or $|N_G(u) \cap S_u| > 1$. It can be observed by the same argument as in the converse part of Theorem 4.2.8 that $u \in V^-$ if $|N_G(u) \cap S_u| = 0$ and $u \in V^+$ if $|N_G(u) \cap S_u| > 1$. Therefore, u is γ-critical and hence, by Theorem 4.2.8, u is in every EDS of G. □

Corollary 4.2.8.2. *Let $G \in \mathscr{E}$ and S be an EDS of G. If $u \in V(G)$ such that $u \in S$ and $G - u \in \mathscr{E}$ and if S_u is an arbitrary EDS of $G - u$, then the following conditions hold:*

(i) $u \in V^0$ *if and only if* $|N_G(u) \cap S_u| = 1$.

(ii) $u \in V^+$ *if and only if* $|N_G(u) \cap S_u| > 1$.

(iii) $u \in V^-$ *if and only if* $|N_G(u) \cap S_u| = 0$.

Remark 4.2.4. *If $u \in V^+$, then it follows from Corollary 4.2.8.2 that $|N(u) \cap S_u| \geq 2$ and hence, it follows that there exist at least two nonadjacent vertices in $N(u)$ so that they can be included in the set S_u.*

Remark 4.2.5. *If $G \in \mathscr{E}$ and $u \in V(G)$, then it follows from Theorem 4.2.8 that $u \in V^0$ if and only if there exists at least one EDS of G which does not contain u. Consequently, if S is an arbitrary EDS of G and if $u \in V - S$, then $u \in V^0$. Hence, $V^0 \supseteq V - S$ and consequently, $V^+ \subseteq S$ and $V^- \subseteq S$. This leads to the following bounds on the size the three critical sets, namely V^0, V^+ and V^-.*

Theorem 4.2.9. *Let $G \in \mathscr{G}_{-v}$ and $|V(G)| = n$. Then the following properties hold.*

(i) $n - \gamma(G) \leq |V^0| \leq n$

(ii) $0 \leq |V^+| \leq \gamma(G)$

(iii) $0 \leq |V^-| \leq \gamma(G)$

Proof. Let S be an EDS of G. Then $|S| = \gamma(G)$. It follows from Theorem 4.2.8 (or Remark 4.2.5) that $V^0 \supseteq V - S$ and hence, $|V^0| \geq |V - S| = n - \gamma(G)$. Further, if a vertex u is in S, then either $u \in V^0$ or u may be γ-critical. Thus, $|V^0| \leq (n - \gamma(G)) + \gamma(G) = n$. Hence, result (i) holds. Next, results (ii) and (iii) hold trivially from the upper and lower bounds on $|V^0|$ in condition (i) and the facts that $V^+ \subseteq S$ and $V^- \subseteq S$. $\square$

It is known that if a graph G has no isolated vertices, then $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$ (Refer to (Haynes et al., 1998)). This fact along with Theorem 4.2.9 leads to the following corollary.

Corollary 4.2.9.1. *If $G \in \mathscr{G}_{-v}$ and G has no isolated vertex, then*

(i) $\lceil \frac{n}{2} \rceil \leq |V^0| \leq n$

(ii) $0 \leq |V^+| \leq \lfloor \frac{n}{2} \rfloor$

(iii) $0 \leq |V^-| \leq \lfloor \frac{n}{2} \rfloor$

Remark 4.2.6. *In general, it is known for an arbitrary graph that, $|V^0| \geq 2|V^+|$ (refer to (Haynes et al., 1998)). But, it is evident from Corollary 4.2.9.1 that for any graph $G \in \mathscr{G}_{-v}$ having no isolated vertex, $|V^0| \geq |V^+|$ and $|V^0| \geq |V^-|$.*

The following proposition shows that if $G \in \mathscr{E}$, then for any $u \in V(G)$ such that $G - u \in \mathscr{E}$ and $u \in V^-$, no neighbor of u can be a pendant vertex.

Proposition 4.2.10. *Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G - u \in \mathscr{E}$. If $u \in V^-$, then $deg_G(x) \geq 2$, for every $x \in N_G(u)$.*

Proof. Let $u \in V^-$. Suppose that x is a pendant vertex adjacent to u. Then, in $G - u$, x becomes an isolated vertex and x must be included in every EDS of $G - u$. Thus, $|N_G(u) \cap S_u| \geq 1$, for every EDS S_u of $G - u$, contradicting that $u \in V^-$. Thus, $deg(x) \geq 2$ for all $x \in N(u)$. $\square$

It follows from Corollary 4.2.8.2 that if $u \in V^+$, then every EDS of $G - u$ will contain at least two neighbors of u and hence, the following proposition follows trivially.

Proposition 4.2.11. *Let $G \in \mathscr{E}$ with $\gamma(G) > 1$. If $deg_G(u) = 1$ and u is in every EDS of G, then $u \notin V^+$.*

Procedure to construct an EDS of $G - u$, knowing an EDS of G: Summarizing the discussions above, to facilitate an easier comparison of $\gamma(G)$ and $\gamma(G-u)$, an EDS of $G-u$ is generated from that of G as follows: Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G-u \in \mathscr{E}$. Let S be an arbitrary EDS of G such that $u \in S$. Then, using Lemma 4.2.7, it is possible to generate an EDS of $G-u$ starting with $S - \{u\}$ using one of the following operations:

$\mathcal{O}_1$: Generate a set $S' \subseteq V(G-u)$ by appending one or more vertices from $N_G(u)$ to $S - \{u\}$ so as to efficiently dominate $N_G(u)$ and thereby, to efficiently dominate $V(G-u)$. (or)

$\mathcal{O}_2$: Generate a set $S' \subseteq V(G-u)$ by deleting one or more vertices in $S - \{u\}$ such that each vertex removed from $S - \{u\}$ is replaced by exactly one of its neighbors in $V(G-u) - (S - \{u\})$. However, it can be noted that for each $x \in N_G(u)$ and for each $y \in S - \{u\}$, $d_G(x,y) = d_{G-u}(x,y) \geq 2$ and hence, none of the vertices in $N_G(u)$ can replace any of the vertices in $S - \{u\}$. Therefore, the set S' generated here may or may not be an EDS of $G-u$.

$\mathcal{O}_3$: If the set S' generated using operation $\mathcal{O}_2$ does not dominate some vertices in $V(G-u)$, then, further addition of one or more suitable vertices from $N_G(u)$ will result in an EDS of $G-u$.

Remark 4.2.7. *It is noted from the above construction that while performing operation $\mathcal{O}_2$, each vertex removed from $S - \{u\}$ is replaced by exactly one of its neighbors. Now, suppose that S_u is an EDS of $G-u$ generated from $S - \{u\}$ using the above procedure. Then, the following properties are inferred.*

(a) If S_u is generated from $S - \{u\}$ using operation $\mathcal{O}_1$ and $|S_u \cap N_G(u)| = 1$, then $\gamma(G-u) = \gamma(G)$. On the other hand, if $|S_u \cap N_G(u)| > 1$, then $\gamma(G-u) > \gamma(G)$.

(b) If S_u is generated from $S - \{u\}$ using operation $\mathcal{O}_2$, then $|S_u| = |S - \{u\}|$ and hence, $\gamma(G-u) < \gamma(G)$.

(c) Suppose that S_u is generated from $S - \{u\}$ using $\mathcal{O}_3$, then, $\gamma(G-u) = \gamma(G)$ if $|S_u \cap N(u)| = 1$ and $\gamma(G-u) > \gamma(G)$ if $|S_u \cap N(u)| > 1$.

Figures 4.1, 4.2, 4.3 and 4.4 are used to illustrate the above construction with a note on the properties listed above. Notice that for the graph G given in Figure 4.1, the set $S = \{2, 6\}$ is an EDS of G. By choosing $u = 2$, the set $S - \{u\}$ $(= \{6\})$ dominates all vertices of $G - \{2\}$ except $N(2)$. Therefore, to dominate $N(2)$ in $G - u$, the set $S - \{u\}$ is extended by adding two neighbors of 2, namely 1 and 3 (operation $\mathcal{O}_1$) so that $\{1, 3, 6\}$ forms an EDS of $G - u$. As mentioned in Remark 4.2.7, in this case, $\gamma(G - u) > \gamma(G)$. Similarly, in Figure 4.2, the set S' is generated as an EDS of $G - \{1\}$ from $S - \{1\}$ using operation $\mathcal{O}_2$ alone and $\gamma(G - \{1\}) < \gamma(G)$. In Figure 4.3, the set S' is obtained as an EDS of $G - \{2\}$ from $S - \{2\}$ using both the operations $\mathcal{O}_3$ such that $|S' \cap N(2)| = 3 > 1$ and hence, $\gamma(G - \{2\}) > \gamma(G)$. Also, in Figure 4.4, S' is generated as an EDS of $G - \{1\}$ using $\mathcal{O}_3$. But $|S' \cap N(1)| = 1$ and hence, $\gamma(G - \{1\}) = \gamma(G)$.

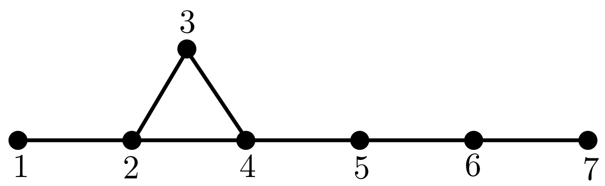


Figure 4.1: A graph $G \in \mathscr{E}$ with $S = \{2, 6\}$ as its EDS; The set $\{1, 3, 6\}$ is obtained as an EDS of $G - \{2\}$ using operation $\mathcal{O}_1$

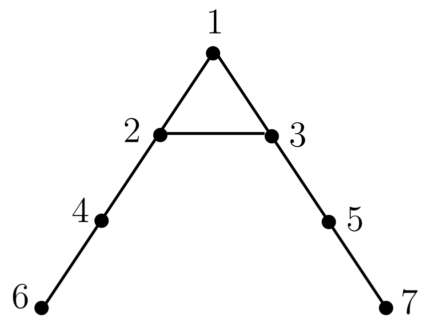


Figure 4.2: A graph $G \in \mathscr{E}$ with $S = \{1, 6, 7\}$ as its EDS; The set $S' = \{4, 5\}$ is obtained as an EDS of $G - \{1\}$ using operation $\mathcal{O}_2$

In general, it is known that a graph $G \in CVR$ if and only if $V(G) = V^-$ (refer to (Haynes et al., 1998)). However, when restricted to the class of efficiently

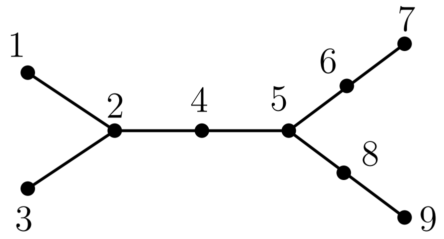


Figure 4.3: A graph $G \in \mathscr{E}$ with $S = \{2, 6, 9\}$ as its EDS; The set $S' = \{1, 3, 4, 7, 9\}$ is got as an EDS of $G - \{2\}$ using $\mathcal{O}_3$

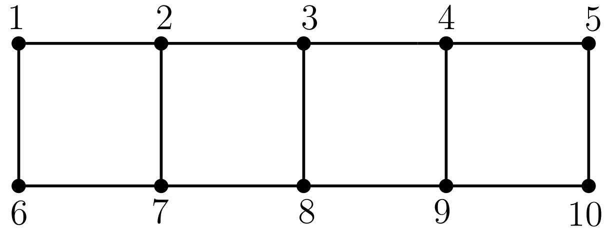


Figure 4.4: $S' = \{3, 6, 10\}$ is got as an EDS of $G - \{1\}$ using $\mathcal{O}_3$ (Replacing every vertex of $S - \{1\}$ by exactly one its neighbors, where $S = \{1, 5, 8\}$)

dominatable graphs, the following characterization is obtained for a graph $G \in \mathscr{G}_{-v}$ to be in $CVR_{\mathscr{E}}$.

Theorem 4.2.12. *Let $G \in \mathscr{G}_{-v}$. Then, $G \in CVR_{\mathscr{E}}$ if and only if $G \cong mK_1$, for $m \geq 1$.*

Proof. Let $G \cong mK_1$, for $m \geq 1$. Then, as every vertex in G is γ^--critical, it follows that $G \in CVR_{\mathscr{E}}$. Conversely, let $G \in CVR_{\mathscr{E}}$ and S be an EDS of G. Then, for all $u \in V(G)$, $\gamma(G - u) \neq \gamma(G)$. That is, $V^0 = \emptyset$ and thus it follows from Remark 4.2.5 that $V - S = \emptyset$. Hence, $S = V(G)$ and this is possible only if $deg(u) = 0$, for all $u \in V(G)$. Equivalently, $G \cong mK_1$, where $m \geq 1$. □

Remark 4.2.8. *Let $G \in \mathscr{G}_{-v}$. Then, it follows from the proof of Theorem 4.2.12 that $V^0 = \emptyset$ if and only if $G \cong mK_1$ if only if $V(G) = V^-$.*

It was discussed in Remark 4.2.5 that if $G \in \mathscr{E}$ and S is an EDS of G, then $V^0 \supseteq V - S$ and hence, $|V^0| \geq n - \gamma(G)$. The following result characterizes those graphs G in $\mathscr{G}_{-v}$ for which $V^0 = V - S$ or equivalently, for which $V^+ \cup V^- = S$, for any EDS S of G.

Theorem 4.2.13. *Let $G \in \mathscr{G}_{-v}$. Then, $|V^0| = n - \gamma(G)$ if and only if G has a unique EDS.*

Proof. Suppose that G has a unique EDS, say S. Then, by Theorem 4.2.8, every vertex $u \in S$ is γ-critical and hence, $S \cap V^0 = \emptyset$. In other words, $V^0 = V - S$. Therefore, $|V^0| = n - \gamma(G)$.
Conversely, let $|V^0| = n - \gamma(G)$. Suppose that S is an arbitrary EDS of G. Then, as $V^0 \supseteq V - S$ and $|S| = \gamma(G)$, it follows from the hypothesis that $V^0 = V - S$. Therefore, every vertex in S must be γ-critical. Hence, by Theorem 4.2.8, each vertex $u \in S$ must be included in every other EDS of G. Since S is arbitrary EDS, this is true for every EDS of G and hence, S is unique. □

Theorem 4.2.14. *Let G be a graph of order n such that $G \in \mathscr{G}_{-v}$ and has a unique vertex, say u, of degree $(n-1)$, then $V^0 \cup V^+ = V(G)$, where $V^+ = \{u\}$.*

Proof. Let u be the unique vertex of degree $n-1$ in G. Then, $S = \{u\}$ is the EDS of G and $\gamma(G) = 1 = |S|$. Clearly, every vertex $v \in V(G) - \{u\}$ must be in V^0. Since, u is the unique vertex of degree $n-1$, for every $v \in V(G-u)$, $deg_{G-u}(v) \leq n-3$. In other words, for each vertex $v \in V(G-u)$, there exist at least two vertices in $G-u$, which are not adjacent to v. Therefore, $\gamma(G-u) > 1 = |S|$ and $u \in V^+$. Hence, the result follows. □

It follows from the definition of critical sets that for any graph G, the sets V^0, V^- and V^+ are disjoint and one or more of these sets together form a partition of $V(G)$. It has been proved in Haynes et al. (1998) that if $u \in V^+$ and $v \in V^-$, then u and v are not adjacent. In Theorem 4.2.15 to follow, it is shown that for any graph $G \in \mathscr{G}_{-v}$, such that $\gamma(G) \leq 2$, $V(G)$ is the union of V^0 and either of V^+ or V^- (but not both). That is, the sets V^+ and V^- do not exist simultaneously. Further, it is shown in Theorem 4.2.16 that, for any connected graph G, if $G \in \mathscr{E}$, then for any $u \in V^+$ and $v \in V^-$, $d_G(u, v) \geq 4$.

Theorem 4.2.15. *Let $G \in \mathscr{G}_{-v}$ such that G is connected and $\gamma(G) \leq 2$. Then either $V(G) = V^0$ or $V(G) = V^0 \cup V^-$ or $V(G) = V^0 \cup V^+$.*

Proof.

Case(i): $\gamma(G) = 1$

Suppose that G has exactly one vertex of degree $n-1$, then it follows from Theorem 4.2.14 that $V(G) = V^0 \cup V^+$, where $|V^+| = 1$. On the other hand, if G has at least two vertices of degree $n-1$, then $V(G) = V^0$. Thus, if $\gamma(G) = 1$, then $V(G) = V^0 \cup V^+$.

Case(ii): $\gamma(G) = 2$

Suppose that $S = \{u, v\}$ is an EDS of G. Then $d_G(u, v) = 3$. Clearly, each vertex in $V - S$ is in V^0. Now, suppose at least one of u and v is in V^0, or both u and v are in V^+ or both u and v are in V^-, then the result follows immediately.

So, suppose that only one of u and v is in V^+ and the other is in V^-. Without loss of generality, let $u \in V^+$ and $v \in V^-$. Then, as $u \in V^+$, $|N_G(u) \cap S_u| \geq 2$, for every EDS S_u of $G - u$. This, in turn implies that, there exist at least two vertices, say, x and y in $N_G(u)$ such that $d_{G-u}(x, y) \geq 3$.

Since $v \in V^-$, $\gamma(G - v) = 1$. Let S_v be an EDS of $G - v$, where $S_v = \{x\}$. Then, it follows from Corollary 4.2.8.2 that $x \notin N_G(v)$ or equivalently, $x \in N_G(u)$. And, x dominates all vertices in $V(G - v)$. Therefore, for each pair of vertices $y, z \in N_G(u)$, $d_{G-v}(y, z) = d_{G-u}(y, z) \leq 2$, which is a contradiction. Hence, the result follows. □

Theorem 4.2.16. *Let $G \in \mathscr{G}_{-v}$ such that G is connected and $\gamma(G) \geq 3$. Then, for any $u \in V^+$ and $v \in V^-$, $d_G(u, v) \geq 4$.*

Proof. Let $u \in V^+$ and $v \in V^-$. Let S be an EDS of G. Then, $u, v \in S$. Suppose that $d_G(u, v) = 3$. Consider the induced subgraph $G^* =< N[u] \cup N[v] >$. As G is connected, G^* is also connected and $G^* \in \mathscr{E}$ with the set $S^* = \{u, v\}$ as its EDS. Then, a similar argument as in case(ii) of Theorem 4.2.15 leads to a contradiction to the hypothesis. Thus, $d_G(u, v) \geq 4$. □

It was shown in Theorem 4.2.12 that a graph $G \in \mathscr{G}_{-v}$ belongs to the class $CVR_{\mathscr{E}}$ if and only if $G \cong mK_1$, for $m \geq 1$. Hence, if $G \in \mathscr{E}$ and $G \ncong mK_1$, then G may or may not be in $UVR_{\mathscr{E}}$. The following section examines the properties of those graphs belonging to the class $UVR_{\mathscr{E}}$.

4.2.3 The $UVR_{\mathscr{E}}$ Class

Every graph has a dominating set. But not all graphs have an efficient dominating set. Hence, it follows from the definition of $UVR_{\mathscr{E}}$ that $UVR_{\mathscr{E}}$ is a subclass of UVR . In Section 4.2.1, some of the well-known graphs belonging to the class $UVR_{\mathscr{E}}$ were identified and discussed. This section attempts to further explore the existence of other graphs belong to the class $UVR_{\mathscr{E}}$ and provides a characterization for those graphs belonging to this class.

Theorem 4.2.17. *Let $G \in \mathscr{E}$ such that G has at least two disjoint efficient dominating sets. Then, $G - u \in \mathscr{E}$, for all $u \in V(G)$.*

Proof. Let $G \in \mathscr{E}$ and S_1 and S_2 be two efficient dominating sets of G such that $S_1 \cap S_2 = \emptyset$. Let $u \in V(G)$.

Case(i): $u \notin S_1 \cup S_2$

Clearly, both S_1 and S_2 are EDS of $G - u$. Hence, $G - u \in \mathscr{E}$. In particular, $\gamma(G - u) = |S_1| = |S_2| = \gamma(G)$ and hence, $u \in V^0$.

Case (ii): Either $u \in S_1$ or $u \in S_2$

Without loss of generality, let $u \in S_1$. Then, as $S_1 \cap S_2 = \emptyset$, $u \notin S_2$. Since $u \notin S_2$, by Case(i), S_2 is an EDS of $G - u$. Thus, $G - u \in \mathscr{E}$. Here, $\gamma(G - u) = \gamma(G)$ and $u \in V^0$.

Hence, it follows from both the cases that $u \in V^0$, for all $u \in V(G)$. □

Remark 4.2.9.

(i) It is evident from the proof of Theorem 4.2.17 that if G has at least two disjoint efficient dominating sets, then $G \in UVR_{\mathscr{E}}$.

(ii) The converse of the Theorem 4.2.17 is not true. For instance, if $n \not\equiv 2 \pmod 3$, $P_n - u \in \mathscr{E}$, for all $u \in V(P_n)$. But, P_n has a unique EDS when $n \not\equiv 2 \pmod 3$.

The following theorem gives a necessary and sufficient condition for a graph $G \in \mathscr{G}_{-v}$ to be in the $UVR_{\mathscr{E}}$ class.

Theorem 4.2.18. *Let G be a graph of order n, where $n \geq 2$. Then, $G \in UVR_{\mathscr{E}}$ if and only if G has k efficient dominating sets $S_1, S_2, \ldots, S_k$ ($k \geq 2$) such that $\cap_{i=1}^{k} S_i = \emptyset$.*

Proof. Let $S_1, S_2, \ldots, S_k$ be k distinct efficient dominating sets of G. Then, $|S_i| = \gamma(G)$, for all $i \in \{1, 2, \ldots, k\}$. Let $u \in V(G)$.

Case (i): $u \notin S_i$, for all $i \in \{1, 2, \ldots, k\}$.

Then, each S_i ($1 \leq i \leq k$) will be an EDS of $G - u$. Hence, $G - u \in \mathscr{E}$ and $\gamma(G - u) = \gamma(G)$. Therefore, $u \in V^0$.

Case (ii): $u \in S_i$ for some i, where $1 \leq i \leq k$.

Then, as $k \geq 2$ and $\cap_{i=1}^{k} S_i = \emptyset$, there exists at least one $j \neq i$ ($1 \leq i, j \leq k$) such that $u \notin S_j$. Since $u \notin S_j$, S_j is an EDS of $G - u$ and hence, $G - u \in \mathscr{E}$. Thus, $\gamma(G - u) = \gamma(G)$ and $u \in V^0$.

Thus, it follows from both the cases that $u \in V^0$, for all $u \in V(G)$ and hence, $G \in UVR_{\mathscr{E}}$.

Conversely, let $G \in UVR_{\mathscr{E}}$. Then, for each $u \in V(G)$, $G - u \in \mathscr{E}$ and $\gamma(G - u) = \gamma(G)$. In other words, $u \in V^0$, for all $u \in V(G)$. Further, $\gamma(G) \geq 1$.

Let S be an EDS of G and $u \in S$. Then, as $n \geq 2$, $V - S \neq \emptyset$. Let $v \in V - S$. Clearly, both u and v are in V^0. Therefore, it follows by Theorem 4.2.8 that, corresponding to the vertex $u \in S$, there exists an EDS of G, say S', which does not contain u. If $S \cap S' = \emptyset$, then the result holds. On the other hand, suppose $S \cap S' \neq \emptyset$, there exists a vertex $w \in S \cap S'$. Then, as $w \in V^0$, by a similar argument as above, there exists another EDS of G, say S'', not containing w. If $S \cap S' \cap S'' = \emptyset$, then the result holds. If not, then as G is finite, continuing the above process will result in at least two efficient dominating sets which satisfy the required conditions. Hence, the result follows. □

Remark 4.2.10. *It is evident from Theorem 4.2.18 that $G \in UVR_{\mathscr{E}}$ if and only if for each vertex $u \in V(G)$, there exists an EDS of G which does not contain u.*

Corollary 4.2.18.1. *Let $G \in \mathscr{G}_{-v}$. If G has at least two vertices of degree $(n-1)$, then $G \in UVR_{\mathscr{E}}$.*

4.3 Edge Removal

Similar to the categorization of graphs with respect to vertex removal, the following classes are defined with respect to the removal of edges:

- $\mathscr{G}_3 = \{G : G \in \mathscr{E} \text{ and } G - e \in \mathscr{E}, \text{ for some } e \in E(G)\}$
- $\mathscr{G}_4$ (or $\mathscr{G}_{-e}$) $= \{G : G \in \mathscr{E} \text{ and } G - e \in \mathscr{E}, \text{ for all } e \in E(G)\}$

In order to study the influence of edge removal on efficient domination, it is required that both G and $G - e$, for some $e \in E(G)$, to be efficiently dominatable. Hence, only those graphs G are considered, for which both $G \in \mathscr{E}$ and $G - e \in \mathscr{E}$, for some $e \in E(G)$. That is, graphs $G \in \mathscr{G}_3$. For $e \in E(G)$, the edge e is

(a) γ-critical if $\gamma(G - e) \neq \gamma(G)$

(b) γ^+-critical if $\gamma(G - e) > \gamma(G)$

(c) γ^--critical if $\gamma(G - e) < \gamma(G)$

Accordingly, the following two categorization of graphs are defined:

(a) $UER_{\mathscr{E}} = UER \cap \mathscr{G}_{-e}$

(b) $CER_{\mathscr{E}} = CER \cap \mathscr{G}_{-e}$

Remark 4.3.1. *(Haynes et al., 1998) Let $G \in \mathscr{G}_{-e}$. Then, the removal of an edge may increase the cardinality of an efficient dominating set by exactly one and in any case, it will not decrease $\gamma(G)$. Hence, if an edge $e \in E(G)$ is γ-critical, then $\gamma(G - e) = \gamma(G) + 1$. In other words, a γ-critical edge is always γ^+-critical and $ER^- = \emptyset$.*

4.3.1 Results on some well-known graphs

Proposition 4.3.1. *For $n \geq 1$, $K_{1,n} \in CER_{\mathscr{E}}$.*

Proof. Let $V(K_{1,n}) = \{u_0, u_1, \ldots, u_n\}$, where u_0 is the central vertex. Then, $S = \{u_0\}$ will be an EDS of $K_{1,n}$. For any edge $e \in E(K_{1,n})$, $e = u_0u_i$, where $i \neq 0$ and $1 \leq i \leq n$. Then, $K_{1,n} - e = K_{1,n-1} \cup \{u_i\}$. Thus, $\gamma(K_{1,n} - e) = \gamma(K_{1,n-1}) + 1 = 2 > \gamma(K_{1,n})$. Hence, $K_{1,n} \in CER_{\mathscr{E}}$. □

Proposition 4.3.2. *For $n \geq 2$, $K_n \in UER_{\mathscr{E}}$.*

Proof. Let $V(K_n) = \{u_1, \ldots, u_n\}$. Then, $S = \{u_i\}$, for any $1 \leq i \leq n$, will be an EDS of K_n. It can be observed that, for any edge $e = u_iu_j$, the set $S = \{u_k\}$, where $k \neq \{i, j\}$ and $k \in \{1, 2, \ldots, n\}$, still forms an EDS of $K_n - e$. Therefore, $\gamma(K_n) = \gamma(K_n - e)$, for all $e \in E(K_n)$. Hence, $K_n \in UER_{\mathscr{E}}$. □

Proposition 4.3.3. *For $n \geq 3$, $C_n \in UER_{\mathscr{E}}$, if and only if $n \equiv 0$ (mod 3).*

Proof. Let $V(C_n) = \{u_1, u_2, \ldots, u_n\}$ and $n \equiv 0$ (mod 3). Then, $C_n \in \mathscr{E}$ and $\gamma(C_n) = \frac{n}{3}$. For any edge $e \in E(C_n)$, and $e = u_iu_j$, for $1 \leq i, j \leq n$ and $i \neq j$, $C_n - e \cong P_{n-1}$. It follows that $n - 1 \equiv 2$ (mod 3) and $\gamma(C_n - e) = \gamma(P_{n-1}) = \lceil \frac{n-1}{3} \rceil = \frac{n}{3}$. Thus, $\gamma(C_n) = \gamma(C_n - e)$, for any $e \in E(C_n)$ and thus, $C_n \in UER_{\mathscr{E}}$. Conversely, Let $C_n \in UER_{\mathscr{E}}$. That is, $C_n \in UER \cap \mathscr{G}_{-e}$. If $n \not\equiv 0$ (mod 3), then $C_n \notin \mathscr{E}$ and hence $C_n \notin UER_{\mathscr{E}}$, which is a contradiction. Thus, $n \equiv 0$ (mod 3). □

Proposition 4.3.4. *For $n \geq 1$, $P_n \in UER_{\mathscr{E}}$ if and only if $n \equiv 1$ (mod 3).*

Proof. *Claim: $P_n \in \mathscr{G}_{-e}$.*

It is known that $P_n \in \mathscr{E}$. For any edge $e \in E(P_n)$, $P_n - e \cong P_i \cup P_{n-i}$ and hence $S' = S_i \cup S_{n-i}$, where S', S_i and S_{n-i} are respectively EDSs of $P_n - e$, P_i and P_{n-i}. Thus, $P_n - e \in \mathscr{E}$ and hence, $P_n \in \mathscr{G}_{-e}$.

For any edge $e \in E(P_n)$ and $e = uv$, the following three cases are discussed: (i) $u \notin S$ and $v \notin S$, (ii) $u \notin S$ and $v \in S$ and (iii) $u \in S$ and $v \notin S$.

Let $n \equiv 1$ (mod 3). Then, $\gamma(P_n) = \frac{n+2}{3}$, where the set $S = \{u_1, u_4, \ldots, u_n\}$ forms an unique EDS of P_n, in which u_1, u_n are pendant vertices. Let $e \in E(P_n)$ and $e = uv$. For any $u \notin S$ and $v \notin S$, S will be an EDS of $P_n - e$ and hence $\gamma(P_n) = \gamma(P_n - e)$. Let $u \notin S$ and $v \in S$. Then, $P_n - e \cong P_i \cup P_{n-i}$. Since,

$n \equiv 1 \pmod 3$, it follows that $i \equiv 0 \pmod 3$ and $n - i \equiv 1 \pmod 3$. Therefore, $\gamma(P_n - e) = \gamma(P_i) + \gamma(P_{n-i}) = \frac{i}{3} + \frac{n-i+2}{3} = \frac{n+2}{3} = \gamma(P_n)$. If $u \in S$ and $v \notin S$, then since, $n \equiv 1 \pmod 3$, it follows that $i \equiv 1 \pmod 3$ and $n - i \equiv 0 \pmod 3$. Therefore, $\gamma(P_n - e) = \gamma(P_i) + \gamma(P_{n-i}) = \frac{i+2}{3} + \frac{n-i}{3} = \frac{n+2}{3} = \gamma(P_n)$. Thus, in all these cases $\gamma(P_n) = \gamma(P_n - e)$, for all $e \in E(G)$ and hence $P_n \in UER_{\mathscr{E}}$.

Conversely, let $P_n \in UER_{\mathscr{E}}$. The following cases are considered:

Case (i): $n \equiv 2 \pmod 3$

In this case, $\gamma(P_n) = \frac{n+1}{3}$, where the set $S = \{u_2, u_5, \ldots, u_n\}$ forms an EDS of P_n. Let $e \in E(P_n)$ and $e = uv$. For $u \notin S$ and $v \notin S$, S will be an EDS of $P_n - e$ and hence $\gamma(P_n) = \gamma(P_n - e)$. Let $u \notin S$ and $v \in S$. Then, $P_n - e \cong P_i \cup P_{n-i}$. Since, $n \equiv 2 \pmod 3$, it follows that $i \equiv 1 \pmod 3$ and $n - i \equiv 1 \pmod 3$. Therefore, $\gamma(P_n - e) = \gamma(P_i) + \gamma(P_{n-i}) = \frac{i+2}{3} + \frac{n-i+2}{3} = \frac{n+4}{3} > \frac{n+1}{3} = \gamma(P_n)$. If $u \in S$ and $v \notin S$, then since, $n \equiv 2 \pmod 3$, it follows that $i \equiv 2 \pmod 3$ and $n - i \equiv 0 \pmod 3$. Therefore, $\gamma(P_n - e) = \gamma(P_i) + \gamma(P_{n-i}) = \frac{i+1}{3} + \frac{n-i}{3} = \frac{n+1}{3} = \gamma(P_n)$.

Case (ii): $n \equiv 0 \pmod 3$

In this case, $\gamma(P_n) = \frac{n}{3}$, where the set $S = \{u_2, u_5, \ldots, u_{n-1}\}$ forms an unique EDS of P_n. Let $e \in E(P_n)$ and $e = uv$. For $u \notin S$ and $v \notin S$, S will be an EDS of $P_n - e$ and hence $\gamma(P_n) = \gamma(P_n - e)$. Let $u \notin S$ and $v \in S$. Then, $P_n - e \cong P_i \cup P_{n-i}$. Since, $n \equiv 0 \pmod 3$, it follows that $i \equiv 1 \pmod 3$ and $n - i \equiv 2 \pmod 3$. Therefore, $\gamma(P_n - e) = \gamma(P_i) + \gamma(P_{n-i}) = \frac{i+2}{3} + \frac{n-i+1}{3} = \frac{n+3}{3} > \frac{n}{3} = \gamma(P_n)$. If $u \in S$ and $v \notin S$, then since, $n \equiv 0 \pmod 3$, it follows that $i \equiv 2 \pmod 3$ and $n - i \equiv 1 \pmod 3$. Therefore, $\gamma(P_n - e) = \gamma(P_i) + \gamma(P_{n-i}) = \frac{i+1}{3} + \frac{n-i+2}{3} = \frac{n+3}{3} > \frac{n}{3} = \gamma(P_n)$.

Hence, if $n \equiv 0 \pmod 3$ and $n \equiv 2 \pmod 3$, it follows that $P_n \notin UER_{\mathscr{E}}$. $\square$

4.3.2 Properties of Critical edges

In this section, some of the properties possessed by the critical edges in an efficiently dominatable graph are discussed and also, a characterization is obtained for any edge in an efficiently dominatable graph to be γ-critical.

In general, if $G \in \mathscr{E}$ and S is an EDS of G, then for any edge $e \in E(G)$, at most one of its end vertices can be in S. That is, if $e = uv$, then exactly one of the following conditions holds: (i) $u \in S$ or $v \in S$ (ii) $u \notin S$ and $v \notin S$. In the results to follow, the properties of a critical edge are analyzed by considering the two cases separately.

If $G \in \mathscr{E}$ and S is an EDS of G, then for each vertex $u \in S$, generate a set S' from $S - \{u\}$ using one of the following operations:

$\mathcal{O}_1'$: Taking $S' = S - \{u\}$ (or)

$\mathcal{O}_2'$: Replacing one or more vertices of $S - \{u\}$ by exactly one of their respective neighbors in $V - S - \{u\}$.

Theorem 4.3.5 guarantees that for any edge $e = uv$ in G, if S_e is an EDS of $G - e$ and S is an EDS of G containing either u or v, then it is always possible to relate S and S_e. Perhaps, this helps in generating an EDS of $G - e$ knowing an EDS of G.

Theorem 4.3.5. *Let $e = uv \in E(G)$. Let S and S_e be an EDS of G and $G - e$ respectively. For a vertex $u \in S$, if $S' = S_e - N_G[u]$, then one of the following holds:*

(i) $S' = \emptyset$

(ii) $S' = S - \{u\}$

(iii) S' *is a set generated from* $S - \{u\}$ *using the operation* $\mathcal{O}_2'$.

Proof. Let $u \in S$ and $S' = S_e - N_G[u]$. Clearly, if $\gamma(G) = 1$, then $S' = \emptyset$. So, assume that $\gamma(G) \geq 2$. Then, $S_e = S' \cup (N_G[u] \cap S_e)$. Since S_e is an EDS, it follows that $(N_G[u] \cap S_e) \cap N_G[S'] = \emptyset$. Suppose that $\gamma(G) \geq 2$ and $S_1 = S - \{u\}$. If $S_e = S_1 \cup (N_G[u] \cap S_e)$, then $S' = S_1 = S - \{u\}$. Otherwise, apply the operation $\mathcal{O}_2'$ repeatedly for elements of S_1 and generate S' until $S' \cup (N_G[u] \cap S_e)$ forms an EDS of $G - e$ and is equal to S_e. Thus in this case, S' is generated from $S - \{u\}$ using the operation $\mathcal{O}_2'$. □

Remark 4.3.2. *Let S' be a set generated from $S - \{u\}$ using the conditions in Theorem 4.3.5.*

(i) It follows from Theorem 4.3.5 that $|S'| = |S_1| = |S| - 1$.

(ii) For any edge $e = uv$ *in* $E(G)$*, either* $|N_G[u] \cap S_e| = 1$ *or* $|N_G[u] \cap S_e| = 2$.

It follows from the above discussion that under the conditions of Theorem 4.3.5, $|S_e| = |S'| + |N_G[u] \cap S_e|$.
If $|N_G[u] \cap S_e| = 1$*, then* $|S_e| = |S'| + 1 = |S|$ *which implies that* $e \in ER^0$.
If $|N_G[u] \cap S_e| = 2$*, then* $|S_e| = |S'| + 2 = |S| + 1$ *which implies that* $e \in ER^+$.

Suppose that $e \in E(G)$, where $e = uv$ and if S is an EDS of G not containing both u and v, then S will also be an EDS of $G - e$ and hence, the following theorem follows.

Theorem 4.3.6. *Let* $e \in E(G)$ *and* $e = uv$*. If there exists an EDS of* G*, say* S*, such that* $u \notin S$ *and* $v \notin S$*, then* $e \in ER^0$.

Proof. Let $S = \{u_1, u_2, \ldots, u_k\}$ be an EDS of G. Let $e \in E(G)$, where $e = uv$ be such that $u \notin S$, $v \notin S$. Then, the vertices u and v are efficiently dominated either by the same element, say u_1, of S or by two different elements of S, say $u \in N_G(u_1)$ and $v \in N_G(u_2)$, where $u_1, u_2 \in S$. In $G - e$, u and v are still dominated by the same elements of S. Thus, S will still remain as an EDS of $G - e$ and $\gamma(G - e) = \gamma(G)$. That is, $e \in ER^0$. □

Theorem 4.3.7. *Let* $G \in \mathscr{E}$ *and* $G - e \in \mathscr{E}$*, for* $e \in E(G)$*. Let* $e = uv$ *such that* $deg(v) = 1$*. Then,* e *is* γ*-critical if and only if* G *has an EDS not containing* v.

Proof. Let $e \in E(G)$, where $e = uv$ and $deg(v) = 1$. Then, $G - e \cong G_1 \cup G_2$, where $G_1 \cong G - v$ and $G_2 \cong K_1$ with $V(G_2) = \{v\}$. Let e be γ-critical and suppose that v is in every EDS of G. Then, clearly $\gamma(G) > 1$ and hence by Corollary 4.2.11, $v \in V^-$. Therefore, $\gamma(G_1) = \gamma(G) - 1$ and hence, $\gamma(G - e) = \gamma(G)$, contradicting that e is γ-critical.
Conversely, let S be an EDS of G not containing v, then S will also be an EDS of $G - v$. Hence, $\gamma(G - e) = \gamma(G) + 1$, which implies that $e \in ER^+$. In other words, e is γ-critical. □

Remark 4.3.3. *It is evident from Theorem 4.3.7 that if G has a unique EDS, say S and $v \in S$ where $deg(v) = 1$, then the edge incident with v is in ER^0.*

Corollary 4.3.7.1. *Let $G \in \mathscr{E}$ and $G - e \in \mathscr{E}$, for $e \in E(G)$. If $e \in ER^0$, where $e = uv$ and if u belongs to an EDS of G, then $deg(v) \geq 2$.*

Next, the properties of those edges with one of the end vertices in an EDS of G are examined. In the theorem to follow, a characterization is obtained for such an edge to be in ER^0.

Theorem 4.3.8. *Let $G \in \mathscr{E}$ and $G - e \in \mathscr{E}$, for $e \in E(G)$ and $e = uv$. Suppose that G has an EDS containing u. Then, $e \in ER^0$ if and only if v is not in any EDS of $G - e$.*

Proof. Suppose that v is not in any EDS of $G - e$. Let S_e be an EDS of $G - e$. Clearly $v \notin S_e$ and $|N_{G-e}[u] \cap S_e| = 1$. Two cases arise: $u \notin S_e$ and $u \in S_e$.
Suppose $u \notin S_e$, then S_e will also be an EDS of G and hence, $e \in ER^0$.
On the other hand, if $u \in S_e$, then as $v \notin S_e$, $|N_G[u] \cap S_e| = 1$. Hence, as discussed in Remark 4.3.2, $e \in ER^0$. Conversely, let $e \in ER^0$. Suppose that $G - e$ has an EDS, say S_e, such that $v \in S_e$. Then, $|N_G[u] \cap S_e| = 2$ and hence by Remark 4.3.2, $e \in ER^+$, which is a contradiction. Therefore, v is not in any EDS of $G - e$. □

Corollary 4.3.8.1. *Let $G \in \mathscr{E}$ and $G - e \in \mathscr{E}$, for $e \in E(G)$ where $e = uv$. If G has an EDS containing u, then $e \in ER^+$ if and only if $G - e$ has an EDS containing v.*

4.3.3 Efficiently Dominatable graphs belonging to the set $\mathscr{G}_{-e}$

In this section, the edge critical sets ER^0, ER^+ in the class of efficiently dominatable graphs are characterized. Also, two classes of graphs namely, $UER_{\mathscr{E}}$ and $CER_{\mathscr{E}}$ are defined and characterized. Throughout this section, it is assumed that every graph G belongs to the class $\mathscr{G}_{-e}$, unless stated otherwise.

Observation 4.3.1. *Let $e \in E(G)$, where $e = uv$. Let S be an EDS of G such that $u \in S$. If S_e, S_u denote EDS of $G - e$ and $G - u$ respectively, then $v \in S_e$ if and only if $v \in S_u$.*

Theorem 4.2.9 says that $V(G) = V^0 \cup V^-$ if and only if $|N_G(u) \cap S_u| \leq 1$, for all $u \in V(G)$. In other words, if there exists a vertex $v \in V(G)$ such that $v \notin N_G(u) \cap S_u$, for any EDS S_u of $G - u$, then $V(G) \neq V^+$.

Theorem 4.3.9. *Let $G \in \mathscr{G}_{-v}$ and $G - e \in \mathscr{E}$, for $e \in E(G)$ and $e = uv$. Suppose that G has an EDS containing u. Then, $e \in ER^0$ if and only if v is not in any EDS of $G - u$.*

Proof. Since $u \in S$, $u \in V^0$ or $u \in V^-$ or $u \in V^+$. Let S_e and S_u be EDS of $G - e$ and $G - u$ respectively. Assume that $v \notin S_u$. Then, it follows from Observation 4.3.1 that $v \notin S_e$. Hence, by Theorem 4.3.8, $e \in ER^0$.

Conversely, let $e \in ER^0$. Suppose that $S = V^0 \cup V^- \cup V^+$. Then, the following cases occur:

Case (i): $u \in V^0$

Then, by Theorem 4.2.9, $|N_G(u) \cap S_u| = 1$ and S_u is also an EDS of G. Suppose that $v \in S_u$. Then, by Observation 4.3.1, $v \in S_e$. Since $N_G[u]$ is efficiently dominated by either u or v, it follows that $u \in S_e$. As $u \in S_e$ and $v \in S_e$, Corollary 4.3.8.1 implies that $e \in ER^+$, which is a contradiction.

Case (ii): $u \in V^-$

Then, $|N_G(u) \cap S_u| = 0$ and $|S_u| = |S| - 1$. Then, it follows trivially that $v \notin S_u$, as $uv \in E(G)$.

Case (iii): $u \in V^+$

Then, $|N_G(u) \cap S_u| \geq 2$ and $|S_u| > |S|$.

Subcase (a): $|N_G(u) \cap S_u| = 2$

Let v, $w \in N_G(u) \cap S_u$. Then, w efficiently dominates u in $G - uv$, and v efficiently dominates u in $G - uw$. Hence, S_u is an EDS of both $G - uv$ and $G - uw$. Thus, $e \in ER^+$ and $e = uw \in ER^+$ and in both the cases $|S_e| = |S_u| = |S| + 1$. For $x \notin N_G(u) \cap S_u$, Observation 4.3.1 implies that $x \notin S_e$. Since u is in every EDS of G, $u \in S_e$. In this case $u \in S_e$ and $x \notin S_e$, and Theorem 4.3.8 implies that

$e = ux \in ER^0$.

Subcase (b): $|N_G(u) \cap S_u| > 2$

Since u is in every EDS of G, $u \in S_e$. If $v \in N_G(u) \cap S_u$, then by Observation 4.3.1, $v \in S_e$ and hence $e \in ER^+$. If $v \notin N_G(u) \cap S_u$, then $v \notin S_e$ and it follows that $e \in ER^0$. $\square$

Remark 4.3.4. *For any graph $G \in \mathscr{G}_{-e}$, it can be concluded from Theorems 4.3.8 and 4.3.9 together with Theorem 4.3.6 that $E(G) = ER^0$ if and only if for every edge $e = uv$ in $E(G)$, v is not in any EDS of $G - u$ if and only if for every $e = uv$ in $E(G)$, v is not in any EDS of $G - e$.*

The Property P:

In the discussions to follow, *a graph G is said to satisfy the property **P**, if for every pair of vertices u, $v \in V(G)$, there exists an EDS of G not containing both u and v.* All graphs having at least three pairwise disjoint efficient dominating sets satisfy Property **P**. For example, cycles C_{3n}, complete graphs K_n.

The characterization for the class $UER_{\mathscr{E}}$ follows.

Theorem 4.3.10. *$G \in UER_{\mathscr{E}}$ if and only if one of the following holds:*

*(i) Graph G satisfies Property **P**.*

(ii) If S is an EDS of G and $e \in E(G)$, where $e = uv$ such that one of its end vertices, say $u \in S$, then for every EDS S_u of $G - u$, either $N_G(u) \cap S_u = \emptyset$ or $N_G(u) \cap S_u$ is not unique.

Proof. The necessary condition follow from Theorems 4.3.6 and 4.3.9.

Conversely, let $G \in UER_{\mathscr{E}}$. Then, $e \in ER^0$, for all $e \in E(G)$. Let $e \in E(G)$, where $e = uv$ and S_e be an EDS of $G - e$. It follows from the Theorem 4.3.9 that $v \notin S_e$.

Case(i): $u \notin S_e$ and $v \notin S_e$.

Then, S_e is an EDS of G also. Therefore, there exists an EDS of G not containing both u and v. As this holds for all $e \in E(G)$, G satisfies Property **P**.

Case(ii): $u \in S_e$ and $v \notin S_e$.

By the discussion in the Theorem 4.3.9, it follows that $u \in S_e$ and $v \notin S_e$ will hold only if $v \notin N_G(u) \cap S_u$, for every EDS S_u of $G - u$. That is, for every EDS S_u of $G - u$, either $N_G(u) \cap S_u = \emptyset$ or $N_G(u) \cap S_u$ is not unique. □

Remark 4.3.5. *Let $G \in UER_{\mathscr{E}}$ and $G \in \mathscr{G}_{-v}$. It follows that if condition (ii) of Theorem 4.3.10 is satisfied, then $V(G) = V^0 \cup V^-$. Equivalently, if $G \in UER_{\mathscr{E}}$, then $V^+ = \emptyset$.*

Theorem 4.3.6 says that for any $e \in E(G)$ and $e = uv$, if $e \in ER^+$, then one its end vertices should be in an EDS of G. If this were to hold for all the edges of G, then $|E(< V - S >)| = 0$, where S is an EDS of G. Also, as G is connected, it follows that $\gamma(G) = 1$. This is stated in the result below.

Theorem 4.3.11. *For any graph G, $G \in CER_{\mathscr{E}}$ if and only if $G \cong K_{1,n}$.*

Next, the effect of edge removal is discussed on $G \in \mathscr{G}$ whose $\gamma(G) = 1$.

Theorem 4.3.12. *Let $G \in \mathscr{G}_{-e}$ with $\gamma(G) = 1$. If G has an unique EDS, then $E(G) = ER^0 \cup ER^+$.*

Proof. Let $e \in E(G)$, where $e = uv$ and S be an unique EDS of G. If $u \notin S$ and $v \notin S$, Theorem 4.3.6 implies that $e \in ER^0$. Without loss of generality, let $S = \{u\}$.

Case(i): $G - e$ is connected

Then, $rad(G - e) = 2$ and hence $\gamma(G - e) \geq 2$. Thus, $e \in ER^+$.

Case(ii): $G - e$ is disconnected

Let G_1 and G_2 be the two components of $G - e$. Since $\gamma(G) = 1$, it follows that $deg(v) = 1$. Let $G_1 = \{v\}$ and $G_2 = G - v$. Then, $\gamma(G_1) = 1$ and $\gamma(G_2) = \gamma(G)$ (since $v \notin S$). Therefore, $\gamma(G - e) = \gamma(G_1) + \gamma(G_2) = 1 + \gamma(G)$, which in turn implies that $e \in ER^+$.

Thus, in all these cases $E(G) = ER^0 \cup ER^+$. □

It is known that for G with $\gamma(G) = 1$, $V(G) = V^0 \cup V^+$. Hence, by Theorems 4.3.10 and 4.3.12, the result follows.

Theorem 4.3.13. *For every graph $G \in \mathscr{G}_{-e}$ with $\gamma(G) = 1$, $G \in UER_{\mathscr{E}}$ if and only if G satisfies Property $\boldsymbol{P}$.*

Proposition 4.3.14. *The property $\boldsymbol{P}$ does not hold for any efficiently dominatable tree.*

Proof. Let T be a tree and $T \in \mathscr{E}$. Since $\delta(T) = 1$, there can be at most two pair wise disjoint efficient dominating sets. Suppose that v_0 is a pendant vertex and $v_0 \in N_T(v_1)$. Then, every EDS of T contains either v_0 or v_1. Hence, property **P** does not hold for the edge $e = v_0v_1$. □

With the observation made in Proposition 4.3.14 and from Theorem 4.3.10, the result follows.

Theorem 4.3.15. *For any tree $T \in \mathscr{G}_{-v}$, $T \in UER_{\mathscr{E}}$ if and only if V^- forms an EDS of T.*

Proof. Let $T \in \mathscr{G}_{-v}$ and S be an EDS of T. Let $e \in E(T)$ and $e = uv$ such that one of its end vertices belong to S, say $u \in S$. Since $\delta(T) = 1$, for every EDS S_u of $G - u$, either $N_T(u) \cap S_u = \emptyset$ or $v \in N_T(u) \cap S_u$. By Proposition 4.3.14, it follows that $T \in UER_{\mathscr{E}}$ if and only if condition (ii) of Theorem 4.3.10 holds. Thus, $T \in UER_{\mathscr{E}}$ if and only if $S = V^-$, or in other words, V^- is an EDS of T. □

4.4 Edge Addition

Analogous to the classes of efficiently dominatable graphs defined with respect to vertex removal and edge removal, the following two classes are defined with respect to edge addition.

- $\mathscr{G}_3 = \{G : G \in \mathscr{E}$ and $G + e \in \mathscr{E}$, for some $e \in E(\overline{G})\}$
- $\mathscr{G}_4$ (or $\mathscr{G}_{+e}$) $= \{G : G \in \mathscr{E}$ and $G + e \in \mathscr{E}$, for all $e \in E(\overline{G})\}$

In order to study the influence of edge addition on efficient domination, it is required that both G and $G + e$ are efficiently dominatable. Hence, only those graphs G are considered where both G and $G+e$ are efficiently dominatable, where $e \in E(\overline{G})$. Equivalently, the graph $G \in \mathscr{G}_3$ is considered.

Similar to the categorization of graphs defined with respect to vertex removal and edge removal, the following categorization of graphs are defined, with respect to edge addition.

(a) $UEA_{\mathscr{E}} = UEA \cap \mathscr{G}_{+e}$

(b) $CEA_{\mathscr{E}} = CEA \cap \mathscr{G}_{+e}$

Remark 4.4.1. *(Haynes et al., 1998) Let $G \in \mathscr{G}_{+e}$. Adding an edge cannot increase the cardinality of an EDS of G, but can decrease $\gamma(G)$ by at most one. Hence, $EA^+ = \emptyset$ and for any graph G in $CEA_{\mathscr{E}}$, $\gamma(G+e) = \gamma(G) - 1$, for all $e \in E(\overline{G})$.*

4.4.1 Results on some well-known graphs

The following classes of graphs belong to the class $UEA_{\mathscr{E}}$:

Proposition 4.4.1. *For $n \geq 1$, $K_{1,n} \in UEA_{\mathscr{E}}$.*

Proof. Let $V(K_{1,n}) = \{u_0, u_1, \ldots, u_n\}$, where u_0 is the central vertex. Then, $S = \{u_0\}$ will be an EDS of $K_{1,n}$. For any edge $e \in E(\overline{G})$, $e = u_iu_j$, where $i \neq j$ and $1 \leq i, j \leq n$. It can be observed that S still forms an EDS of $G+e$. Therefore, $\gamma(G) = \gamma(G+e)$ and $e \in EA^0$, for any edge $e \in E(\overline{G})$. Hence, $K_{1,n} \in UEA_{\mathscr{E}}$. □

Proposition 4.4.2. *For $n \geq 1$, $C_{3n} \in UEA_{\mathscr{E}}$.*

Proof. Let $V(C_{3n}) = \{u_1, u_2, \ldots, u_{3n}\}$. Clearly, C_{3n} has three pairwise disjoint EDSs, namely, $S_1 = \{u_1, u_4, \ldots, u_{3n-2}\}$, $S_2 = \{u_2, u_5, \ldots, u_{3n-1}\}$ and $S_3 = \{u_3, u_6, \ldots, u_{3n}\}$. For any $e \in E(\overline{C_{3n}})$, $e = u_iu_j$, where $i \neq j$ and $1 \leq i, j \leq 3n$. Further, $j \neq i+1$ and $j \neq i-1$. Now, the following two cases are considered:

Case (i): $|i-j| \equiv 0 \pmod 3$

In this case, both u_i and u_j belong to the same EDS of C_{3n}. Without loss of generality, let u_i, $u_j \in S_1$. Then, as the sets S_2 and S_3 are EDSs of C_{3n}, not containing u_i and u_j, both S_2 and S_3 dominate $V(C_{3n}+e)$. Further, if u_i' and u_j' are the vertices in S_2 (or S_3) which dominate u_i and u_j, respectively, then, in $C_{3n}+e$, $d(u_i', u_j') = 3$ and $d(x, y) \geq 3$, for every other pair of vertices, $x, y \in S_2$ (or S_3). Therefore, in this case, both S_2 and S_3 are EDS of $C_{3n}+e$ and hence, $\gamma(C_{3n}) = \gamma(C_{3n}+e)$, for every $e \in E(\overline{C_{3n}})$.

Case (ii): $|i-j| \not\equiv 0 \pmod 3$

In this case, both u_i and u_j belong to different EDSs of C_{3n}. Without loss of generality, let $u_i \in S_1$ and $u_j \in S_2$. Then, the set S_3 is an EDS of C_{3n} not containing both u_i and u_j. By a similar argument as in Case (i), it can be observed that S_3 forms an EDS of $C_{3n}+e$ and hence, $\gamma(C_{3n}) = \gamma(C_{3n}+e)$, for every $e \in E(\overline{C_{3n}})$.

Thus, in both the cases, $e \in EA^0$, for every $e \in E(\overline{C_{3n}})$ and hence, $C_{3n} \in UEA_{\mathscr{E}}$.

□

Remark 4.4.2. *It can be observed in Propositions 4.4.1 and 4.4.2 that in both $K_{1,n}$ and C_{3n}, the existence of an EDS not containing the end vertices of any newly added edge, guarantees that their domination number does not alter due to edge addition and hence, they belong to the class $UEA_{\mathscr{E}}$. This property is generalized in Theorem 4.4.4 (or Remark 4.4.4) and is proved to be true for an arbitrary graph. This in turn results in the identification of few other well known graphs belonging to the class $UAE_{\mathscr{E}}$, as listed in Observation 4.4.1.*

4.4.2 Main Results

In this section, investigation is made on some properties of edges that are critical with respect to edge addition. Also, such critical edges are characterized.

In the following theorem, a constructive procedure is given to relate an EDS of G with an EDS of $G+e$, which helps further in comparing the $\gamma(G)$ value of G and $G+e$.

Theorem 4.4.3. *Let $G \in \mathscr{E}$ and $e \in E(\overline{G})$, where $e = uv$. If G has an EDS containing both u and v and if S' is an EDS of $G+e$, then $|S' - (N_G[u] \cup N_G[v])| = \gamma(G) - 2$.*

Proof. Let $\gamma(G) = k$ and $S = \{u, v, u_1, u_2, \ldots, u_{k-2}\}$ be an EDS of G. Let $S_1 = S - \{u, v\}$ and S' be an EDS of $G + e$. Then, as S is a 2-packing of G, $N_G[u] \cap N_G[u_i] = \emptyset$ and $N_G[v] \cap N_G[u_i] = \emptyset$, for $1 \leq i \leq k - 2$. Let $T = S' \cap (N_G[u] \cup N_G[v])$. Clearly, $T \subseteq S'$ and $T \neq \emptyset$. Further, $S_1 \neq S'$. Now, by using one of the following two operations, we generate a set S'_1 from S_1:
(i) If $S_1 \subset S'$, then, take $S'_1 = S_1$.
(ii) Else, for each vertex $x \in S_1 - S'$, replace x by the unique vertex in $N_G[x] \cap S'$. (*As S' is an EDS of $G + e$, for each $x \in S_1 - S'$, the existence of such a unique neighbor is guaranteed in S'.*) Let the new set generated be S'_1.
Clearly, in either case, $|S'_1| = |S_1| = |S| - 2 = \gamma(G) - 2$. Further, as S' is a 2-packing of $G + e$, $T \cap N_G[S'] = \emptyset$ or precisely, $T \cap S'_1 = \emptyset$. Also, $S' \supseteq S'_1 \cup T$.
Claim: $S' = S'_1 \cup T$
Let $S^* = S'_1 \cup T$. Suppose there exists a vertex $w \in S' - S^*$. Then, as S' is a 2-packing of $G + e$, $N_G[w] \cap N_G[S^*] = \emptyset$. As $S \subseteq N[S^*]$, it follows that $N_G[w] \cap S = \emptyset$, contradicting that S is an EDS of G. Thus, $S' = S'_1 \cup T$. Hence, $|S'| = |S'_1| + |T| = \gamma(G) - 2 + |T|$. This implies that $|S'| - |T| = \gamma(G) - 2$. That is, $|S' - (N_G[u] \cup N_G[v])| = \gamma(G) - 2$. □

Remark 4.4.3. *If $T = S' \cap (N_G[u] \cup N_G[v])$, then as S' is a 2-packing of $G + e$, $|T|$ is either 1 or 2. Thus, it follows from the discussion in Theorem 4.4.3 that if $|T| = 1$, then $|S'| = |S'_1| + 1 = \gamma(G) - 1$. On the other hand, if $|T| = 2$, then $|S'| = |S'_1| + 2 = \gamma(G)$.*

Corollary 4.4.3.1. *Let $G \in \mathscr{E}$ and let $e \in E(\overline{G})$, where $e = uv$. If G has an EDS containing both u and v and S' is an EDS of $G + e$, then $e \in EA^0$ if and only if $|S' \cap (N_G[u] \cup N_G[v])| = 2$.*

Suppose $G \in \mathscr{E}$ and S is an EDS of G, then for any nonadjacent vertex pairs, say u and v in G, the following cases arise: (i) $u \notin S$ and $v \notin S$ (ii) $u \in S$ and $v \in S$ (iii) $u \in S$ and $v \notin S$. Based on these cases, the study is done on the effect of adding an edge between a pair of vertices, which are not adjacent in G. Each of these cases are discussed and characterizations are obtained for critical edges.

Initially, in Theorem 4.4.4 and Corollary 4.4.4.1, the study is made on the effect of adding an edge between a pair of nonadjacent vertices, both of which either belong to or do not belong to at least one EDS of G (cases (i) and (ii) stated above) and obtain a characterization for such an edge to be in the critical sets EA^0 and EA^-. Later in Theorem 4.4.5 and Corollary 4.4.5.1, the discussion is made on the effect of adding an edge falling under Case (iii) stated above and obtain an independent characterization for such an edge to be in EA^0 and EA^-.

Theorem 4.4.4. *Let $G \in \mathscr{E}$, $G + e \in \mathscr{E}$, for $e \in E(\overline{G})$ and $e = uv$. If either both u and v belong to an EDS of G, or both do not belong to an EDS of G, then, $e \in EA^0$ if and only if $G + e$ has an EDS not containing both u and v.*

Proof. Let S' be an EDS of $G+e$, not containing both u and v, then there exist a pair of vertices, say u' and v' in S', (where u' and v' may or may not be distinct) such that $u, v \in N_{G+e}(u') \cup N_{G+e}(v')$. Clearly, $d_{G+e}(u', v')$ is either 0 or 3, which implies that $d_G(u', v') = 0$ or $d_G(u', v') \geq 3$. Further, the remaining vertices can be efficiently dominated in G, by the same vertices as in $G + e$. Thus, S' will be an EDS of G also. Hence, $|S'| = \gamma(G)$ and consequently, $e \in EA^0$.
Conversely, let $e \in EA^0$. Suppose that one of the end vertices of e belong to S'. Without loss of generality, let $u \in S'$. Since $d_{G+e}(u, N_G[v]) \leq 2$, $N_G[v] \cap S' = \emptyset$. Therefore, $|S' \cap (N_G[u] \cup N_G[v])| = 1$ and hence, by Theorem 4.4.3, $|S' - (N_G[u] \cup N_G[v])| = |S'| - 1 = \gamma(G) - 2$. That is, $|S'| = \gamma(G) - 1$ which implies that $uv \in EA^-$, a contradiction. Hence, the result follows. □

Remark 4.4.4. *Precisely, it follows from Theorem 4.4.4 that for any nonadjacent vertex pairs u and v in G, if G has an EDS, say S, not containing both u and v, then $uv \in EA^0$. For, S itself will be an EDS of $G + uv$, as well.*

Corollary 4.4.4.1. *Let $G \in \mathscr{E}$, $G + e \in \mathscr{E}$, for $e \in E(\overline{G})$ and $e = uv$. If G has either an EDS containing both u and v or an EDS not containing both u and v, then $e \in EA^-$ if and only if every EDS of $G + e$ contains either u or v (but not both).*

Theorem 4.4.5. *Let $G \in \mathscr{E}$ and $G + e \in \mathscr{E}$, for $e \in E(\overline{G})$, where $e = uv$. If S is any EDS of G such that $u \in S$ and $v \notin S$, then $e \in EA^0$ if and only if $G + e$ also has an EDS, say S', such that $v \notin S'$.*

Proof. Let S be an EDS of G. Without loss of generality, let $u \in S$ and $v \notin S$. Then there exists say $v' \in S$, such that $v \in N_G(v')$.
Let S' be an EDS of $G + e$ such that $v \notin S'$. Then, the following cases arise:
Case (i): $u \in S'$
In this case, in $G + e$, u will efficiently dominate $N_G[u]$ and v. As $d_{G+e}(u, v') = 2$, the vertex v' must be efficiently dominated in $G+e$ by exactly one of its neighbors in $N_G(v')$ other than v. Therefore, $|S' \cap (N_G[u] \cup N_G[v'])| = 2$ and hence by Corollary 4.4.3.1, $e \in EA^0$.
Case (ii): $u \notin S'$
In this case, since S' is an EDS of $G + e$ and as $u \notin S'$, $v \notin S'$, it follows that u is dominated by exactly one of its neighbors in $G + e$, other than v. Similarly, v is dominated by exactly one of its neighbors in $G + e$, other than u. That is, $|S' \cap N_{G+e}[u]| = |S' \cap N_{G+e}[v]| = 1$. Further, $|S' \cap (N_G[u] \cup N_G[v])| = 2$. Therefore, by Corollary 4.4.3.1, $e \in EA^0$.
Conversely, let $e \in EA^0$. Suppose that S' is an EDS of $G+e$ such that $v \in S' - S$. Then, in $G+e$, v will efficiently dominate u and v'. Further, for each $x \in N_G(u) \cup (N_G(v') - \{v\})$, $d_{G+e}(x, v) = 2$. Therefore, except for v, none of the other vertices in $N_G[u] \cup N_G[v']$ will belong to S'. In other words, $|S' \cap (N_G[u] \cup N_G[v'])| = 1$. Hence, it follows from Theorem 4.4.3 that $|S'| = \gamma(G) - 1$. Therefore, $e \in EA^-$, which contradicts our hypothesis and hence, $v \notin S'$. □

Corollary 4.4.5.1. *Let $G \in \mathscr{E}$ and $G + e \in \mathscr{E}$, for $e \in E(\overline{G})$, where $e = uv$. If S is any EDS of G such that $u \in S$ and $v \notin S$, then $e \in EA^-$ if and only if $G + e$ has an EDS containing v.*

In Section 4.4.2, the edge critical properties in graphs G are discussed, where $G \in \mathscr{E}$ and $G + e \in \mathscr{E}$, for some $e \in E(\overline{G})$. That is, those graphs in $\mathscr{G}_3 \cup \mathscr{G}_{+e}$ were analyzed. In the next section, the study is done exclusively on those graphs belonging to the class $\mathscr{G}_{+e}$.

4.4.3 Changing and Unchanging domination in graphs belonging to the class $\mathscr{G}_{+e}$

In this section, the classes $UEA_{\mathscr{E}}$ and $CEA_{\mathscr{E}}$ are investigated. It is clear from the definition that to study the two classes $UEA_{\mathscr{E}}$ and $CEA_{\mathscr{E}}$, it is necessary for every edge in $E(\overline{G})$ to preserve the efficient domination property. Thus, it is assumed throughout this section that $G \in \mathscr{G}_{+e}$, unless specified otherwise.

Theorem 4.4.6. *If $G \in \mathscr{G}_{+e}$, then $G \in CEA_{\mathscr{E}}$ if and only if $G \cong mK_1$, for $m \geq 1$.*

Proof. Let $G \in \mathscr{G}_{+e}$. If $G \cong mK_1$ $(m \geq 1)$, then for every $e \in E(\overline{G})$, $\gamma(G+e) < \gamma(G)$ and hence, $G \in CEA_{\mathscr{E}}$. Conversely, suppose that $G \in CEA_{\mathscr{E}}$ and S be an EDS of G. Suppose $G \neq mK_1$, for any $m \geq 1$. Without loss of generality, let G be connected. Then, $G \neq K_1$ and $S \subsetneq V(G)$. Hence, there always exist a pair of nonadjacent vertices u and v in G such that $u \notin S$ and $v \notin S$. It follows from Theorem 4.4.4 (or Remark 4.4.4) that the edge $uv \in EA^0$, contradicting that $G \in CEA_{\mathscr{E}}$. Hence, $G \cong mK_1$, where $m \geq 1$. □

The following theorem by Carrington et al. (1991); Haynes et al. (1998) gives a characterization for the class UEA.

Theorem 4.4.7. *(Haynes et al., 1998) $G \in UEA$ if and only if $V^- = \emptyset$.*

Next, the class $UEA_{\mathscr{E}}$ is characterized when $\gamma(G) = 1$ and when $\gamma(G) \geq 2$; some necessary/sufficient conditions are obtained for which G lies in the class $UEA_{\mathscr{E}}$.

Theorem 4.4.8. *If $\gamma(G) = 1$, then $G \in UEA_{\mathscr{E}}$.*

Proof. Let $\gamma(G) = 1$. Then, $G \in \mathscr{E}$. Let $S = \{x\}$ be an EDS of G. Let $e \in E(\overline{G})$, where $e = uv$. Clearly, $u \neq x$ and $v \neq x$. That is, $u, v \in V - S$. Therefore, it follows from Theorem 4.4.4 (or Remark 4.4.4) that $e \in EA^0$. Since e is arbitrary, $e \in EA^0$, for all $e \in E(\overline{G})$ and thus, $G \in UEA_{\mathscr{E}}$. □

Proposition 4.4.9. *Let $G \in \mathscr{G}_{-v}$. Let $e \in E(\overline{G})$, where $e = uv$ and S' be an EDS of $G + e$. If $u \in V^+$, then $u \in S'$.*

Proof. Let $u \in V^+$. Then, it follows from Theorem 4.2.8 that u is in every EDS of G. Let $e = uv$ and S' be an EDS of $G + e$. Suppose S' does not contain both u and v, then S' will be an EDS of G also, contradicting that u is in every EDS of G. Then, S' must contain either u or v.

Now, suppose $u \notin S'$. Then, $v \in S'$ and as $d(v, x) = 2$, for all $x \in N_G[u]$, $S' \cap N_G[u] = \emptyset$. Further, S' efficiently dominates all except $N_G[u]$ in $V(G)$. Hence, S' will be an EDS of $G - u$ also and thus, $u \in V^-$ which is a contradiction. Thus, $u \in S'$. □

Theorem 4.4.10. *Let $G \in \mathscr{G}_{-v}$ and $V^+ \neq \emptyset$. Then, $G \in UEA_{\mathscr{E}}$ if and only if $\gamma(G) = 1$.*

Proof. The sufficient part follows from Theorem 4.4.8. Conversely, let $G \in UEA_{\mathscr{E}}$ and S be an EDS of G. Suppose that $|S| = \gamma(G) = k$, where $k > 1$. As $V^+ \neq \emptyset$, there exists, say $u \in V^+$. Then, it follows from Theorem 4.2.8 that $u \in S$. Also, since $\gamma(G) > 1$, there exists $v \in S$ such that u and v are nonadjacent in G. Now, consider the graph $G + uv$, in which $u, v \in S$. If S' is any EDS of $G + uv$, then by Proposition 4.4.9, $u \in S'$ and by Corollary 4.4.5.1, $uv \in EA^-$, which is a contradiction. Thus, $\gamma(G) = 1$. □

Theorem 4.4.11. *Let $G \in \mathscr{G}_{-v}$. If $\gamma(G) \geq 2$ and $G \in UEA_{\mathscr{E}}$, then $V^+ = \emptyset$ and $V^- = \emptyset$. Equivalently, $V(G) = V^0$.*

Proof. Let $G \in UEA_{\mathscr{E}}$ and $\gamma(G) \geq 2$. Suppose $V^+ \cup V^- \neq \emptyset$. Let $u \in V^+ \cup V^-$. Then, either $u \in V^-$ or $u \in V^+$ or u lies in both. The following cases are considered:

Case (i): $u \in V^-$

Then, as $V^- \neq \emptyset$, by Theorem 4.4.7, $G \notin UEA$, contradicting our assumption that $G \in UEA_{\mathscr{E}}$.

Case (ii): $u \in V^+$

Let S be an EDS of G containing v and S' be any EDS of $G+uv$. As $u \in V^+ \cup V^-$, by Theorem 4.2.8, $u \in S$. Further, as $u \in V^+$, by Proposition 4.4.9, $u \in S'$. Since S' is arbitrary, it follows from Corollary 4.4.4.1 that $uv \in EA^-$, contradicting that $G \in UEA_{\mathscr{E}}$.

Hence, from both the cases it follows that if $G \in UEA_{\mathscr{E}}$, then $V^{-} = \emptyset$ and $V^{+} = \emptyset$. Equivalently, $V(G) = V^{0}$. □

If $G \in \mathscr{E}$ and satisfies property **P**, then for any edge $e \in E(\overline{G})$, it follows from Theorem 4.4.4 (or Remark 4.4.4) that $e \in EA^{0}$. This fact leads to the following theorem.

Theorem 4.4.12. *Let $G \in \mathscr{E}$. If G satisfies property **P**, then $G \in UEA_{\mathscr{E}}$.*

Similar to the well known graphs identified to be in the class $UEA_{\mathscr{E}}$ in Section 4.4.1, by using Theorem 4.4.12, a few more well known graphs belonging to the class $UAE_{\mathscr{E}}$ are identified and listed them in Observation 4.4.1.

Observation 4.4.1. *The following are some of the well known graphs satisfying the hypotheses of Theorem 4.4.12 and hence, belong to the class $UEA_{\mathscr{E}}$:*

1. *Wheel graphs: W_n $(= C_{n-1} \circ K_1)$*
2. *Fan graphs: F_n $(= P_{n-1} \circ K_1)$*
3. *Hypercube: Q_3 $(= \Box_{i=1}^{3} K_2)$*
4. *$K_m \Box K_{1,m}$, for $m \geq 3$*
5. *$C_{3n} \Box K_{1,3}$, for $n \geq 1$*
6. *$K_{m,n} \Box K_m$, for $m \geq 3$*
7. *$K_{m,n} \Box K_n$, for $n \geq 3$*

4.4.4 The Classes of graph $G \notin \mathscr{G}_{+e}$

As mentioned earlier, not all efficiently dominatable graphs belong to the class $\mathscr{G}_{+e}$. For instance, $P_n \notin \mathscr{G}_{+e}$, for all n. In this section, some classes of graphs that do not belong to the class $\mathscr{G}_{+e}$ are explored.

Theorem 4.4.13. *Let $G \in \mathscr{G}_{-v}$ and $\gamma(G) \geq 2$. If $S = V^{+}$, then $G \notin \mathscr{G}_{+e}$.*

Proof. Let S be an EDS of G and $S = V^+$. Suppose that $G \in \mathscr{G}_{+e}$. Let $u, v \in S$. Then, $G + uv \in \mathscr{E}$. Let S' be any EDS of $G + uv$. As $S = V^+$, both u and v are in V^+, which in turn implies by Proposition 4.4.9 that $u \in S'$ and $v \in S'$. But, this leads to a contradiction that S' is a 2-packing of $G + uv$. Hence, $G \notin \mathscr{G}_{+e}$. □

Proposition 4.4.14. *For any tree T with $\gamma(T) = 2$ and $T \in \mathscr{E}$, $T \notin \mathscr{G}_{+e}$.*

Proof. Any tree $T \in \mathscr{E}$ having $\gamma(T) = 2$ will be isomorphic to the graph in Figure 4.5. Let $S = \{u, v\}$ be an EDS of T. Suppose that $T \in \mathscr{G}_{+e}$ and S' is an EDS of $T + uv$. Since there exists no EDS of $T + uv$ not containing both u and v, it follows that either $u \in S'$ or $v \in S'$. Suppose that $u \in S'$. Then, as $d_G(x, u) = 2$, for all $x \in N_G(v)$, all the vertices in $N_G(v)$ are not dominated efficiently by S', contradicting that S' is an EDS of $T + uv$. A similar contradiction arises when $v \in S'$. Hence, $T \notin \mathscr{G}_{+e}$. □

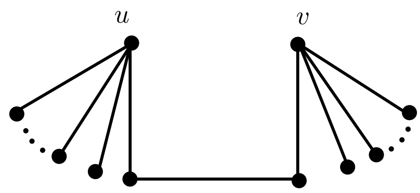


Figure 4.5: An efficiently dominatable tree with an EDS $S = \{u, v\}$

Let $G \in \mathscr{E}$ and S be any EDS of G. If the induced subgraph $G^* \cong \langle V - S \rangle$ is complete, then $diam(G) = 3$ and therefore $\gamma(G) \geq 2$. Also, all the vertices in G^* are of eccentricity two and hence cannot be in S. Thus, S is unique in this case.

Theorem 4.4.15. *Let $G \in \mathscr{E}$ and $\gamma(G) \geq 2$. If S is an EDS of G and the induced subgraph $\langle V - S \rangle$ is complete, then $G \notin \mathscr{G}_{+e}$.*

Proof. As the induced subgraph $\langle V - S \rangle$ is complete, for any pair of nonadjacent vertices $u, v \in V(G)$, at least one of u or v must be in S. Suppose that $G \in \mathscr{G}_{+e}$. Let $uv \in E(\overline{G})$ and S' be an EDS of $G + uv$. The following cases are considered:
Case (i): $\gamma(G) = 2$.
Subcase(a): $u \in S$ and $v \notin S$
Let $S = \{u, w\}$ be an EDS of G. Then, $v \in N_G(w)$. In $G + uv$, $deg_{G+uv}(v) = n - 1$

and thus $S' = \{v\}$ is an EDS of $G + uv$. Thus, $\gamma(G + uv) = 1 < \gamma(G)$, which implies that $uv \in EA^{-}$.

Subcase(b): $u \in S$, $v \in S$

Then, $diam(G + uv) = 2$ and thus $G + uv \notin \mathscr{E}$.

Case (ii): $\gamma(G) > 2$

In this case, any pair of vertices in S are mutually at a distance three from each other. Also, every vertex in $V - S$ is of eccentricity two and hence none of them will belong to S'. The following subcases are considered:

Subcase(a): $u \in S$, $v \notin S$

Let $v \in N_G(u')$, where $u' \in S$. To dominate u, either $u \in S'$ or $v \in S'$ or any one of the vertices of $N_G(u)$ should be a member of S'. If $u \in S'$, then u will dominate $N_G[u]$ and v. Since $d_{G+uv}(u, u') = 2$, $u' \notin S'$. To dominate u', one of the vertices in $N_G(u')$ other than v must belong to S'. But this is not possible as $N_G(u') \subset V - S$. Hence u' is left undominated efficiently by S'. Therefore $u \notin S'$. By a similar argument, it can be shown that $v \notin S'$. Also, by the above discussion, no vertex of $N_G(u)$ will be a member of S', contradicting that S' is an EDS of $G + uv$. Thus, $G + uv \notin \mathscr{E}$.

Subcase(b): $u \in S$, $v \in S$

Clearly, $u \notin S'$. Because, if $u \in S'$, then all the vertices of $N_G(v)$ will be left undominated efficiently. Similarly, $v \notin S'$. Also, no vertex of $V - S$ can be a member of S', contradicting that S' is an EDS of $G + uv$. Thus, $G + uv \notin \mathscr{E}$. □

4.5 Relationship among the classes

In this section, throughout it is assumed that $G \neq K_n$ and $G \in \mathscr{G}_{-v} \cap \mathscr{G}_{-e} \cap \mathscr{G}_{+e}$. Here, the relationship is discussed among the classes arising from the changing/unchanging efficient domination with respect to vertex removal, edge removal and edge addition and represent through the Venn diagram.

4.5.1 Results on some well-known graphs

1. $K_{1,n} \in UEA_{\mathscr{E}} \cap CER_{\mathscr{E}}$ and $V(K_{1,n}) = V^0 \cup V^+$.

2. $K_n \in UVR_{\mathscr{E}} \cap UER_{\mathscr{E}}$, for $n \geq 3$.

3. $C_{3n} \in UVR_{\mathscr{E}} \cap UER_{\mathscr{E}} \cap UEA_{\mathscr{E}}$.

4. $P_n \in UER_{\mathscr{E}}$ and $V(P_n) = V^0 \cup V^-$, when $n \equiv 1 \pmod 3$.
   When $n \equiv 2 \pmod 3$, $P_n \in UVR_{\mathscr{E}}$ and $E(P_n) = ER^0 \cup ER^+$.
   When $n \equiv 0 \pmod 3$, $E(P_n) = ER^0 \cup ER^+$ and $V(P_n) = V^0 \cup V^+$.
   But, $P_n \notin \mathscr{G}_{+e}$.

Proposition 4.5.1. *Let $G \in \mathscr{G}_{-v} \cap \mathscr{G}_{-e} \cap \mathscr{G}_{+e}$. Then the following conditions hold.*

(i) $CER_{\mathscr{E}} \subset UEA_{\mathscr{E}}$

(ii) $CER_{\mathscr{E}} \cap UVR_{\mathscr{E}} = \emptyset$

(iii) $UVR_{\mathscr{E}} \subset UEA_{\mathscr{E}}$

Proof. (i) If $G \in CER_{\mathscr{E}}$, then by Theorem 4.3.11, $G \cong K_{1,n}$. By Theorem 4.4.10, as $\gamma(K_{1,n}) = 1$, $G \in UEA_{\mathscr{E}}$. Hence, $CER_{\mathscr{E}} \subseteq UEA_{\mathscr{E}}$. Since all graphs G with $\gamma(G) = 1$ and $G \neq K_{1,n}$ belong to $UEA_{\mathscr{E}}$ class but does not belong to the $CER_{\mathscr{E}}$ class, $CER_{\mathscr{E}} \subset UEA_{\mathscr{E}}$.
(ii) If $G \in CER_{\mathscr{E}}$, then by Theorem 4.3.11, $G \cong K_{1,n}$. Since $V(K_{1,n}) = V^0 \cup V^+$, it follows that $G \notin UVR_{\mathscr{E}}$. Thus, the classes $CER_{\mathscr{E}}$ and $UVR_{\mathscr{E}}$ are disjoint. That is, $CER_{\mathscr{E}} \cap UVR_{\mathscr{E}} = \emptyset$.
(iii) Let S' be an EDS of $G + uv$. Suppose that $G \notin UEA_{\mathscr{E}}$. Then, there exists $e \in E(\overline{G})$, where $e = uv$, such that $\gamma(G + uv) = \gamma(G) - 1$. Since $uv \in EA^-$, one of the following cases hold (by Corollaries 4.4.4.1, 4.4.5.1).
Case (i): If $u \in S$ and $v \in S$, then either $u \in S'$ or $v \in S'$.
Suppose that $u \in S'$. Then, for all $x \in N_G[v]$, $d_{G+uv}(u, x) \leq 2$ and $N[v] \cap S' = \emptyset$. Thus, S' is an EDS of $G - v$, where $|S'| < |S|$. Hence, $v \in V^-$.
Case (ii): If $v \notin S$, then $v \in S'$.
In this case, for all $x \in N_G[u]$, $d_{G+uv}(v, x) \leq 2$ and hence $N_G[u] \cap S' = \emptyset$. Thus, S' is an EDS of $G - u$, where $|S'| < |S|$ and hence $u \in V^-$.
Thus, in both of the cases it is observed that if $G \notin UEA_{\mathscr{E}}$, then $V^- \neq \emptyset$, which

in turn implies that $G \notin UVR_{\mathscr{E}}$. That is, $UVR_{\mathscr{E}} \subseteq UEA_{\mathscr{E}}$. But, as an instance, $G \cong K_{1,n} \in UEA_{\mathscr{E}}$ and $K_{1,n} \notin UVR_{\mathscr{E}}$. Thus, $UVR_{\mathscr{E}} \subset UEA_{\mathscr{E}}$. □

Theorem 4.5.2. *For any graph G, G has at least three pairwise disjoint efficient dominating sets if and only if $G \in UVR_{\mathscr{E}} \cap UER_{\mathscr{E}} \cap UEA_{\mathscr{E}}$.*

Proof. Let $S_1, S_2, \ldots, S_k$, for $k \geq 3$, be EDSs of G such that $S_i \cap S_j = \emptyset$, for $1 \leq i, j \leq k$ and $i \neq j$. Since G satisfies property **P** (by Proposition 3.1.18), Theorems 4.2.18, 4.3.10, 4.4.12 imply that $G \in UVR_{\mathscr{E}} \cap UER_{\mathscr{E}} \cap UEA_{\mathscr{E}}$.
Conversely, let $G \in UVR_{\mathscr{E}} \cap UER_{\mathscr{E}} \cap UEA_{\mathscr{E}}$. Then, $V(G) = V^0$. By Proposition 4.5.1, $UVR_{\mathscr{E}} \subset UEA_{\mathscr{E}}$. Thus, $G \in UVR_{\mathscr{E}} \cap UER_{\mathscr{E}}$. Let $e \in E(G)$, where $e = uv$. Since $G \in UER_{\mathscr{E}}$ and $V(G) = V^0$, Theorem 4.3.10 implies that either G satisfies Property **P** or $N_G(u) \cap S_u$ is not unique, where S_u is an EDS of $G - u$. That is, for every u in S, at least two neighbors exist, say $v, w \in N(u)$, so that v and w are in distinct EDS of G. Since, this is true for all the vertices in S, G must have at least three pairwise efficient dominating sets. Hence, the result follows. □

4.5.2 Representation of different classes

Motivated by the representation in Haynes and Henning (2003), an attempt is made to represent the different classes of the efficiently dominatable graphs through Venn diagram.
To represent graph classes as in Figure 4.6, it is assumed that graphs G considered are connected and not complete, $G \in \mathscr{E}$ and $G \in \mathscr{G}_{-v} \cap \mathscr{G}_{-e} \cap \mathscr{G}_{+e}$.

If $G \in \mathscr{E}$, then $G \notin CVR_{\mathscr{E}}$ and $G \notin CEA_{\mathscr{E}}$. Also, $UVR_{\mathscr{E}} \subset UEA_{\mathscr{E}}$ and $UVR_{\mathscr{E}} \cap CER_{\mathscr{E}} = \emptyset$. Thus, an efficiently dominatable graph is represented in only four classes, as in Venn diagram given in Figure 4.6. The regions of the Venn diagram are labeled from R_1 to R_7, as in Figure 4.7.

The following observations are made:

(a) **The Region R_6:**

For any graph G, $G \in R_6$ if and only if $G \in UEA_{\mathscr{E}} \cap UER_{\mathscr{E}} \cap UVR_{\mathscr{E}}$. Equivalently, $G \in R_6$ if and only if G satisfies property **P**, that is, if and

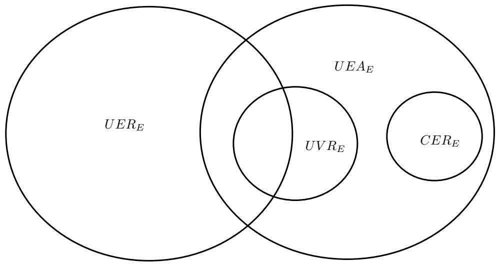


Figure 4.6: The classes of changing and unchanging efficiently dominatable graphs

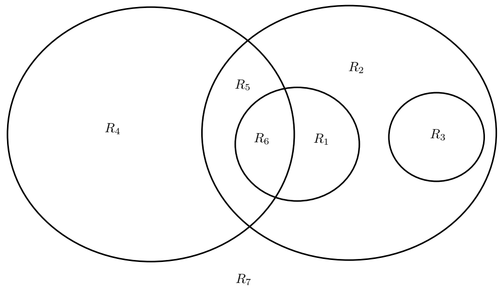


Figure 4.7: Representations of Regions

only if G has at least three pairwise disjoint efficient dominating sets. For example, for any n, $C_{3n} \in R_6$.

(b) **The Region R_3:**

$G \in CER_{\mathscr{E}}$ if and only if $G \cong K_{1,n}$, for $n \geq 2$. Thus, $R_3 = \{K_{1,n} : n \geq 2\}$.

(c) **The Region** R_2:

For any graph G, $G \in R_2$ if and only if $G \in UEA_{\mathscr{E}}$ and $G \notin UVR_{\mathscr{E}} \cap UER_{\mathscr{E}} \cap CER_{\mathscr{E}}$. In this region, $V(G) = V^0 \cup V^+$, where $V^+ \neq \emptyset$. Thus, $G \in UEA_{\mathscr{E}}$ if and only if $\gamma(G) = 1$ and $G \neq K_{1,n}$. For example, the graph G obtained by adding/appending one or more pendant edges to exactly one vertex of K_n, for $n \geq 3$, belongs to this region.

(e) **The Region** R_5:

Theorem 4.5.3. *For any connected graph G and $G \in \mathscr{E}$, the subset R_5 is empty.*

Proof. For any graph G, $G \in R_5$ if and only $G \in UER_{\mathscr{E}} \cap UEA_{\mathscr{E}}$, but $G \notin UVR_{\mathscr{E}}$. Since $G \notin UVR_{\mathscr{E}}$, $V^- \cup V^+ \neq \emptyset$. If $G \in UER_{\mathscr{E}}$, then $V^+ = \emptyset$ and if $G \in UEA_{\mathscr{E}}$, then $V^- = \emptyset$. Hence, this is not possible. Thus, $R_5 = \emptyset$. □

(f) **The Region** R_4:

For any graph G, $G \in R_4$ if and only if $G \in UER_{\mathscr{E}}$ and $G \notin UVR_{\mathscr{E}} \cup UEA_{\mathscr{E}}$.

Thus, the graphs belonging to this region contain $V(G) = V^0 \cup V^-$, where $V^- \neq \emptyset$ and $\gamma(G) \geq 2$. For example, the graph in Figure 4.8 belongs to R_4.

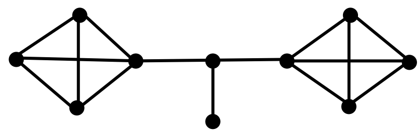


Figure 4.8: A Graph $G \in R_4$

(g) **The Region** R_1:

For any graph G, $G \in R_6$ if and only if $G \in UVR_{\mathscr{E}} \cap UEA_{\mathscr{E}}$ and $G \notin UER_{\mathscr{E}}$. Here $V(G) = V^0$ and $\gamma(G) \geq 2$. Let S be an EDS of G. Since $G \notin UER_{\mathscr{E}}$, for some $u \in S$, $N(u) \cap S_u$ is unique, where S_u is an EDS of $G - u$.

(h) **The Region** R_7:

Not all efficiently dominatable graphs fall in one of the four classes $UVR_{\mathscr{E}}$, $UER_{\mathscr{E}}$, $CER_{\mathscr{E}}$ and $UEA_{\mathscr{E}}$ and hence, $R_7 \neq \emptyset$. The graphs G belonging to R_7 have $V(G) = V^0 \cup V^- \cup V^+$, where $V^- \cup V^+ \neq \emptyset$ and $\gamma(G) \geq 2$.

Table 4.1: A Comparision of properties possessed by any arbitrary graph and a graph $G \in \mathscr{E}$ with respect to Vertex Removal

Properties possessed by a graph	**Properties possessed by a graph** $G \in \mathscr{E}$
$V^0 = \{u \in V : \gamma(G-u) = \gamma(G)\}$ $V^+ = \{u \in V : \gamma(G-u) > \gamma(G)\}$ $V^- = \{u \in V : \gamma(G-u) < \gamma(G)\}$	$V^0 = \{(V-S) \cup S' : S' \subseteq S$ and $\gamma(G-u) = \gamma(G)$, for every $u \in S'\}$ $V^+ = \{u \in S' : S' \subseteq S$ and $\gamma(G-u) > \gamma(G)$, for every $u \in S'\}$ $V^- = \{u \in S' : S' \subseteq S$ and $\gamma(G-u) < \gamma(G)$, for every $u \in S'\}$, for any EDS S and S' of G and $G-u$ respectively.
Every vertex in V^+ lies in every dominating set of G. If $v \in V^-$, then there exists a γ-set D of G such that $v \notin D$.	Every vertex $u \in V^-$ or V^+ if and only if u belongs to every EDS of G.
$\gamma(G) \neq \gamma(G-v)$, for all $v \in V(G)$ if and only if $V(G) = V^-$.	$\gamma(G) \neq \gamma(G-v)$, for all $v \in V(G)$ if and only if $V(G) = V^- \cup V^+$.
For any connected graph G and for $u \in V^-$, $v \in V^+$, $d_G(u,v) \geq 2$.	For any connected graph $G \in \mathscr{E}$ and for $u \in V^-$, $v \in V^+$, $d_G(u,v) \geq 4$.
The class CVR exists.	For any connected graph $G \in \mathscr{E}$, the class $CVR_{\mathscr{E}}$ does not exist.
A graph $G \in UVR$ if and only if G has no isolated vertices and for each vertex v either (a) there is an γ-set D such that $v \in D$, $pn[v,S]$ contains at least one vertex from $V-S$, or (b) v is in every γ-set and there is a subset of $\gamma(G)$ vertices in and $G-N[v]$ that dominates $G-v$.	$G \in UVR_{\mathscr{E}}$ if and only if G has k efficient dominating sets $S_1, S_2, \ldots, S_k$ $(k \geq 2)$ such that $\cap_{i=1}^{k} S_i = \emptyset$.

Table 4.2: A Comparision of properties possessed by any arbitrary graph and a graph $G \in \mathscr{E}$ with respect to Edge Removal

Properties possessed by a graph	**Properties possessed by a graph** $G \in \mathscr{E}$
$G \in UER$ if and only if $V(G) = V^0 \cup V^- \cup V^+$.	If $G \in UER_{\mathscr{E}}$, then $V(G) = V^0 \cup V^-$.
A graph $G \in UER$ if and only if, for each $e = uv \in E(G)$, there exists a γ-set D such that one of the following conditions is satisfied: (a) $u, v \in D$. (b) $u, v \in V - D$. (c) $u \in D$ and $v \in V - D$ implies $\lvert N(v) \cap D \rvert \geq 2$.	A graph $G \in UER_{\mathscr{E}}$ if and only if one of the following conditions hold: (a) G satisfies property **P**. (b) For $e = uv \in E(G)$ and $u \in S$, $N_G(u) \cap S_u = \emptyset$ or not unique, where S and S_u are EDS of G and $G - u$ respectively.

Table 4.3: A Comparision of properties possessed by any arbitrary graph and a graph $G \in \mathscr{E}$ with respect to Edge Addition

Properties possessed by a graph	**Properties possessed by a graph** $G \in \mathscr{E}$
The class CEA exists.	For any connected graph $G \in \mathscr{E}$, the class $CEA_{\mathscr{E}}$ does not exist.
$G \in UEA$ if and only if $V(G) = V^0 \cup V^+$.	For $\gamma(G) \geq 2$, if $G \in UEA_{\mathscr{E}}$, then $V(G) = V^0$.

Conclusion

In this chapter, the study of the concept of criticality is initiated for the class of efficiently dominatable graphs. The behaviour of an efficiently dominatable graph is analyzed with respect to vertex removal, edge removal and edge addition. Some properties of critical vertices are discussed and the necessary and sufficient conditions for a vertex to be γ-critical are obtained. The vertex critical sets V^0, V^+ and V^- and the classes $UVR_{\mathscr{E}}$, $CVR_{\mathscr{E}}$ are characterized. An attempt is made to characterize the critical edges, edge critical sets: ER^0, ER^+ and the classes $UER_{\mathscr{E}}$, $CER_{\mathscr{E}}$ obtained from them. Further, with respect to edge addition, the critical edges, edge critical sets EA^0, EA^- and the two classes $UEA_{\mathscr{E}}$ and $CEA_{\mathscr{E}}$ are characterized. Finally, the relationship among all the classes arising out of vertex removal, edge removal and edge addition are discussed.

Chapter 5

Efficient Domination in Cartesian Product of Graphs

In this chapter, the concept of efficient domination is discussed for the cartesian product of graphs. In the literature, significant interest is shown to study the structural properties of the cartesian product of graphs with respect to different graph parameters. Also, it is one of the widely used multi-dimensional architectures in distributed computing, making the problem to be of sufficient interest from both Graph theoretic as well as application perspective.

In this chapter, the structural properties of the cartesian product of graphs are studied in terms of its factors. Initially, few basic properties of the product $G\Box H$ are discussed in terms of its factors. Next, the notion of efficient domination is studied for the cartesian product of $K_{1,p}$ with some well-known graphs, namely, the star graph $K_{1,n}$, path P_n, complete graph K_n and cycle C_n, for all n and for an arbitrary p. Similarly, the problem is studied for the cartesian product of K_p with each of the aforesaid well-known graphs, for an arbitrary p. Later, the study is extended to the products $G\Box K_{1,p}$ and $G\Box K_p$, where G is arbitrary. Also, the necessary and sufficient conditions are derived for the products $G\Box K_{1,p}$ and $G\Box K_p$ to be efficiently dominatable, for an arbitrary G. Further, it is known that the problem of deciding whether or not a graph G is efficiently dominatable is $\mathcal{NP}$-complete and so also, for the products $G\Box K_{1,p}$ and $G\Box K_p$. Hence, an attempt is made in this chapter to provide an exact exponential time solution for the efficient domination problem in the Cartesian product $G\Box K_{1,p}$ and $G\Box K_p$, for an arbitrary graph G. Finally, the study is generalized to the cartesian product of two or more complete graphs, with a special focus on Hamming graphs.

5.1 Efficient Domination in the cartesian product of two arbitrary graphs

Throughout this chapter, the basic notations and terminologies with reference to cartesian product of graphs are followed as in (Imrich and Klavžar, 2000).

Definition 5.1.1. *(Imrich and Klavžar, 2000) The cartesian product of two graphs* $G = (V_1, E_1)$ *and* $H = (V_2, E_2)$*, denoted by* $G\Box H$*, is the graph with vertex set* $V_1 \times V_2$ *in which two vertices* (u_1, v_1) *and* (u_2, v_2) *are adjacent if and only if either (i)* $u_1 = u_2$ *and* $v_1v_2 \in E_2$ *or (ii)* $u_1u_2 \in E_1$ *and* $v_1 = v_2$*.*

The graphs G and H are called the *factors* of $G\Box H$. For $v \in V(H)$, the induced subgraph $G^{(v)}$ of $G\Box H$, defined as $G^{(v)} =< \{(u, v) \in V(G\Box H) : u \in V(G)\} >$ is called the *G-layer with respect to v* in $G\Box H$. Analogously, for $u \in V(G)$, the induced subgraph $H^{(u)}$ of $G\Box H$, defined as $H^{(u)} =< \{(u, v) \in V(G\Box H) : v \in V(H)\} >$ is called the *H-layer with respect to u* in $G\Box H$. The subgraph of $G\Box H$ induced by any G-layer (or H-layer) is isomorphic to G (or H).

The structure of the cartesian product of two graphs G and H and the layers $G^{(v)}$ and $H^{(u)}$ are illustrated in Figure 5.1.

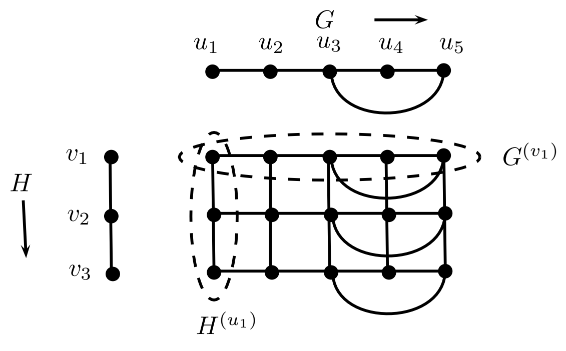


Figure 5.1: The Structure of $G\Box H$ and $G^{(v_j)}$ and $H^{(u_i)}$ layers

Definition 5.1.2. *(Imrich and Klavžar, 2000) The mapping* $p_G : (u, v) \rightarrowtail u$ *(or* $p_H : (u, v) \rightarrowtail v$*) from* $V(G\Box H)$ *to* $V(G)$ *(or* $V(H)$*) is called the* ***projection*** *from* $G\Box H$ *onto the factor* G *(or* H*).*

It can be observed that if the product graph is efficiently dominatable, then its factors may or may not be efficiently dominatable and vice versa. Also, for any graph G, $1 \leq F(G) \leq n$ and a graph $G \in \mathscr{E}$ if and only if $F(G) = n$. The following proposition is deduced from this fact.

Proposition 5.1.1. *For any two graphs* G *and* H*, the following properties hold:*

(i) If G*,* H *and* $G\Box H$ *are all efficiently dominatable, then* $F(G\Box H) = F(G)F(H)$*.*

(ii) If $G\Box H$ *is efficiently dominatable and at least one of* G *and* H *is not efficiently dominatable, then* $F(G\Box H) > F(G)F(H)$*.*

(iii) If both G *and* H *are efficiently dominatable, but* $G\Box H$ *is not efficiently dominatable, then* $F(G\Box H) < F(G)F(H)$*.*

Let G and H be two graphs of order n and p respectively and let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_p\}$. For any $x \in V(G\Box H)$, $x = (u_i, v_j)$ for some i and j, where $1 \leq i \leq n, 1 \leq j \leq p$. Moreover, x is identified as a vertex in the j^{th} row and i^{th} column. Further, $\deg_{G\Box H}(x) = \deg_G(p_G(x)) + \deg_H(p_H(x)) = \deg_G(u_i) + \deg_H(v_j)$. Also, for any $u_i \in V(G)$, $p_G(u_i, v_j) = u_i$, for all j, $(1 \leq j \leq p)$. Similarly, for any $v_j \in V(H)$, $p_H(u_i, v_j) = v_j$, for all i, $(1 \leq i \leq n)$. The following results are obtained based on this fact.

Proposition 5.1.2. *For any nonempty subset* S' *of* $V(G\Box H)$*,* $I_{G\Box H}(S') \geq I_G(S_1) + I_H(S_2) - |S'|$*, where* $S_1 = p_G(S')$ *and* $S_2 = p_H(S')$*. The equality holds if and only if* $|S'| = |S_1| = |S_2|$*.*

Proof. Let $S_1 = p_G(S')$ and $S_2 = p_H(S')$. Since there may exist two (or more) vertices, say, x and y in S' such that $p_G(x) = p_G(y)$, it follows that $|S'| \geq |S_1|$. Similarly, $|S'| \geq |S_2|$. Hence,

$$
\begin{aligned}
I_{G\Box H}(S') &= \sum_{(u_i,v_j)\in S'} [\deg_{G\Box H}(u_i,v_j)] + |S'| \\
&= \sum_{(u_i,v_j)\in S'} [\deg_G(p_G(u_i,v_j)) + \deg_H(p_H(u_i,v_j))] + |S'| \\
&\geq \sum_{u_i\in S_1} [\deg_G(u_i)] + \sum_{v_j\in S_2} [\deg_H(v_j)] + |S_1| + |S_2| - |S'|
\end{aligned}
$$

Thus, $I_{G\Box H}(S') \geq I_G(S_1) + I_H(S_2) - |S'|$.

If $|S'| = |S_1| = |S_2|$, then

$$
\begin{aligned}
I_G(S_1) + I_H(S_2) &= [|S_1| + \sum_{u_i\in S_1} \deg_G(u_i)] + [|S_2| + \sum_{v_j\in S_2} \deg_H(v_j)] \\
&= \sum_{(u_i,v_j)\in S'} [\deg_G(p_G(u_i,v_j)) + \deg_H(p_H(u_i,v_j))] + (|S_1| + |S_2|) \\
&= \sum_{(u_i,v_j)\in S'} [\deg_{G\Box H}(u_i,v_j)] + 2|S'| \\
&= I_{G\Box H}(S') + |S'|
\end{aligned}
$$

Thus, $I_{G\Box H}(S') = I_G(S_1) + I_H(S_2) - |S'|$.

Conversely, let $I_{G\Box H}(S') = I_G(S_1) + I_H(S_2) - |S'|$. Since $|S'| \geq 1$, $I_{G\Box H}(S') < I_G(S_1) + I_H(S_2)$. Let $|S'| = k$, $|S_1| = l$ and $|S_2| = p$. Clearly, $k \geq l$ and $k \geq p$.

Claim: $k = l = p$.

Suppose $l < k$ and $p = k$. Then, there exist at least two vertices, say, x and y in S' such that $p_G(x) = p_G(y) = u$, where $u \in V(G) \cap S_1$ and hence, $\deg(u)$ is counted at least twice in $I_{G\Box H}(S')$. Since $p = k$, for every $v \in V(H) \cap S_2$, $\deg(v)$ is counted only once in $I_{G\Box H}(S')$. Consequently, $I_{G\Box H}(S') - I_H(S_2) = I_G(S_1) + k'$, where $k' > 0$. That is, $I_{G\Box H}(S') - I_H(S_2) > I_G(S_1)$ or $I_{G\Box H}(S') > I_G(S_1) + I_H(S_2)$, which is a contradiction. A similar discussion holds when $l = k$, $p < k$ and $l < k$, $p < k$. Thus, for the equality $I_{G\Box H}(S') = I_G(S_1) + I_H(S_2) - |S'|$ to hold, we must have $l = k = p$. That is, $|S'| = |S_1| = |S_2|$. □

As $G^{(v)}$ is isomorphic to G, for all $v \in V(H)$, at most $\rho(G)$ elements belong to an EDS of $G\Box H$ (or an $F(G\Box H)$-set), from each of these G-layers. A similar argument holds for $H^{(u)}$, for all $u \in V(G)$. This leads to the following upper bound on the domination number of the product.

Proposition 5.1.3. *If $G\Box H \in \mathscr{E}$, where G and H are graphs of order n and p respectively, then $\gamma(G\Box H) \leq \min\{p \times \rho(G), n \times \rho(H)\}$.*

5.2 Efficient Domination in the cartesian product of some well-known graphs

In this section, the notion of efficient domination is discussed for the product $G\Box K_{1,p}$, when G is isomorphic to one of the following graphs: K_n, $K_{1,n}$, P_n and C_n. Further, the conditions under which these products are efficiently dominatable are identified; the exact values of their respective efficient domination numbers are also computed.

The Cartesian product $K_n\Box K_{1,p}$:

Theorem 5.2.1. *For $n > 1$, $K_n\Box K_{1,p} \in \mathscr{E}$ if and only if $p = n$. When $p \neq n$,*

$$F(K_n\Box K_{1,p}) = \begin{cases} n+p; & \text{if } (i,0) \in S' \text{ (for } 1 \leq i \leq n \text{)} \\ p(n+1); & \text{if } p \leq n \\ n(n+1); & \text{if } p > n \end{cases}$$

where S' is a maximal 2-packing of $K_n\Box K_{1,p}$.

Proof. Let $n > 1$ and $V(K_n\Box K_{1,p}) = \{(i,j) : 1 \leq i \leq n, 0 \leq j \leq p\}$, where (i,j) corresponds to a vertex in the i^{th} column and j^{th} row (refer to Figure 5.2). In general, an $F(K_n\Box K_{1,p})$ or an EDS of $K_n\Box K_{1,p}$ either contains one or more

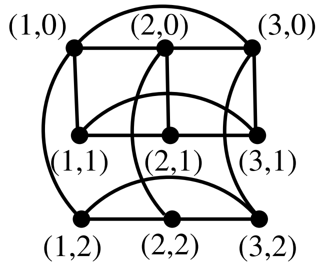


Figure 5.2: $K_3\Box K_{1,2}$

vertices from the layer $K_n^{(0)}$ or may not contain any vertex from $K_n^{(0)}$. So, based on this fact, if S' is any $F(K_n\Box K_{1,p})$ or an EDS of $K_n\Box K_{1,p}$, then the following two cases arise:

Case(i): $S' \cap V(K_n^{(0)}) \neq \emptyset$

Every vertex in $V(K_n^{(0)})$ (that is, in the first row) is of eccentricity two. Hence, if a vertex from $V(K_n^{(0)})$ is included in S', then no other vertex can be included and in such a case, at most $n+p$ vertices are efficiently dominated by S'.

Case(ii): $S' \cap V(K_n^{(0)}) = \emptyset$

In this case, exactly one vertex can be chosen from every other row. Without loss of generality, choosing $S' = \{(1,1),(2,2),(3,3),\ldots,(p,p)\}$, when $p \leq n$ and $S' = \{(1,1),(2,2),(3,3),\ldots,(n,n)\}$, when $p > n$, it can be observed that at most $p(n+1)$ vertices are dominated by S', when $p \leq n$ and at most $n(n+1)$ vertices are dominated by S', when $p > n$. Hence,

$$F(K_n \Box K_{1,p}) = \begin{cases} n+p; & \text{if } (i,0) \in S' \text{ (for } 1 \leq i \leq n) \\ p(n+1); & \text{if } p \leq n \\ n(n+1); & \text{if } p > n \end{cases}$$

which implies that $F(K_n \Box K_{1,p})$ is equal to $n+p$ or $p(n+1)$ or $n(n+1)$. Further, $K_n \Box K_{1,p} \in \mathscr{E}$ if and only if $F(K_n \Box K_{1,p}) = n(p+1)$.

But, as $n > 1$, $n+p \neq n(p+1)$. Hence, $F(K_n \Box K_{1,p}) = (n+1)$ or $n(n+1)$, as the case may be. Now, suppose $F(K_n \Box K_{1,p}) = p(n+1)$, then $K_n \Box K_{1,p} \in \mathscr{E}$ if and only if $p(n+1) = n(p+1)$ if and only if $p = n$. Similarly, if $F(K_n \Box K_{1,p}) = n(n+1)$, then $K_n \Box K_{1,p} \in \mathscr{E}$ if and only if $p = n$. Hence, the result follows. $\Box$

The Cartesian product $K_{1,n} \Box K_{1,p}$:

Let $V(K_{1,n} \Box K_{1,p}) = \{(i,j) : 0 \leq i \leq n, 0 \leq j \leq p\}$, where the vertices $(0,j)$ and $(i,0)$ represent the central vertices of $K_{1,n}$ and $K_{1,p}$ respectively (refer to Figure 5.3). In $K_{1,n} \Box K_{1,p}$, $\deg(0,0) = n+p$, $\deg(0,j) = n+1$, $\deg(i,0) = p+1$ and $\deg(i,j) = 2$, for all i,j, where $1 \leq i \leq n$, $1 \leq j \leq p$.

Theorem 5.2.2. *For $n \geq 2$ and $p \geq 2$, $K_{1,n} \Box K_{1,p} \notin \mathscr{E}$. If S' is a maximal 2-packing of $K_{1,n} \Box K_{1,p}$, then*

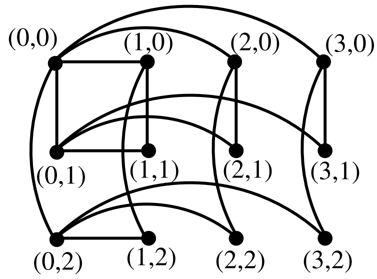


Figure 5.3: $K_{1,3}\square K_{1,2}$

$$F(K_{1,n}\square K_{1,p}) = \begin{cases} n+p+1; & \text{if } (0,0)\in S' \\ 4p-1; & \text{if } n=p \\ \max\{4n+2, 3n+p-1\}; & \text{if } n<p \\ \max\{4p+2, 3p+n-1\}; & \text{if } n>p \end{cases}$$

Proof. Let S' be a maximal 2-packing of $K_{1,n}\square K_{1,p}$.

Case(i): $(0,0)\in S'$

The vertex $(0,0)$ is of eccentricity two and hence if $(0,0)\in S'$, then no other vertex can be included in S'. Thus, if $(0,0)\in S'$, then $|S'|=1$ and S' can efficiently dominate $n+p+1$ vertices.

Case(ii): $(0,0)\notin S'$

For any n and l, $K_{1,n}^{(l)}\cong K_{1,n}$. Therefore, for each $i,\ j$ $(0\le i\le n,\ 0\le j\le p)$, at most one vertex from the layers $K_{1,p}^{(i)}$ and $K_{1,n}^{(j)}$ can be included in S'.

Subcase(i): $n=p$

If any vertex from the first column (or the layer $K_{1,p}^{(0)}$), say, $(0,1)$ is included in S' (excluding the vertex $(0,0)$), then there are p other possible choices of vertices to include in S' (that is, one vertex from each of the layers $K_{1,p}^{(i)}$, $1\le i\le p$). Having chosen $(0,1)$ from the layer $K_{1,p}^{(0)}$, without loss of generality, if the vertices $(1,2),(2,3),\ldots,(p-1,p)$ are chosen from the layers $K_{1,p}^{(i)}$, $(1\le i\le p-1)$ respectively, then it can be observed that no vertex from $K_{1,p}^{(p)}$ can be included in S', as each vertex in $K_{1,p}^{(p)}$ is at distance two from at least one vertex included already in S'. Hence, in such a case, $I(S')=(p+2)+3(p-1)=4p-1$. Further, this is the maximum influence among all 2-packings which include a vertex from the first column, as the maximum possible number of vertices have been chosen from each column. On the other hand, if no vertex is chosen from the first column to include in S', then in such a case also, it can be shown by a similar argument that the maximum influence among all such 2-packings which do not include a vertex from the first column is $4p-1$.

Subcase(ii): $n < p$

If a vertex from the layer $K_{1,p}^{(0)}$, say, $(0,1)$ is included in S', then there are at most n other possible choices of vertices to include in S' (choosing at most one vertex from each of the other layers $K_{1,p}^{(i)}$, for $1 \leq i \leq n$). Thus, upon choosing $(0,1)$, without loss of generality, if the vertices $(1,2),(2,3),\ldots,(n,n+1)$ are chosen, then $I(S') = (n+2)+3n = 4n+2$. It can be observed that the influence obtained as above is the maximum among those 2-packings which include a vertex from $K_{1,p}^{(0)}$, as $n < p$ and the maximum possible vertices (that is, one vertex) have been chosen from each of the n columns.

On the other hand, if no vertex is chosen from the layer $K_{1,p}^{(0)}$, then start choosing vertices from $K_{1,p}^{(1)}$. If vertex, say, $(1,0)$ is chosen from $K_{1,p}^{(1)}$ to include in S', then there are at most $n-1$ other possible choices of vertices to include in S' (choosing at most one vertex from each of the other layers $K_{1,p}^{(i)}$, for $2 \leq i \leq n$). So, having chosen $(1,0)$, without loss of generality, choosing the vertices $(2,1),(3,2),\ldots,$ $(n,n-1)$ to include in S', $I(S') = (p+2)+3(n-1) = 3n+p-1$. As discussed earlier, it can be observed that the influence so obtained is the maximum among those 2-packings which do not include any vertex from $K_{1,p}^{(0)}$.

Hence, comparing the above possible influences, it can be observed that whenever $n < p$, for any 2-packing S' of $K_{1,n} \Box K_{1,p}$,

$$\max\{I(S')\} = \begin{cases} 4n+2; & \text{if } n < p \leq n+3 \\ 3n+p-1; & \text{if } p > n+3 \end{cases}$$

Subcase(iii): $n > p$

By a similar argument as above, it can be shown that if S' includes a vertex from the first row, then S' can efficiently dominate at most $4p+2$ vertices and $\{(1,0),(2,1),\ldots,(p+1,p)\}$ is one such set. On the other hand, if S' does not include a vertex from the first row, then S' can efficiently dominate at most $3p+n-1$ vertices and $\{(0,1),(1,2),\ldots,(p-1,p)\}$ is one such set. Thus, comparing the above possible influences, it can be observed that whenever $n > p$, for any 2-packing S' of $K_{1,n}\square K_{1,p}$,

$$\max\{I(S')\} = \begin{cases} 4p+2; & \text{if } p < n \leq p+3 \\ 3p+n-1; & \text{if } n > p+3 \end{cases}$$

Hence, it follows from cases (i) and (ii) that

$$F(K_{1,n}\square K_{1,p}) = \begin{cases} n+p+1; & \text{if } (0,0) \in S' \\ 4p-1; & \text{if } n = p \\ \max\{4n+2, 3n+p-1\}; & \text{if } n < p \\ \max\{4p+2, 3p+n-1\}; & \text{if } n > p \end{cases}$$ □

The Cartesian product $P_n\square K_{1,p}$:

Let $V(P_n\square K_{1,p}) = \{(i,j) : 1 \leq i \leq n, 0 \leq j \leq p\}$, where the vertex $(i,0)$ represents the central vertex of $K_{1,p}$ (refer to Figure 5.4). Then, $deg_{P_n\square K_{1,p}}(1,0) = p+1 = deg_{P_n\square K_{1,p}}(n,0)$ and $deg_{P_n\square K_{1,p}}(i,0) = p+2$, for $2 \leq i \leq n-1$. For $1 \leq j \leq p$, $deg_{P_n\square K_{1,p}}(1,j) = deg_{P_n\square K_{1,p}}(n,j) = 2$. For all the other vertices, $deg_{P_n\square K_{1,p}}(i,j) = 3$.

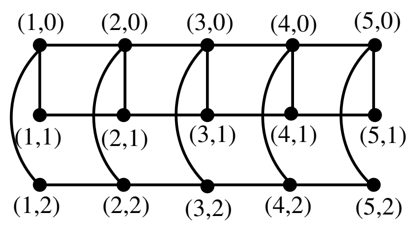


Figure 5.4: $P_5\square K_{1,2}$

Let S' be a maximal 2-packing of $P_n\square K_{1,p}$ and let $|S' \cap V(P_n^{(j)})| = l_j$, for $j \in \{0,1,\ldots,p\}$. The vertices in $P_n^{(j)}$, excluding $(1,j)$ and (n,j), are called the internal vertices of $P_n^{(j)}$, for all j, where $0 \leq j \leq p$. When $l_0 \geq 1$, then, for each i, where $2 \leq i \leq n-1$, if $(i,0) \in S'$, then no other vertex from $K_{1,p}^{(i)}$ and its neighboring layers, namely, $K_{1,p}^{(i-1)}$ and $K_{1,p}^{(i+1)}$ can belong to S'. Further, if $(1,0)$ is included in S', then no other vertex from $K_{1,p}^{(1)}$ and its neighboring layer, namely,

$K_{1,p}^{(2)}$ can be included in S'. Similarly, if $(n, 0)$ is in S', then no other vertex from $K_{1,p}^{(n-1)}$ and $K_{1,p}^{(n)}$ can belong to S'.

Theorem 5.2.3. *$P_n \square K_{1,2} \notin \mathscr{E}$, for $n \geq 3$ and*

$$F(P_n \square K_{1,2}) = \begin{cases} \frac{8n}{3}; & \text{if } n \equiv 0 \ (mod\ 3) \\ \frac{8n}{3} + \frac{1}{3}; & \text{if } n \equiv 1 \ (mod\ 3) \\ \frac{8n}{3} + \frac{2}{3}; & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}.$$

Proof. Let S' be a maximal 2-packing of $P_n \square K_{1,2}$ and let $|S' \cap V(P_n^{(j)})| = l_j$, for $j \in \{0, 1, 2\}$. As S' is a 2-packing, it can include at most one element from each layers $K_{1,2}^{(i)}$, for $i \in \{1, 2, \ldots, n\}$. Hence, $|S'| = \sum_{j=0}^{2} l_j \leq n$. Also, S' either contains one or more vertices from the layer $P_n^{(0)}$ or may not contain any vertex from $P_n^{(0)}$. Based on this, the following cases are considered:

Case(i): $n \equiv 0 \pmod 3$

If $l_0 = 0$, then S' must include vertices only from the two layers $P_n^{(j)}$, where $1 \leq j \leq 2$. In addition, for each j $(1 \leq j \leq 2)$, $S' \cap V(P_n^{(j)})$ is a 2-packing of $P_n^{(j)}$ and hence, $|S' \cap V(P_n^{(j)})| \leq \rho(P_n^{(j)}) = \lceil \frac{n}{3} \rceil$. Thus,

$$|S'| = \sum_{j=1}^{2} l_j \leq 2 \left\lceil \frac{n}{3} \right\rceil \tag{5.1}$$

As $n \equiv 0 \pmod 3$, $|S'| \leq 2\left(\frac{n}{3}\right) = \frac{2n}{3}$. Clearly, $|S'| < n$ and hence S' may or may not include the vertices from $\{(1, j) : 1 \leq j \leq 2\} \cup \{(n, j) : 1 \leq j \leq 2\}$, where degree of each vertex is two. The remaining (internal) vertices have degree 3. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1, 2\}$, it can be observed that only one vertex belongs to $\{(1, j) : 1 \leq j \leq 2\} \cup \{(n, j) : 1 \leq j \leq 2\}$ (refer to Figure 5.5). Thus, $I(S') \leq 4(\sum_{j=1}^{2} l_j - 1) + 3 = \frac{8n}{3} - 1$.

For the case $l_0 \geq 1$, the following subcases arise:

Subcase(i): S' includes neither $(1, 0)$ or $(n, 0)$.

Then, having chosen l_0 (internal) vertices from the layer $P_n^{(0)}$, no vertex from the corresponding column and its neighboring columns can be considered for subsequent choices of vertices from the remaining rows, to include in S'. Thus,

$$|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\left\lceil \frac{n-3l_0}{3} \right\rceil \tag{5.2}$$

As $n \equiv 0 \pmod 3$, $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2(\frac{n-3l_0}{3}) = \frac{2n}{3} - 2l_0$. Thus, $|S'| \leq \frac{2n}{3} - l_0$. In this case, a maximum of $n - 2l_0 + 1$ vertices can be included in S'. It follows from (5.2) that $|S'| < n - 2l_0 + 1$ and hence S' may or may not include vertices from $\{(1, j) : 1 \leq j \leq 2\} \cup \{(n, j) : 1 \leq j \leq 2\}$. If $(i, 0)$ is chosen in S', where $3 \leq i \leq n-2$, then $S' \cap V(P_n^{(j)})$, for $j \in \{1, 2\}$, includes exactly two vertices from $\{(1, j) : 1 \leq j \leq 2\} \cup \{(n, j) : 1 \leq j \leq 2\}$, each of degree three. On the other hand if $(i, 0) \in S'$, for $i = 2$ or $i = n-1$, then all the vertices included in $S' \cap V(P_n^{(j)})$, for $j \in \{1, 2\}$, will have degree four, hence giving the maximum influence (refer to Figure 5.6). Thus, in this case, $I(S') = 5l_0 + 4\sum_{j=1}^{2} l_j \leq 5l_0 + 4(\frac{2n}{3} - 2l_0)$. It can be observed that $I(S')$ is maximum when l_0 is minimum. Thus, for $l_0 = 1$, $|S'| \leq \frac{2n}{3} - 1$ and $I(S') \leq 5 + 4(\frac{2n}{3} - 2) = \frac{8n}{3} - 3$.

Subcase(ii): S' includes either $(1, 0)$ or $(n, 0)$

Then, two columns for each choice of vertices $(1, 0)$ or $(n, 0)$ and three columns corresponding to each (internal) vertex in $S' \cap V(P_n^{(0)})$ cannot be considered, while choosing vertices from the second and subsequent rows, for inclusion in S'. Thus,

$$|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\left\lceil \frac{n-3l_0+1}{3} \right\rceil \tag{5.3}$$

But, it can be observed that after choosing a maximal 2-packing with maximum influence of cardinality $\lceil \frac{n-3l_0+1}{3} \rceil$ from the layer $P_n^{(1)}$, a maximal 2-packing with maximum influence of cardinality $\lceil \frac{n-3l_0+1}{3} \rceil - 1$ can be chosen from the layer $P_n^{(2)}$. Thus, $|S'| - l_0 \leq 2\lceil \frac{n-3l_0+1}{3} \rceil - 1 \leq \frac{2n}{3} - 2l_0 + 1$ and hence $|S'| \leq \frac{2n}{3} - l_0 + 1$. In this case, a maximum of $n - 2l_0 + 1$ vertices can be included in S'. It follows from (5.3) that $|S'| < n - 2l_0 + 1$ and hence S' may or may not include vertices from $\{(1, j) : 1 \leq j \leq 2\} \cup \{(n, j) : 1 \leq j \leq 2\}$. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1, 2\}$, it is observed that only one vertex belongs to $\{(1, j) : 1 \leq j \leq 2\} \cup \{(n, j) : 1 \leq j \leq 2\}$ (refer to Figure 5.7). Thus, $I(S') \leq 5(l_0 - 1) + 4 + 4(\frac{2n}{3} - 2l_0 + 1 - 1) + 3 = \frac{8n}{3} - 3l_0 + 2$. Since, $I(S')$ is maximum when l_0 is minimum, choose $l_0 = 1$. Thus, $|S'| \leq \frac{2n}{3}$ and

$I(S') \leq \frac{8n}{3} - 1$.

Subcase(iii): S' includes both $(1,0)$ and $(n,0)$

Then, two columns for each choice of vertices $(1,0)$ and $(n,0)$ and three columns for each choice of (internal) vertex in $S' \cap V(P_n^{(0)})$ cannot be considered, while choosing vertices from the second and subsequent rows, for inclusion in S'. Thus,

$$|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\left\lceil \frac{n - 3l_0 + 2}{3} \right\rceil \tag{5.4}$$

As $n \equiv 0 \pmod 3$, $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0+2}{3} \rceil = \frac{2n}{3} - 2l_0 + 2$ and hence $|S'| \leq \frac{2n}{3} - l_0 + 2$. In this case, at most $n - 2l_0 + 2$ vertices can be included in S'. But, from (5.4) it follows that $|S'| < n - 2l_0 + 2$ and hence, $|S'|$ may or may not include vertices from $\{(1,j) : 1 \leq j \leq 2\} \cup \{(n,j) : 1 \leq j \leq 2\}$. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1,2\}$, it is observed that all the vertices included in S' from $P_n^{(j)}$, for $j \in \{1,2\}$, have degree three each (refer to Figure 5.8). Thus, $I(S') \leq 5(l_0 - 2) + 4(2) + 4(\frac{2n}{3} - 2l_0 + 2) = \frac{8n}{3} - 3l_0 + 6$. As $I(S')$ is maximum when l_0 is minimum, choose $l_0 = 2$. Thus, $|S'| \leq \frac{2n}{3}$ and $I(S') \leq \frac{8n}{3}$.

Thus, comparing the influences when $l_0 = 0$ and $l_0 \geq 1$, it can be observed that Subcase (iii) gives the maximum influence. The set $S' = \{(1,0),(n,0)\} \cup \{(3,1),(6,1),\ldots,(n-3,1)\} \cup \{(4,2),(7,2),\ldots,(n-2,2)\}$ is a maximal 2-packing of $P_n \Box K_{1,2}$ of cardinality $\frac{2n}{3}$ having influence $\frac{8n}{3}$. Therefore, when $n \equiv 0 \pmod 3$, $F(P_n \Box K_{1,2}) = \frac{8n}{3}$.

Case(ii): $n \equiv 1 \pmod 3$

If $l_0 = 0$, then as $n \equiv 1 \pmod 3$, $|S'| \leq 2\lceil \frac{n}{3} \rceil = 2(\frac{n+2}{3})$ (using 5.1). After choosing a maximal 2-packing with maximum influence of cardinality $(\frac{n+2}{3})$ with maximum influence from the layer $P_n^{(1)}$, it can be observed that a maximal 2-packing of maximum influence with cardinality $(\frac{n+2}{3}) - 1$ can be chosen from the layer $P_n^{(2)}$. Thus, $|S'| \leq 2(\frac{n+2}{3}) - 1 = \frac{2n+1}{3}$ can be chosen. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1,2\}$, it is noted that two vertices belongs to $\{(1,j) : 1 \leq j \leq 2\} \cup \{(n,j) : 1 \leq j \leq 2\}$. Thus, $I(S') \leq 4(\frac{2n+1}{3} - 2) + 3(2) = \frac{8n}{3} - \frac{2}{3}$.

For the case $l_0 \geq 2$, the following subcases arise:

Subcase(i): S' includes neither $(1,0)$ or $(n,0)$

As $n \equiv 1 \pmod 3$, using (5.2), we get $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0}{3} \rceil = 2(\frac{n-3l_0+2}{3})$. After choosing a maximal 2-packing with maximum influence of cardinality $(\frac{n-3l_0+2}{3})$ from the layer $P_n^{(1)}$, it can be observed that a maximal 2-packing with maximum influence of cardinality $(\frac{n-3l_0+2}{3}) - 1$ can be chosen from the layer $P_n^{(2)}$. Thus, $|S'| - l_0 \leq 2(\frac{n-3l_0+2}{3}) - 1 = \frac{2n}{3} - 2l_0 + \frac{1}{3}$ and hence, $|S'| \leq \frac{2n}{3} - l_0 + \frac{1}{3}$. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1,2\}$, it is observed that only one vertex belongs to $\{(1,j) : 1 \leq j \leq 2\} \cup \{(n,j) : 1 \leq j \leq 2\}$. Hence, $I(S') \leq 5l_0 + 4(\frac{2n}{3} - 2l_0 + \frac{1}{3} - 1) + 3 = \frac{8n}{3} - 3l_0 + \frac{1}{3}$. Since, $I(S')$ is maximum when l_0 is minimum, choose $l_0 = 1$. Thus, $|S'| \leq 2(\frac{n-1}{3})$ and $I(S') \leq \frac{8n}{3} - \frac{8}{3}$.

Subcase(ii): S' includes either $(1,0)$ or $(n,0)$

Using (5.3), as $n \equiv 1 \pmod 3$, $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0+1}{3} \rceil = \frac{2n}{3} - 2l_0 + \frac{4}{3}$ and hence $|S'| \leq \frac{2n}{3} - l_0 + \frac{4}{3}$. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1,2\}$, it is observed that only one vertex belongs to $\{(1,j) : 1 \leq j \leq 2\} \cup \{(n,j) : 1 \leq j \leq 2\}$. Hence, $I(S') \leq 5(l_0 - 1) + 4 + 4(\frac{2n}{3} - 2l_0 + \frac{4}{3} - 1) + 3 = \frac{8n}{3} - 3l_0 + \frac{10}{3}$. Since, $I(S')$ is maximum when l_0 is minimum, choose $l_0 = 1$. Thus, $|S'| \leq \frac{2n+1}{3}$ and $I(S') \leq \frac{8n}{3} + \frac{1}{3}$.

Subcase(iii): S' includes both $(1,0)$ and $(n,0)$

As $n \equiv 1 \pmod 3$, using (5.4), $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0+2}{3} \rceil = 2(\frac{n-3l_0+2}{3}) = \frac{2n}{3} - 2l_0 + \frac{4}{3}$. Thus, $|S'| \leq \frac{2n}{3} - l_0 + \frac{4}{3}$. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1,2\}$, it can be observed that all the vertices included in S' from $P_n^{(j)}$, for $j \in \{1,2\}$, have degree three each. Hence, $I(S') \leq 5(l_0 - 2) + 4(2) + 4(\frac{2n}{3} - 2l_0 + \frac{4}{3}) = \frac{8n}{3} - 3l_0 + \frac{10}{3}$. Since, $I(S')$ is maximum when l_0 is minimum, choose $l_0 = 2$. Thus, $|S'| \leq \frac{2n-2}{3}$ and $I(S') \leq \frac{8n}{3} - \frac{8}{3}$.

Comparing the influences obtained when $l_0 = 0$ and $l_0 \geq 1$, it can be seen that the influence obtained in Subcase(ii) gives the maximum influence. The set

$S' = \{(1,0)\} \cup \{(3,1),(6,1),\ldots,(n-1,1)\} \cup \{(4,2),(7,2),\ldots,(n,2)\}$ is a maximal 2-packing of $P_n \Box K_{1,2}$ of cardinality $\frac{2n+1}{3}$ having influence $\frac{8n}{3}+\frac{1}{3}$. Thus, when $n \equiv 1 \pmod 3$, $F(P_n \Box K_{1,2}) = \frac{8n}{3}+\frac{1}{3}$.

Case(iii): $n \equiv 2 \pmod 3$

If $l_0 = 0$, then as $n \equiv 2 \pmod 3$, $|S'| \leq 2\lceil \frac{n}{3} \rceil = 2(\frac{n+1}{3})$ (using (5.1)). After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1,2\}$, it is observed that two vertices belongs to $\{(1,j) : 1 \leq j \leq 2\} \cup \{(n,j) : 1 \leq j \leq 2\}$. Thus, $I(S') \leq 4(2(\frac{n+1}{3})-2)+3(2) = \frac{8n}{3}+\frac{2}{3}$.

For the case $l_0 \geq 1$, the following subcases arise:

Subcase(i): S' includes neither $(1,0)$ or $(n,0)$

Using (5.2), as $n \equiv 2 \pmod 3$, $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0}{3} \rceil = \frac{2n}{3} - 2l_0 + \frac{2}{3}$ and hence $|S'| \leq \frac{2n}{3} - l_0 + \frac{2}{3}$. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1,2\}$, it can be observed that one vertex belongs to $\{(1,j) : 1 \leq j \leq 2\} \cup \{(n,j) : 1 \leq j \leq 2\}$. Hence, $I(S') \leq 5l_0 + 4(\frac{2n}{3} - 2l_0 + \frac{2}{3} - 1) + 3 = \frac{8n}{3} - 3l_0 + \frac{5}{3}$. Since, $I(S')$ is maximum when l_0 is minimum, choose $l_0 = 1$. Thus, $|S'| \leq \frac{2n-1}{3}$ and $I(S') \leq \frac{8n}{3} - \frac{4}{3}$.

Subcase(ii): S' includes either $(1,0)$ or $(n,0)$

As $n \equiv 2 \pmod 3$, using (5.3), $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0+1}{3} \rceil = \frac{2n}{3} - 2l_0 + \frac{2}{3}$. Thus, $|S'| \leq \frac{2n}{3} - l_0 + \frac{2}{3}$. After choosing maximal 2-packings having maximum influence one from each layer $P_n^{(j)}$, for $j \in \{1,2\}$, it can be observed that all the vertices included in S' from $P_n^{(j)}$, for $j \in \{1,2\}$, have degree three each. Hence, $I(S') \leq 5(l_0-1) + 4 + 4(\frac{2n}{3} - 2l_0 + \frac{2}{3}) = \frac{8n}{3} - 3l_0 + \frac{5}{3}$. Since, $I(S')$ is maximum when l_0 is minimum, choose $l_0 = 1$. Thus, $|S'| \leq \frac{2n-1}{3}$ and $I(S') \leq \frac{8n}{3} - \frac{4}{3}$.

Subcase(iii): S' includes both $(1,0)$ and $(n,0)$

As $n \equiv 2 \pmod 3$, using (5.4), $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0+2}{3} \rceil = 2(\frac{n-3l_0+4}{3})$. But, after choosing a maximal 2-packing with maximum influence of cardinality $(\frac{n-3l_0+4}{3})$ from the layer $P_n^{(1)}$, it can be observed that a maximal 2-packing with maximum influence of cardinality $(\frac{n-3l_0+4}{3}) - 1$ can be chosen from the layer $P_n^{(2)}$. Thus, $|S'| - l_0 \leq 2(\frac{n-3l_0+4}{3}) - 1 = \frac{2n}{3} - 2l_0 + \frac{5}{3}$ and hence, $|S'| \leq \frac{2n}{3} - l_0 + \frac{5}{3}$. After choosing maximal 2-packings having maximum influence one from each layer

$P_n^{(j)}$, for $j \in \{1,2\}$, it is noted that all the vertices included in S' from $P_n^{(j)}$, for $j \in \{1,2\}$, have degree three each. Hence, $I(S') \leq 5(l_0-2)+4(2)+4(\frac{2n}{3}-2l_0+\frac{5}{3}) = \frac{8n}{3} - 3l_0 + \frac{14}{3}$. Since, $I(S')$ is maximum when l_0 is minimum, choose $l_0 = 2$. Thus, $|S'| \leq \frac{2n-1}{3}$ and $I(S') \leq \frac{8n}{3} - \frac{4}{3}$.
Comparing influences obtained when $l_0 = 1$ and $l_0 \geq 1$, it is observed that the influence obtained when $l_0 = 0$ gives the maximum influence. The set $S' = \{(1,1),(4,1),\ldots,(n-1,1)\} \cup \{(2,2),(5,2),\ldots,(n,2)\}$ is a maximal 2-packing of $P_n \Box K_{1,2}$ of cardinality $2(\frac{n+1}{3})$ having influence $\frac{8n}{3}+\frac{2}{3}$. Thus, when $n \equiv 2 \pmod 3$, $F(P_n \Box K_{1,2}) = \frac{8n}{3} + \frac{2}{3}$. □

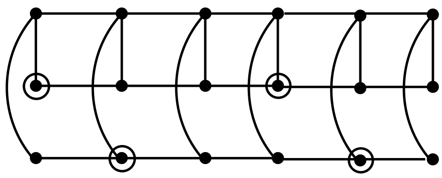


Figure 5.5: $P_6 \Box K_{1,2}$, when $l_0 = 0$

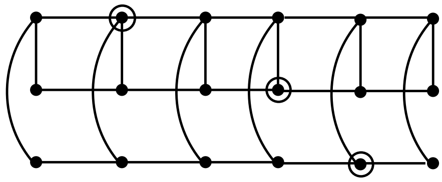


Figure 5.6: $P_6 \Box K_{1,2}$ - An example for Subcase(i)

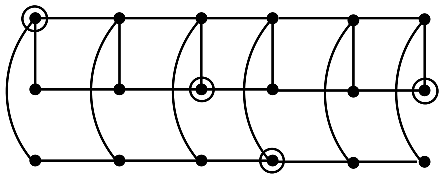


Figure 5.7: $P_6 \Box K_{1,2}$ - An example for Subcase(ii)

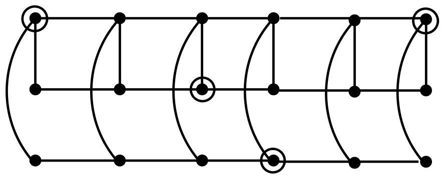


Figure 5.8: $P_6 \Box K_{1,2}$ - An example for Subcase(iii)

Lemma 5.2.4. *For $n \geq 3$ and $p \geq 3$, if S' is a maximal 2-packing of $P_n \Box K_{1,p}$ and $l_0 = |S' \cap V(P_n^{(0)})|$, then*

$$I(S') \leq \begin{cases} 4n-2; & \text{if } l_0 = 0 \\ 4n + (p-9)l_0 + 6; & \text{if } l_0 \geq 1 \end{cases}$$

Proof. Let S' be a maximal 2-packing of $P_n \Box K_{1,p}$ and let $|S' \cap V(P_n^{(j)})| = l_j$, for $j \in \{0,1,\ldots,p\}$. As S' is a 2-packing, it can include at most one element from each column, (that is from each layer $K_{1,p}^{(i)}$, for $i \in \{1,2,\ldots,n\}$). Hence, $|S'| = \sum_{j=0}^{p} l_j \leq n$. Also, S' either contains one or more vertices from the layer $P_n^{(0)}$ or may not contain any vertex from $P_n^{(0)}$. Based on this, the following cases are considered:

Case(i): $l_0 = 0$

As $l_0 = 0$, S' includes at most two vertices from $\{(1,j) : 1 \le j \le p\} \cup \{(n,j) : 1 \le j \le p\}$ (that is, at most one from each of the two layers $K_{1,p}^{(0)}$ and $K_{1,p}^{(n)}$, excluding $(1,0)$ and $(n,0)$) and those vertices are of degree 2 each. The remaining (internal) vertices have degree 3. Thus,

$$|S'| = \sum_{j=1}^{p} l_j \le n \quad \text{and} \tag{5.5}$$

$$I(S') \le 4(\sum_{j=1}^{p} l_j - 2) + 2(3)$$

$$\le 4n - 2 \tag{5.6}$$

Case(ii): $l_0 \ge 1$

The following subcases are considered:

Subcase(i): S' includes neither $(1,0)$ nor $(n,0)$

Then, as discussed above, having chosen l_0 internal vertices from the first row (that is, $P_n^{(0)}$), no vertex from the corresponding column and its neighboring columns can be considered for subsequent choices of vertices from the remaining rows, to include in S'. Thus,

$$|S'| - l_0 = \sum_{j=1}^{p} l_j \le n - 3l_0 \quad \text{and} \tag{5.7}$$

$$I(S') \le (p+3)l_0 + 4(\sum_{j=1}^{p} l_j - 2) + 3(2)$$

$$\le 4n + (p-9)l_0 - 2 \tag{5.8}$$

Subcase(ii): S' includes either $(1,0)$ nor $(n,0)$

Then, as discussed earlier, two columns for each choice of vertices $(1,0)$ or $(n,0)$ and three columns corresponding to each internal vertex in $S' \cap V(P_n^{(0)})$ cannot be considered, while choosing vertices from the second and subsequent rows, for inclusion in S'. Further, as $p \ge 3$, if $(1,0) \in S'$ and $(n,0) \notin S'$, then exactly one vertex from $\{(n,j) : 1 \le j \le p\}$ will be included in S'. Similar is the case, when $(n,0) \in S'$ and $(1,0) \notin S'$. Therefore,

$$
\begin{aligned}
|S'| - l_0 &= \sum_{j=1}^{p} l_j \\
&\leq n - [3(l_0 - 1) + 2] = n - 3l_0 + 1 \\
&\Rightarrow |S'| = n - 2l_0 + 1 \;\; \text{and} \qquad (5.9) \\
I(S') &\leq (p+3)(l_0 - 1) + (p+2) + 4(\sum_{j=1}^{p} l_j - 1) + 3 \\
&\leq 4n + (p-9)l_0 + 2 \qquad (5.10)
\end{aligned}
$$

Subcase(iii): S' includes both $(1, 0)$ and $(n, 0)$

Then, as discussed earlier, two columns for each choice of the vertices $(1, 0)$ and $(n, 0)$ and three columns for each choice of the internal vertices in $S' \cap V(P_n^{(0)})$ cannot be considered, while choosing vertices from the remaining rows, for inclusion in S'. Hence,

$$
|S'| - l_0 = \sum_{j=1}^{p} l_j \leq n - [3(l_0 - 2) + 4] = n - 3l_0 + 2 \qquad (5.11)
$$

$$
\Rightarrow |S'| \leq n - 2l_0 + 2 \qquad (5.12)
$$

$$
\begin{aligned}
\text{and } I(S') &\leq (p+3)(l_0 - 2) + 2(p+2) + 4(\sum_{j=1}^{p} l_j) \\
&\leq 4n + (p-9)l_0 + 6 \qquad (5.13)
\end{aligned}
$$

Comparing the three subcases (i), (ii) and (iii), the influence obtained in Subcase(iii) is found to be maximum. Hence, it is concluded from cases (i) and (ii) that $I(S')$ is at most $4n - 2$, if $l_0 = 0$ and at most $4n + (p-9)l_0 + 6$, if $l_0 \geq 1$. □

Remark 5.2.1. *It is noted from the discussion in Lemma 5.2.4 that whenever* $p \geq 3$ *and* $l_0 \geq 1$*, a maximal* 2*-packing has maximum influence only when it includes both the vertices* $(1, 0)$ *and* $(n, 0)$*. Hence, in such a case,* l_0 *must be at least two.*

Using these facts and Lemma 5.2.4, the efficient domination number of $P_n \Box K_{1,p}$ is obtained for $n \geq 3$ and $p \geq 3$, in Theorems 5.2.5 and 5.2.6.

Theorem 5.2.5. *For* $n \geq 3$ *and* $3 \leq p \leq 5$, $P_n \Box K_{1,p} \notin \mathscr{E}$ *and* $F(P_n \Box K_{1,p}) = 4n - 2$.

Proof. Let S' be a maximal 2-packing of $P_n \Box K_{1,p}$. It can be observed from Lemma 5.2.4 and Remark 5.2.1 that S' can attain maximum influence only when either $l_0 = 0$ or $l_0 \geq 2$ and S' includes both $(1,0)$ and $(n,0)$. Since $4n + (p-9)l_0 + 6 \leq 4n - 2$, when $3 \leq p \leq 5$ and $l_0 \geq 2$, it follows that $F(P_n \Box K_{1,p})$ is at most $4n - 2$. Hence, it is required to search for a maximal 2-packing of cardinality at most n and having influence at most $4n - 2$, if one such exists. The following three cases are considered:

Case(i): $n \equiv 0 \pmod 3$

The set $S' = \{(1,1),(4,1),\ldots,(n-2,1)\} \cup \{(2,2),(5,2),\ldots,(n-1,2)\} \cup \{(3,3),(6,3),\ldots,(n,3)\}$ is a maximal 2-packing of $P_n \Box K_{1,p}$ with cardinality n and having influence $4n - 2$.

Case(ii): $n \equiv 1 \pmod 3$

The set $S' = \{(1,1),(4,1),\ldots,(n,1)\} \cup \{(2,2),(5,2),\ldots,(n-2,2)\} \cup \{(3,3),(6,3),\ldots,(n-1,3)\}$ is a maximal 2-packing of $P_n \Box K_{1,p}$ with cardinality n and having influence $4n - 2$.

Case(iii): $n \equiv 2 \pmod 3$

The set $S' = \{(1,1),(4,1),\ldots,(n-1,1)\} \cup \{(2,2),(5,2),\ldots,(n-3,2)\} \cup \{(3,3),(6,3),\ldots,(n-2,3)\}$ is a maximal 2-packing of $P_n \Box K_{1,p}$ with cardinality n and having influence $4n - 2$.

As in all the three cases, it is possible to find a maximal 2-packing of cardinality n and having influence $4n - 2$, it follows that $F(P_n \Box K_{1,p}) = 4n - 2$. □

Theorem 5.2.6. *For $n \geq 3$ and $p \geq 6$, $P_n \Box K_{1,p} \notin \mathscr{E}$ and*

$$F(P_n \Box K_{1,p}) = \begin{cases} 4n + 2p - 12; & \text{if } 6 \leq p \leq 9 \\ 4n + p\lceil \frac{n}{3} \rceil - 9\lceil \frac{n-6}{3} \rceil - 12; & \text{if } p \geq 10 \end{cases}$$

Proof. Let S' be a maximal 2-packing of $P_n \Box K_{1,p}$. Following the discussion in Remark 5.2.1, either $l_0 = 0$ or $l_0 \geq 2$.

But, it follows from Lemma 5.2.4 that if $l_0 = 0$, then $|S'| = \sum_{j=1}^{p} l_j \leq n$ and $I(S') \leq 4n-2$; if $l_0 \geq 2$ and S' includes both $(1,0)$ and $(n,0)$, then $|S'| \leq n - 2l_0 + 2$ and $I(S') \leq 4n + (p-9)l_0 + 6$. In particular, the following observations are made:

- When $l_0 \geq 2$ and $p \geq 6$, $4n - 2 < 4n + (p-9)l_0 + 6$. Hence, in such a case,

S' may attain maximum influence when $l_0 \geq 2$ and it includes both $(1,0)$ and $(n,0)$.

- For $6 \leq p \leq 9$, the quantity $4n + (p-9)l_0 + 6$ is maximum when l_0 is minimum.
- For $p \geq 10$, the quantity $4n+(p-9)l_0+6$ is maximum when l_0 is maximum.
- After choosing the vertices $(1,0)$ and $(n,0)$ from $P_n^{(0)}$ to include in S', from the remaining $(n-2)$ vertices in $P_n^{(0)}$, the vertices $(2,0)$, $(3,0)$, $(n-2,0)$ and $(n-1,0)$ cannot be included in S', as S' is a 2-packing. Hence, from the remaining $(n-6)$ vertices, at most $\lceil \frac{n-6}{3} \rceil$ vertices can be chosen from $P_n^{(0)}$ for possible inclusion in S'. Hence, $2 \leq l_0 \leq 2 + \lceil \frac{n-6}{3} \rceil$.

Based on these observations, it is required to search for a maximal 2-packing of cardinality at most $n - 2l_0 + 2$ which follows the above conditions. The following three cases are considered:

Case(i): $n \equiv 0 \pmod 3$

For $6 \leq p \leq 9$, with the minimum value of l_0, that is, $l_0 = 2$, the set $S' = \{(1,0),(n,0)\} \cup \{(3,1),(6,1),\ldots,(n-3,2)\} \cup \{(4,2),(7,2),\ldots,(n-2,2)\} \cup \{(5,3),(8,3),\ldots,(n-4,3)\}$ is a maximal 2-packing of cardinality $n-2$ and having influence $4n+2p-12$.

For $p \geq 10$, as $n \equiv 0 \pmod 3$, the maximum value of l_0 is $2 + \lceil \frac{n-6}{3} \rceil = \frac{n}{3}$. The set $S' = \{(1,0),(n,0)\} \cup \{(4,0),(7,0),\ldots,(n-5,0)\} \cup \{(n-4,1),(n-3,2)\}$ is a maximal 2-packing of $P_n \square K_{1,p}$ such that $l_0 = \frac{n}{3}$, $|S'| = \frac{n+6}{3}$ and having influence $4n + (p-9)\frac{n}{3} + 6 = 4n + p\lceil \frac{n}{3} \rceil - 9\lceil \frac{n-6}{3} \rceil - 12$.

Case(ii): $n \equiv 1 \pmod 3$

For $6 \leq p \leq 9$, with the minimum value of l_0, that is, $l_0 = 2$, the set $S' = \{(1,0),(n,0)\} \cup \{(3,1),(6,1),\ldots,(n-4,2)\} \cup \{(4,2),(7,2),\ldots,(n-3,2)\} \cup \{(5,3),(8,3),\ldots,(n-2,3)\}$ is a maximal 2-packing of cardinality $n-2$ and having influence $4n+2p-12$.

For $p \geq 10$, as $n \equiv 1 \pmod 3$, the maximum value of l_0 is $2 + \lceil \frac{n-6}{3} \rceil = \frac{n+2}{3}$. The set $S' = \{(1,0),(n,0)\} \cup \{(4,0),(7,0),\ldots,(n-3,0)\}$ is a maximal 2-packing of

$P_n \Box K_{1,p}$ such that $l_0 = \frac{n+2}{3}$, $|S'| = \frac{n+2}{3}$ and having influence $4n+(p-9)(\frac{n+2}{3})+6 = 4n + p\lceil \frac{n}{3} \rceil - 9\lceil \frac{n-6}{3} \rceil - 12$.

Case(iii): $n \equiv 2 \pmod 3$

For $6 \leq p \leq 9$, with the minimum value of l_0, that is, $l_0 = 2$, the set $S' = \{(1,0),(n,0)\} \cup \{(3,1),(6,1),\ldots,(n-2,2)\} \cup \{(4,2),(7,2),\ldots,(n-4,2)\} \cup \{(5,3),(8,3),\ldots,(n-3,3)\}$ is a maximal 2-packing of cardinality $n-2$ and having influence $4n+2p-12$.

For $p \geq 10$, as $n \equiv 2 \pmod 3$, the maximum value of l_0 is $2 + \lceil \frac{n-6}{3} \rceil = \frac{n+1}{3}$. The set $S' = \{(1,0),(n,0)\} \cup \{(4,0),(7,0),\ldots,(n-4,0)\} \cup \{(n-3,1)\}$ is a maximal 2-packing of $P_n \Box K_{1,p}$ such that $l_0 = \frac{n+1}{3}$, $|S'| = \frac{n+4}{3}$ and having influence $4n + (p-9)(\frac{n+1}{3}) + 6 = 4n + p\lceil \frac{n}{3} \rceil - 9\lceil \frac{n-6}{3} \rceil - 12$.

Hence, the result follows. □

The Cartesian product $C_n \Box K_{1,p}$:

Let $V(C_n \Box K_{1,p}) = \{(i,j) : 1 \leq i \leq n, 0 \leq j \leq p\}$, where (i,j) represents a vertex in i^{th} column and j^{th} row (refer to Figure 5.9). The vertex $(i,0)$ corresponds to the central vertex of $K_{1,p}^{(i)}$, for each $i \in \{1,2,\ldots,n\}$. Then, $\deg_{C_n \Box K_{1,p}}(i,0) = p+2$, for $1 \leq i \leq n$ and $\deg_{C_n \Box K_{1,p}}(i,j) = 3$ for $1 \leq i \leq n$, $1 \leq j \leq p$. For $0 \leq j \leq p$, a vertex in the j^{th} layer (that is, in $C_n^{(j)}$) is dominated either by itself or by any of its neighbors in $C_n^{(j)}$ or by the neighbor in $C_n^{(0)}$ (that is, its copy in $C_n^{(0)}$).

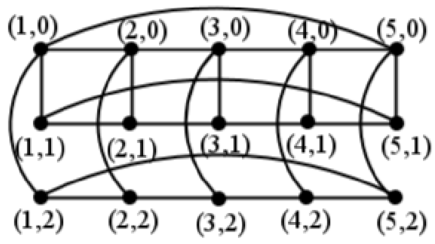


Figure 5.9: $C_5 \Box K_{1,2}$

Let S' be a maximal 2-packing of $C_n \Box K_{1,p}$ and let $|S' \cap V(C_n^{(j)})| = l_j$, for $j \in \{0,1,\ldots,p\}$. Since S' can include at most one element from each of the layers $K_{1,p}^{(i)}$, for $i \in \{1,2,\ldots,n\}$, it follows that $|S'| = \sum_{j=0}^{p} l_j \leq n$. Also, S' either contains one or more vertices from the layer $C_n^{(0)}$ or may not contain any vertex from $C_n^{(0)}$. Hence, in general $I(S') = (p+3)l_0 + 4\sum_{j=1}^{p} l_j$. Moreover, the following observations are made:

1. If $l_0 = 0$, then
$$|S'| = \sum_{j=1}^{p} l_j \leq n \text{ and } I(S') \leq 4n \tag{5.14}$$

2. If $l_0 \geq 1$, then for each choice of vertices, say, $(i, 0)$ from $C_n^{(0)}$, no vertex from its neighboring two layers, namely, $K_{1,p}^{(i-1)}$ and $K_{1,p}^{(i+1)}$ can belong to S'. Thus,
$$|S'| - l_0 = \sum_{j=1}^{p} l_j \leq n - 3l_0 \text{ and}$$
$$I(S') \leq (p+3)l_0 + 4(n - 3l_0) = 4n + (p-9)l_0 \tag{5.15}$$

Based on these facts, the following results are obtained for $C_n \square K_{1,p}$.

Theorem 5.2.7. *$C_n \square K_{1,2} \notin \mathscr{E}$, for $n \geq 3$ and*
$$F(C_n \square K_{1,2}) = \begin{cases} \frac{8n}{3}; & \text{if } n \equiv 0 \ (mod\ 3) \\ \frac{8n-5}{3}; & \text{if } n \equiv 1 \ (mod\ 3) \\ \frac{8n-1}{3}; & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

Proof. Let S' be a maximal 2-packing of $C_n \square K_{1,2}$. It follows from (5.14) and (5.15) that if $l_0 = 0$, then $I(S') \leq 4n$. Otherwise, $I(S') \leq 4n - 7l_0$.
Case(i): $n \equiv 0 \pmod 3$
If $l_0 = 0$, then it follows from (5.14) that $|S'| \leq n$ and $I(S') \leq 4n$. But, as $l_0 = 0$, S' must include vertices only from the two layers $C_n^{(j)}$, where $1 \leq j \leq 2$. In addition, for each j $(1 \leq j \leq 2)$, $S' \cap V(C_n^{(j)})$ is a 2-packing of $C_n^{(j)}$ and hence, $|S' \cap V(C_n^{(j)})| \leq \rho(C_n^{(j)}) = \lfloor \frac{n}{3} \rfloor$. Hence, as $n \equiv 0 \pmod 3$, $|S'| \leq 2\lfloor \frac{n}{3} \rfloor = 2\left(\frac{n}{3}\right) = \frac{2n}{3}$. The set $\{(1,1), (4,1), \ldots, (n-2,1), (2,2), (5,2), \ldots, (n-1,2)\}$ is a 2-packing of $C_n \square K_{1,2}$ with cardinality $\frac{2n}{3}$ and having influence $\frac{8n}{3}$. Since all the vertices excluding the vertices from $C_n^{(0)}$ have degree three, it follows that any 2-packing of cardinality $\frac{2n}{3}$ will have influence $\frac{8n}{3}$. Therefore, when $l_0 = 0$, the maximum influence is $\frac{8n}{3}$ and is attained by a maximal 2-packing of cardinality $\frac{2n}{3}$.
Next, let us consider the case when $l_0 \geq 1$. Using (5.15), $|S'| \leq n - 2l_0 \leq n - 2$.

Having chosen l_0 vertices from $C_n^{(0)}$, the remaining vertices chosen from the $n-3l_0$ columns to include in S' is given by $\sum_{j=1}^{2} l_j$. Further, for every vertex chosen from $C_n^{(0)}$, no vertex can be chosen from the corresponding column and its two neighboring columns, while choosing vertices in the remaining rows (that is, in $C_n^{(j)}$, for $j > 0$). Hence, from each $C_n^{(j)}$ ($j > 0$), we are left with $n - 3l_0$ vertices. As S' is a 2-packing of $C_n \Box K_{1,2}$ and the induced subgraph of the remaining vertices of each layer $C_n^{(j)}$ ($j > 0$) is either P_{n-3l_0} or disjoint copies of P_l, where $l \leq n-3l_0$. Hence, among the remaining $n - 3l_0$ vertices, at most $\lceil \frac{n-3l_0}{3} \rceil$ vertices can be chosen from each row. Therefore, $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0}{3} \rceil = 2(\frac{n}{3}) - 2l_0$, as $n \equiv 0 \pmod 3$. That is, $|S'| \leq \frac{2n}{3} - l_0$ and the influence of any such set is at most $5l_0 + 4(\frac{2n}{3} - 2l_0) = \frac{8n}{3} - 3l_0$, which is less than $\frac{8n}{3}$. Hence, when $n \equiv 0 \pmod 3$, $F(C_n \Box K_{1,2}) = \frac{8n}{3}$.

Case(ii): $n \equiv 1 \pmod 3$

If $l_0 = 0$, then it follows from (5.14) that $|S'| = \sum_{j=1}^{2} l_j \leq n$. But, as $n \equiv 1 \pmod 3$, it can be shown by a similar argument as in Case(i) that $|S'| \leq 2\lfloor \frac{n}{3} \rfloor = 2\left(\frac{n-1}{3}\right) = \frac{2n-2}{3}$ and hence, $I(S') \leq 4(\frac{2n-2}{3}) = \frac{8n-8}{3}$.

On the other hand, let $l_0 \geq 1$. Then, by a similar argument as in Case(i) it can be shown that $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0}{3} \rceil = \frac{2n-6l_0+4}{3}$, as $n \equiv 1 \pmod 3$. Thus, $|S'| \leq \frac{2n-6l_0+4}{3} + l_0$. But, now it will be shown by a constructive proof that $|S'| \lneq \frac{2n-6l_0+4}{3} + l_0$, for any 2-packing S' of $C_n \Box K_{1,2}$ and it will also be proved that there exists a 2-packing in $C_n \Box K_{1,2}$ with cardinality $(\frac{2n-6l_0+4}{3} - 1 + l_0)$, having maximum influence.

Suppose that $l_0 = 1$. Without loss of generality, choose $(1, 0)$ from the layer $C_n^{(0)}$ to include in S'. Next, start choosing vertices from the layer $C_n^{(1)}$ to include in S'. The vertex $(1, 0)$ dominates itself and dominates the vertices $(2, 0), (n, 0)$ and $(1, j)$, for $1 \leq j \leq 2$. Next, $(2, 1)$ can be dominated by itself or $(1, 1)$ or $(2, 0)$ or $(3, 1)$. Since S' is a 2-packing, it cannot include $(2, 1)$ or $(1, 1)$ or $(2, 0)$. Therefore, $(3, 1) \in S'$. Continuing from $(3, 1)$, choose vertices $\{(3, 1), (6, 1), \ldots, (n-1, 1)\}$ from the layer $C_n^{(1)}$. It can be observed that $\lceil \frac{n-3l_0}{3} \rceil = \frac{n+2-3l_0}{3}$ vertices from the layer $C_n^{(1)}$ are included in S'. Next, choose vertices from the layer $C_n^{(2)}$. None of

the vertices $(i,2)$, for $1 \leq i \leq 3$ can be included in S', as they are either adjacent or at a distance two from the vertices already included in S'. Hence, excluding the vertices lying in the first three columns (that is, $K_{1,2}^{(i)}$, for $i \in \{1,2,3\}$) of the layer $C_n^{(2)}$, choose from the remaining vertices in $C_n^{(2)}$ to include in S'. Without loss of generality, choose $(4,2)$ to include in S'. Having chosen $(4,2)$, choose vertices $\{(4,2),(7,2),\ldots,(n-3,2)\}$ to include in S'. It can be observed that the vertices $(n-1,2)$ and $(n,2)$ cannot be included in S', since they are at a distance two from the vertices already included in S'. Hence, from the layer $C_n^{(2)}$, it is possible to choose only $\lceil \frac{n-3l_0}{3} \rceil - 1$ vertices. Similarly, for any choice of the initial vertex from $C_n^{(2)}$ other than $(4,2)$, it can be observed that at most $\lceil \frac{n-3l_0}{3} \rceil - 1$ vertices can be chosen. Thus, $\sum_{j=1}^{2} l_j \leq \frac{2n-6l_0+4}{3} - 1 = \frac{2n-6l_0+1}{3}$. Hence, $|S'| \leq \frac{2n-6l_0+1}{3} + l_0 = \frac{2n-3l_0+1}{3} \leq \frac{2n-2}{3}$ and $I(S') \leq 5l_0 + 4(\frac{2n-6l_0+1}{3}) = \frac{8n}{3} - (3l_0 - \frac{4}{3}) \leq \frac{8n-5}{3}$.
The set $S' = \{(0,0)\} \cup \{(3,1),(6,1),\ldots,(n-1,1)\} \cup \{(4,2),(7,2),\ldots,(n-3,2)\}$ is a 2-packing of $C_n \square K_{1,2}$ with cardinality $\frac{2n-2}{3}$ and having influence $\frac{8n-5}{3}$. It can shown that for any other 2-packing T' of $C_n \square K_{1,2}$, where $|T' \cap V(C_n^{(0)})| = 1$ and $|T'| = \frac{2n-2}{3}$, has influence $\frac{8n-5}{3}$. Moreover, it follows from (5.15) that any other 2-packing of $C_n \square K_{1,2}$ such that $l_0 > 1$ will have influence less than $\frac{8n-5}{3}$. Hence, when $n \equiv 1 \pmod 3$, $F(C_n \square K_{1,2}) = \frac{8n-5}{3}$.
Case(iii): $n \equiv 2 \pmod 3$
If $l_0 = 0$, then it follows from (5.14) that $|S'| = \sum_{j=1}^{2} l_j \leq n$. But, as $n \equiv 2$ (mod 3), it can be shown by a similar argument as in Case(i) that $|S'| \leq 2\lfloor \frac{n}{3} \rfloor = 2\left(\frac{n-2}{3}\right) = \frac{2n-4}{3}$ and hence, $I(S') \leq 4(\frac{2n-4}{3}) = \frac{8n-16}{3}$.
On the other hand, if $l_0 \geq 1$, then by a similar argument as in Case(i) it can be shown that $|S'| - l_0 = \sum_{j=1}^{2} l_j \leq 2\lceil \frac{n-3l_0}{3} \rceil = \frac{2n-6l_0+2}{3}$, as $n \equiv 2 \pmod 3$. Hence, $|S'| \leq \frac{2n-6l_0+2}{3} + l_0 = \frac{2n-3l_0+2}{3} \leq \frac{2n-1}{3}$ and $I(S') \leq 5l_0 + 4(\frac{2n-6l_0+2}{3}) = \frac{8n}{3} - (3l_0 - \frac{8}{3}) \leq \frac{8n-1}{3}$. The set $S' = \{(0,0)\} \cup \{(3,1),(6,1),\ldots,(n-2,1)\} \cup \{(4,2),(7,2),\ldots,(n-1,2)\}$ is a 2-packing of $C_n \square K_{1,2}$ with cardinality $\frac{2n-1}{3}$ and having influence $\frac{8n-1}{3}$. By choosing vertices row by row, in a similar fashion as for S', it can shown that any other 2-packing T' of $C_n \square K_{1,2}$ such that $|T' \cap V(C_n^{(0)})| = 1$ and $|T'| = \frac{2n-1}{3}$ has the same influence $\frac{8n-1}{3}$. Moreover, it follows from (5.15)

that any other 2-packing of $C_n \Box K_{1,2}$ such that $l_0 > 1$ will have influence less than $\frac{8n-1}{3}$. Hence, when $n \equiv 2 \pmod 3$, $F(C_n \Box K_{1,2}) = \frac{8n-1}{3}$. □

Theorem 5.2.8. *For $n \geq 3$, $C_n \Box K_{1,3} \in \mathscr{E}$ if and only if $n \equiv 0 \ (mod\ 3)$.*
When $n \not\equiv 0 \ (mod\ 3)$, the following holds:

$$F(C_n \Box K_{1,3}) = \begin{cases} 4n-4; & \text{if } n \equiv 1 \ (mod\ 3) \\ 4n-6; & \text{if } n \equiv 2 \ (mod\ 3) \end{cases}$$

Proof. Let S' be a maximal 2-packing of $C_n \Box K_{1,3}$. It follows from (5.14) and (5.15) that if $l_0 = 0$, then $I(S') \leq 4n$. Otherwise, $I(S') \leq 4n - 6l_0$. Hence, if $l_0 \neq 0$, then S' has maximum influence when l_0 is minimum.

Case(i): $n \equiv 0 \pmod 3$

For $S' = \{(1,1),(4,1),\ldots,(n-2,1)\} \cup \{(2,2),(5,2),\ldots,(n-1,2)\} \cup \{(3,3),(6,3),\ldots,(n,3)\}$, it can be seen that $I(S') = 4n$. Hence, $C_n \Box K_{1,3} \in \mathscr{E}$, if $n \equiv 0 \pmod{3}$.

Case(ii): $n \equiv 1 \pmod 3$

If $l_0 = 0$, then it follows from (5.14) that $|S'| = \sum_{j=1}^{3} l_j \leq n$ and hence, $I(S') \leq 4n$. But, as $l_0 = 0$, S' must include vertices only from the three layers $C_n^{(j)}$, where $1 \leq j \leq 3$. In addition, for each j $(1 \leq j \leq 3)$, $S' \cap V(C_n^{(j)})$ is a 2-packing of $C_n^{(j)}$ and hence, $|S'| \leq 3\lfloor \frac{n}{3} \rfloor = 3\left(\frac{n-1}{3}\right)$, as $n \equiv 1 \pmod 3$. That is, $|S'| \leq n-1$. The set $\{(1,1),(4,1),\ldots,(n-3,1)\} \cup \{(2,2),(5,2),\ldots,(n-2,2)\} \cup \{(3,3),(6,3),\ldots,(n-1,3)\}$ is a 2-packing of $C_n \Box K_{1,3}$ with cardinality $n-1$ and having influence $4(n-1)$. Infact, as $l_0 = 0$ and all the vertices excluding those in $C_n^{(0)}$ have degree three, any 2-packing of cardinality $n-1$ which does not include vertices from $C_n^{(0)}$ will have influence $4(n-1)$ and hence, in this case, the maximum influence is $4n-4$.

On the other hand, if $l_0 \geq 1$, then it follows from (5.15) that the maximum influence of a maximal 2-packing of $C_n \Box K_{1,3}$ is $4n - 6l_0$, which is less than $4n-4$. Hence, when $n \equiv 1 \pmod 3$, $F(C_n \Box K_{1,3}) = 4n-4$.

Case(iii): $n \equiv 2 \pmod 3$

If $l_0 = 0$, then $|S'| = \sum_{j=1}^{3} l_j \leq n$. But, as $n \equiv 2 \pmod 3$, it can be shown by a similar argument as in Case(ii) that $|S'| \leq 3\lfloor \frac{n}{3} \rfloor = n-2$ and hence, $I(S') \leq 4(n-2)$.

On the other hand, if $l_0 \geq 1$, then using (5.15), $|S'| \leq n - 2l_0 \leq n - 2$. The set $S' = \{(0,0)\} \cup \{(3,1),(6,1),\ldots,(n-2,1)\} \cup \{(4,2),(7,2),\ldots,(n-1,2)\} \cup \{(5,3),(8,3),\ldots,(n-3,3)\}$ is a 2-packing of $C_n \Box K_{1,3}$ with cardinality $n-2$ and having influence $4n-6$. Moreover, $|S' \cap V(C_n^{(0)})| = 1$ (that is, $l_0 = 1$). By a similar argument as in Case(ii), it can be seen that if T' is any other 2-packing of $C_n \Box K_{1,3}$ such that $|T' \cap V(C_n^{(0)})| = 1$ and $|T'| = n-2$, then $I(T') = 4n-6 = I(S')$. Further, it follows from (5.15) that any other 2-packing of $C_n \Box K_{1,3}$ such that $l_0 > 1$ will have influence less than $4n-6$. Hence, when $n \equiv 2 \pmod 3$, $F(C_n \Box K_{1,3}) = 4n-6$. Also, it follows from all the above three cases that $C_n \Box K_{1,3} \in \mathscr{E}$ if and only if $n \equiv 0 \pmod 3$. □

Theorem 5.2.9. *For $p \geq 4$ and $n \geq 3$, $C_n \Box K_{1,p} \notin \mathscr{E}$ and*

$$F(C_n \Box K_{1,p}) = \begin{cases} \max\{4n-4, 4n+p-9\}; & \text{for } n \equiv 2 \ (mod\ 3) \text{ and } p = 4 \\ \max\{4n, 4n+p-9\}; & \text{otherwise} \end{cases}$$

Proof. Let S' be a maximal 2-packing of $C_n \Box K_{1,p}$.

If $l_0 = 0$, then it follows from (5.14) that $|S'| = \sum_{j=1}^{p} l_j \leq n$ and hence, $I(S') \leq 4n$. But, as $l_0 = 0$, S' must include vertices only from the p layers $C_n^{(j)}$, where $1 \leq j \leq p$. In addition, for each j $(1 \leq j \leq p)$, $S' \cap V(C_n^{(j)})$ is a 2-packing of $C_n^{(j)}$. Having chosen 2-packings from the layers $C_n^{(j)}$, for $j \in \{1,2,3\}$ to include in S' (as discussed in Theorem 5.2.8), it follows that

$$\sum_{j=1}^{3} l_j \leq \begin{cases} n; & \text{if } n \equiv 0 \pmod 3 \\ n-1; & \text{if } n \equiv 1 \pmod 3 \\ n-2; & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

For $n \equiv 1 \pmod 3$, S' can include the remaining one vertex from the layer $C_n^{(4)}$ and for $n \equiv 2 \pmod 3$, remaining two vertices can be chosen one each from the layers $C_n^{(4)}$ and $C_n^{(5)}$ to include in S'. But, when $p = 4$, the layer $C_n^{(5)}$ does not exists. Hence,

$$|S'| \leq \begin{cases} n-1; & \text{for } n \equiv 2 \pmod 3 \text{ and } p = 4 \\ n; & \text{otherwise} \end{cases}$$

When $n \equiv 0 \pmod 3$, the set $S' = \{(1,1),(4,1),\ldots,(n-2,1)\} \cup \{(2,2),(5,2),\ldots,(n-1,2)\} \cup \{(3,3),(6,3),\ldots,(n,3)\}$ is of cardinality n and has influence $4n$. When

$n \equiv 1 \pmod 3$, the set $S' = \{(1,1),(4,1),\ldots,(n-4,1)\} \cup \{(2,2),(5,2),\ldots,(n-2,2)\} \cup \{(3,3),(6,3),\ldots,(n-1,3)\}$ is of cardinality n and has influence $4n$. When $n \equiv 2 \pmod 3$ and $p = 4$, the set $S' = \{(1,1),(4,1),\ldots,(n-4,1)\} \cup \{(2,2),(5,2),\ldots,(n-3,2)\} \cup \{(3,3),(6,3),\ldots,(n-2,3)\} \cup \{(n-1,4)\}$ is of cardinality $n-1$ and has influence $4(n-1)$. For $n \equiv 2 \pmod 3$ and $p > 4$, the set $S' = \{(1,1),(4,1),\ldots,(n-4,1)\} \cup \{(2,2),(5,2),\ldots,(n-3,2)\} \cup \{(3,3),(6,3),\ldots,(n-2,3)\} \cup \{(n-1,4)\} \cup \{(n,5)\}$ is of cardinality n and has influence $4n$. Furthermore, when $l_0 = 0$, any maximal 2-packing of $C_n \square K_{1,p}$ with cardinality n (or $n-1$) will have influence $4n$ (or $4n-1$).

On the other hand, if $l_0 \geq 1$, then using (5.15), $|S|' \leq n - 2l_0 \leq n-2$. Also, it follows that, the maximum influence of a maximal 2-packing of $C_n \square K_{1,p}$ is $4n + (p-9)l_0$, that is, at most $4n+p-9$. This value exceeds $4n$, whenever $p > 9$. Thus, if $p > 9$ and $n \equiv 0 \pmod 3$, then the set $S' = \{(0,0)\} \cup \{(3,1),(6,1),\ldots,(n-3,1)\} \cup \{(4,2),(7,2),\ldots,(n-2,2)\} \cup \{(5,3),(8,3),\ldots,(n-1,3)\}$ is of cardinality $n-2$ and has influence $4n+p-9$. Similarly, if $p > 9$ and $n \equiv 1 \pmod 3$, then the set $S' = \{(0,0)\} \cup \{(3,1),(6,1),\ldots,(n-1,1)\} \cup \{(4,2),(7,2),\ldots,(n-3,2)\} \cup \{(5,3),(8,3),\ldots,(n-2,3)\}$ is of cardinality $n-2$ and has influence $4n+p-9$ and when $p > 9$ and $n \equiv 2 \pmod 3$, the set $S' = \{(0,0)\} \cup \{(3,1),(6,1),\ldots,(n-2,1)\} \cup \{(4,2),(7,2),\ldots,(n-1,2)\} \cup \{(5,3),(8,3),\ldots,(n-3,3)\}$ is of cardinality $n-2$ has influence $4n+p-9$. Moreover, any 2-packing T' of $C_n \square K_{1,p}$ where $|T' \cap V(C_n^{(0)})| = 1$ and $|T'| = n-2$ has influence $4n+p-9$. Further, it follows from (5.15) that any other 2-packing of $C_n \square K_{1,p}$ such that $l_0 > 1$ will have influence less than $4n+p-9$. Hence,

$$F(C_n \square K_{1,p}) = \begin{cases} \max\{4n-4, 4n+p-9\}; & \text{for } n \equiv 1 \pmod 3 \text{ and } p = 4 \\ \max\{4n, 4n+p-9\}; & \text{otherwise} \end{cases}$$

□

The Cartesian product $K_n \square K_p$:

For any positive integer p, it is known that $F(K_p \square K_p) = 2p-1$ (Goddard et al., 2000). In general, $K_n \square K_p$ is a regular graph of diameter two. Therefore, the product is efficiently dominatable if and only if its radius is one. The following result supports this fact and in addition, it computes the exact value of the efficient domination number of the product when it is not efficiently dominatable.

Theorem 5.2.10. *$K_n \square K_p \in \mathscr{E}$ if and only if either $n = 1$ or $p = 1$. Whenever $n \geq 2$ and $p \geq 2$, $F(K_n \square K_p) = n + p - 1$.*

Proof. Let $V(K_n \square K_p) = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq p\}$, where (i, j) corresponds to a vertex in the i^{th} column and j^{th} row (refer to Figure 5.10). If either $n = 1$ or $p = 1$, then it is evident that $K_n \square K_1 \in \mathscr{E}$, as $K_n \square K_1 \cong K_n$ and $K_1 \square K_p \cong K_p$.

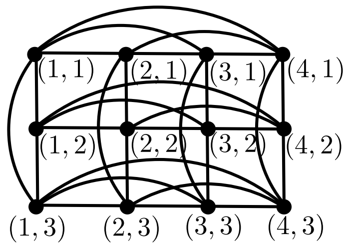


Figure 5.10: $K_4 \square K_3$

Conversely, let $n > 1$ and $p > 1$. Then, it can be observed that $K_n \square K_p$ is a regular graph of degree $n + p - 2$ and is of diameter two. Hence, if S' is a maximal 2-packing of $K_n \square K_p$, then $|S'| = 1$ and $I(S') \leq n + p - 1$. The set $S' = \{(1, 1)\}$ is a maximal 2-packing of $K_n \square K_p$ with cardinality one and having influence $n + p - 1$. Thus, $F(K_n \square K_p) = n + p - 1$, for n, $p \geq 2$. □

The Cartesian product $P_n \square K_p$:

Let $V(P_n \square K_p) = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq p\}$, where (i, j) corresponds to a vertex in the i^{th} column and j^{th} row (refer to Figure 5.11). Then, for $j \in \{1, 2, \ldots, p\}$, $deg_{P_n \square K_p}(1, j) = p = deg_{P_n \square K_p}(n, j)$ and for $2 \leq i \leq n - 1$ and $1 \leq j \leq p$, $deg_{P_n \square K_p}(i, j) = p + 1$.

Let S' be a maximal 2-packing of $P_n \square K_p$. For each i, j, where $2 \leq i \leq n - 1$ and $1 \leq j \leq p$, if $(i, j) \in S'$, then no other vertex in $V(K_p^{(i)})$ and its neighboring layers (or columns), namely, $V(K_p^{(i-1)}) \cup V(K_p^{(i+1)})$ can belong to S'. Furthermore, it can be observed that if $(1, j) \in S'$, for some $j \in \{1, \ldots, p\}$, then no other vertex from $V(K_p^{(1)}) \cup V(K_p^{(2)})$ can be included in S'. Similarly, if $(n, j) \in S'$, for some $j \in \{1, \ldots, p\}$, then no other vertex from $V(K_p^{(n)}) \cup V(K_p^{(n-1)})$ can be included

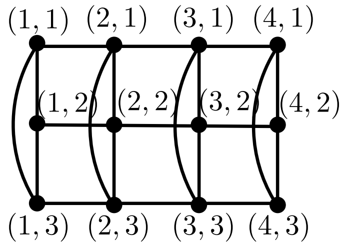


Figure 5.11: $P_4 \square K_3$

in S'. Also, S' can include at most one element from each of the layers $K_p^{(i)}$, for $i \in \{1, 2, \ldots, n\}$. In other words, S' can include elements only from the alternating columns. Thus,

$$|S'| \leq \left\lceil \frac{n}{2} \right\rceil \text{ and} \tag{5.16}$$

$$I(S') \leq (p+2)\left\lceil \frac{n}{2} \right\rceil \tag{5.17}$$

In particular, S' may or may not contain the vertices $(1, j)$ and (n, j), for some j ($j \in \{1, 2, \ldots, p\}$). Accordingly, for any $j \in \{1, 2, \ldots, p\}$, the following cases arise:

Case(i): S' includes neither $(1, j)$ nor (n, j)

Then as discussed above, for each choice of vertices, say (i, j), where $2 \leq i \leq n-1$ and $1 \leq j \leq p$, no vertex from the i^{th} column (that is, from $V(K_p^{(i)})$) and its neighboring columns (that is, $V(K_p^{(i-1)}) \cup V(K_p^{(i+1)})$) can be considered for subsequent choices of vertices from the remaining rows, to include in S'. And, all the vertices included in S' are of degree $p+1$. Thus,

$$|S'| \leq \left\lceil \frac{n}{2} \right\rceil - 1 \text{ and}$$

$$I(S') \leq (p+2)(\left\lceil \frac{n}{2} \right\rceil - 1) = (p+2)\left\lceil \frac{n}{2} \right\rceil - (p+2) \tag{5.18}$$

Case(ii): S' includes either $(1, j)$ or (n, j)

Then as discussed above, for each choice of vertices, say (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq p$, no vertex from $V(K_p^{(i)})$ and its neighboring column(s) can be considered for subsequent choices of vertices from the remaining rows, to include in S'. And, all the vertices included in S' are of degree $p+1$, except $(1, j)$ (or (n, j)), where $(1, j)$ (or (n, j)) is of degree p. Thus,

$$|S'| \leq \left\lceil \frac{n}{2} \right\rceil \text{ and}$$

$$\begin{aligned} I(S') &\leq (p+1) + (p+2)(\left\lceil \frac{n}{2} \right\rceil - 1) \\ &\leq (p+2)\left\lceil \frac{n}{2} \right\rceil - 1 \end{aligned} \tag{5.19}$$

Case(iii): S' includes both $(1, j)$ and (n, j)

Then as discussed above, for each choice of vertices, say (i, j), where $1 \leq i \leq n$ and $1 \leq j \leq p$, no vertex from the corresponding column and the neighboring column(s) can be considered for subsequent choices of vertices from the remaining rows, to include in S'. And, all the vertices included in S', except $(1, j)$ and (n, j) (which are of degree p), are of degree $p + 1$. Thus,

$$|S'| \leq \left\lceil \frac{n}{2} \right\rceil \text{ and}$$

$$\begin{aligned} I(S') &\leq 2(p+1) + (p+2)(\left\lceil \frac{n}{2} \right\rceil - 2) \\ &\leq (p+2)\left\lceil \frac{n}{2} \right\rceil - 2 \end{aligned} \tag{5.20}$$

Using the above facts, the following result is proved.

It is already known that $P_1 \square K_1 \cong P_1 \in \mathscr{E}$ and $P_1 \square K_2 \cong K_2 \in \mathscr{E}$, hence the Theorem 5.2.11 is discussed for remaining values of n and p.

Theorem 5.2.11. *If $n \geq 2$ and $p \geq 3$, then $P_n \square K_p \notin \mathscr{E}$ and*

$$F(P_n \square K_p) = \begin{cases} \frac{pn+2n-2}{2}; & \text{if } n \text{ is even} \\ \frac{pn+2n+p-2}{2}; & \text{if } n \text{ is odd} \end{cases}$$

Proof. Let $n \geq 2$ and $p \geq 3$. Suppose that S' is a maximal 2-packing of $P_n \square K_p$. Then, the following two cases are considered:

Case(i): n is even

Since n is even, it is noted from the above discussion that $|S'| \leq \lceil \frac{n}{2} \rceil = \frac{n}{2}$. And, $I(S') \leq (p+2)\lceil \frac{n}{2} \rceil - 1 = (p+2)(\frac{n}{2}) - 1 = \frac{(pn+2n-2)}{2}$. Since $I(S') \neq np$, $P_n \square K_p \notin \mathscr{E}$. It is required to find a 2-packing of $P_n \square K_p$ having the maximum influence. It can be observed that for $n \equiv 0 \pmod 4$, the set $S' = \{(1, 1), (5, 1), \ldots, (n-3, 1)\} \cup \{(3, 2), (7, 2), \ldots, (n-1, 2)\}$ is a maximal 2-packing of $P_n \square K_p$ with cardinality $(\frac{n}{2})$ and having influence $\frac{(pn+2n-2)}{2}$.

And, for $n \equiv 2 \pmod 4$, the set $S' = \{(1, 1), (5, 1), \ldots, (n-1, 1)\} \cup \{(3, 2), (7, 2),$

$\ldots, (n-3, 2)\}$ is a maximal 2-packing of $P_n \Box K_p$ with cardinality $(\frac{n}{2})$ and having influence $\frac{pn+2n-2}{2}$.

Case(ii): n is odd

Claim: S' *includes both* $(1, j)$ *and* (n, j)

Since n is odd, if S' includes $(1, j)$, say without loss of generality, let $(1, 1) \in S'$, then $S' = \{(1, 1), (5, 1), \ldots, (n, 1)\} \cup \{(3, 2), (7, 2), \ldots, (n-2, 2)\}$, when $n \equiv 1 \pmod 4$ and $S' = \{(1, 1), (5, 1), \ldots, (n-2, 1)\} \cup \{(3, 2), (7, 2), \ldots, (n, 2)\}$, whenever $n \equiv 3 \pmod 4$. Thus, it can be observed that if S' includes $(1, j)$, then it also includes (n, j) and vice versa.

Thus, in both of these cases $|S'| \leq \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ and $I(S') \leq (p+2)\lceil \frac{n}{2} \rceil - 2 = (p+2)(\frac{n+1}{2}) - 2 = \frac{(pn+2n+p-2)}{2}$.

Thus, $F(P_n \Box K_p) = \begin{cases} \frac{pn+2n-2}{2}; & \text{if } n \text{ is even} \\ \frac{pn+2n+p-2}{2}; & \text{if } n \text{ is odd} \end{cases}$ □

The Cartesian product $C_n \Box K_p$:

Let $V(C_n \Box K_p) = \{(i, j) : 1 \leq i \leq n, 1 \leq j \leq p\}$, where (i, j) corresponds to a vertex in the i^{th} column and j^{th} row (refer to Figure 5.12). Then, for $1 \leq i \leq n$ and $1 \leq j \leq p$, $deg_{C_n \Box K_p}(i, j) = p + 1$.

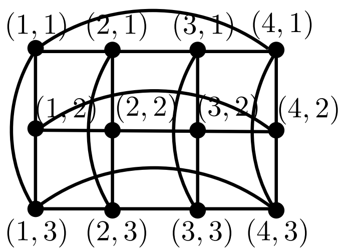


Figure 5.12: $C_4 \Box K_3$

Let S' be a maximal 2-packing of $C_n \Box K_p$. It can be observed that S' can include at most one element from each of the layers $K_p^{(i)}$, for $i \in \{1, 2, \ldots, n\}$. Also, for each i, j, where $1 \leq i \leq n$ and $1 \leq j \leq p$, if $(i, j) \in S'$, then no other vertex from $V(K_p^{(i)})$ and its neighboring layers, namely, $V(K_p^{(i-1)}) \cup V(K_p^{(i+1)})$ can belong to S'. Hence, S' can include elements only from the alternating columns.

Thus,

$$|S'| \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and} \tag{5.21}$$

$$I(S') \leq (p+2)\left\lfloor \frac{n}{2} \right\rfloor \tag{5.22}$$

Theorem 5.2.12. *If $n \geq 2$ and $m \geq 3$, then $C_n \Box K_p \notin \mathscr{E}$ and*

$$F(C_n \Box K_p) = \begin{cases} \frac{pn+2n}{2}; & \textit{if } n \textit{ is even} \\ \frac{pn+2n-p-2}{2}; & \textit{if } n \textit{ is odd} \end{cases}$$

Proof. Suppose that S' is a maximal 2-packing of $C_n \Box K_p$. Then, two cases arise:

Case(i): n is even

Since n is even, by choosing vertices from the alternating columns as discussed above and using (5.21) and (5.22), $|S'| \leq \lfloor \frac{n}{2} \rfloor = \frac{n}{2}$. And, $I(S') \leq (p+2)\lfloor \frac{n}{2} \rfloor = (p+2)(\frac{n}{2}) = \frac{(pn+2n)}{2}$.

If $n \equiv 0 \pmod 4$, then the set $S' = \{(1,1),(5,1),\ldots,(n-3,1)\} \cup \{(3,2),(7,2),\ldots,(n-1,2)\}$ is a maximal 2-packing of $C_n \Box K_p$ with cardinality $(\frac{n}{2})$ and having influence $\frac{(pn+2n)}{2}$.

For $n \equiv 2 \pmod 4$, the set $S' = \{(1,1),(5,1),\ldots,(n-5,1)\} \cup \{(3,2),(7,2),\ldots,(n-3,2)\} \cup \{(n-1,3)\}$ is a maximal 2-packing of $C_n \Box K_p$ with cardinality $(\frac{n}{2})$ and having influence $\frac{(pn+2n)}{2}$.

Case(ii): n is odd

Since n is odd, it follows from the above discussion and (5.21) and (5.22) that $|S'| \leq \lfloor \frac{n}{2} \rfloor = \frac{n-1}{2}$. And $I(S') \leq (p+2)\lfloor \frac{n}{2} \rfloor = (p+2)(\frac{n-1}{2}) = \frac{(pn+2n-p-2)}{2}$.

If $n \equiv 1 \pmod 4$, the set $S' = \{(1,1),(5,1),\ldots,(n-4,1)\} \cup \{(3,2),(7,2),\ldots,(n-2,2)\}$ is a maximal 2-packing of $C_n \Box K_p$ with cardinality $(\frac{n-1}{2})$ and having influence $\frac{(pn+2n-p-2)}{2}$.

For $n \equiv 3 \pmod 4$, the set $S' = \{(1,1),(5,1),\ldots,(n-2,1)\} \cup \{(3,2),(7,2),\ldots,(n-4,2)\}$ is a maximal 2-packing of $C_n \Box K_p$ with cardinality $(\frac{n-1}{2})$ and having influence $\frac{(pn+2n-p-2)}{2}$. Thus,

$$F(C_n \Box K_p) = \begin{cases} \frac{pn+2n}{2}; & \text{if } n \text{ is even} \\ \frac{pn+2n-p-2}{2}; & \text{if } n \text{ is odd} \end{cases}$$

□

5.3 Efficient Domination in the cartesian Product $G\square K_{1,p}$

It is known that $K_1\square K_{1,p} \cong K_{1,p}$ and is efficiently dominatable. Hence, from now on, it is assumed that the factor G in the product $G\square K_{1,p}$ is connected and $G \neq K_1$.

In this section, with the motivation of identifying the class of efficiently dominatable graphs having $K_{1,p}$ as one of the factors, initially some conditions are derived for any vertex subset of $G\square K_{1,p}$ to be an $F(G\square K_{1,p})$-set. Then, efficiently dominatable product graphs $G\square K_{1,p}$ are characterized.

Throughout the discussions to follow, the following notations are used, unless specified otherwise:

Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(K_{1,p}) = \{v_0, v_1, \ldots, v_p\}$, where v_0 represents the central vertex. Then, $|V(G\square K_{1,p})| = n(p+1)$. For any vertex $(u_i, v_j) \in V(G\square K_{1,p})$, where $1 \leq i \leq n$ and $1 \leq j \leq p$, $\deg_{G\square K_{1,p}}(u_i, v_j) = \deg_G(u_i) + 1$ and $\deg_{G\square K_{1,p}}(u_i, v_0) = \deg_G(u_i) + p$. Clearly, for any set $S' \subseteq V(G\square K_{1,p})$, if S' is an $F(G\square K_{1,p})$-set, then the set S'_0 may or may not be empty, where $S'_0 = S' \cap V(G^{(v_0)})$.

Notation 5.3.1.

- $|V(G)| = n$
- *For any set* $S' \subseteq V(G\square K_{1,p})$, S *denotes* $p_G(S')$
- *For* $0 \leq j \leq p$, $S'_j = V(G^{(v_j)}) \cap S'$ *and* $S_j = p_G(S'_j)$

Fact 5.3.1. *Let* S' *be an* $F(G\square K_{1,p})$*-set and* $S = p_G(S')$*. Then the following properties are noted:*

1. *For any* $i \in \{1, 2, \ldots, n\}$, $|V(K_{1,p}^{(u_i)}) \cap S'| \leq 1$ *and hence,* $|S'| \leq n$*. Further,* $|S'| = |S|$.
2. $I_G(S) + |S| \leq F(G\square K_{1,p}) \leq I_G(S) + p|S|$.

Proof. For any $(u_i, v_j) \in V(G\Box K_{1,p})$,

$$\deg_{G\Box K_{1,p}}(u_i, v_j) = \begin{cases} \deg_G(u_i) + p; & \text{if } j = 1 \\ \deg_G(u_i) + 1; & \text{otherwise.} \end{cases}$$

Further, as $|S'| = |S|$,

$$\sum_{u_i \in S} [\deg_G(u_i) + 1] \leq \sum_{(u_i,v_j)\in S'} \deg_{G\Box H}(u_i, v_j) \leq \sum_{u_i \in S} [\deg_G(u_i) + p]$$

$$\Rightarrow |S| + \sum_{u_i \in S} \deg_G(u_i) \leq \sum_{(u_i,v_j)\in S'} \deg_{G\Box H}(u_i, v_j) \leq p|S| + \sum_{u_i \in S} \deg_G(u_i),$$

for all $(u_i, v_j) \in V(G\Box K_{1,p})$.

Therefore, $|S| + I_G(S) \leq I_{G\Box K_{1,p}}(S') \leq p|S| + I_G(S)$. Equivalently, $I_G(S) + |S| \leq F(G\Box K_{1,p}) \leq I_G(S) + p|S|$. □

3. $F(G\Box K_{1,p}) = I_G(S) + |S|$ *if and only if* $S'_0 = \emptyset$.

4. $F(G\Box K_{1,p}) = I_G(S) + p|S|$ *if and only if* $S'_j = \emptyset$, *for all* j, *where* $1 \leq j \leq p$.

Proposition 5.3.1. *Let* G *be a graph of order* n, *where* $n \geq 2$. *If* $G\Box K_{1,p} \in \mathscr{E}$ *and* S' *is its EDS, then either* $p \leq \delta(G) + 1$ *or* $p \leq n - \Delta'(G) - 1$, *where* $\Delta'(G) = \max\{\deg(u) : u \in p_G(S'_0)\}$.

Proof. Let S' be an EDS of $G\Box K_{1,p}$. As discussed earlier, for any $u \in V(G)$, $|V(K_{1,p}^{(u)}) \cap S'| \leq 1$. Also, for any j $(0 \leq j \leq p)$, exactly one vertex is chosen from $N_G[u] \times \{v_j\}$ to efficiently dominate (u, v_j). Two cases arise: $S'_0 = \emptyset$ and $S'_0 \neq \emptyset$.

Case(i): $S'_0 = \emptyset$

In this case, the maximum number of copies of u dominated efficiently by $\cup_{j=1}^{p}(N_G[u] \times \{v_j\})$ is $\deg(u) + 1$. Hence, $p \leq \deg(u) + 1$. Since u is arbitrary, $p \leq \delta(G) + 1$.

Case(ii): $S'_0 \neq \emptyset$

Let $u \in p_G(S'_0)$. Then, (u, v_0) dominates $V(K_{1,p}^{(u)}) \cup [N_G[u] \times \{v_0\}]$. Let $x \in N_G(u)$. Then, $V(K_{1,p}^{(x)}) \cap S' = \emptyset$. Hence, to efficiently dominate each of the p vertices in $V(K_{1,p}^{(x)}) - \{(x, v_0)\}$, p distinct vertices are needed, one from each set $[V(G) - N[u]] \times \{v_j\}$, $(1 \leq j \leq p)$. Hence, $p \leq |V(G) - N[u]|$. That is,

$p \leq n - \deg_G(u) - 1$. Since, u is arbitrary, this is true for every vertex in $p_G(S'_0)$. Hence, $p \leq n - \Delta'(G) - 1$, where $\Delta'(G) = \max\{\deg(u) : u \in p_G(S'_0)\}$. $\square$

Suppose $S' = \cup_{j=0}^{p} S'_j$, where $S'_j \subseteq V(G^{(v_j)})$ and $S_j = p_G(S'_j)$, for $0 \leq j \leq p$, then it is observed that S' is a 2-packing of $G \square K_{1,p}$ if and only if S'_j, for each $j \in \{0, 1, \ldots, p\}$, is a 2-packing in $G^{(v_j)}$ if and only if S_j is a 2-packing in G, for each $j \in \{0, 1, \ldots, p\}$. Also, $I(S') = \sum_{j=0}^{p} I(S'_j)$. Based on this fact the following theorem gives a necessary and sufficient condition for an arbitrary subset of $V(G \square K_{1,p})$ to be an $F(G \square K_{1,p})$-set.

Theorem 5.3.2. *Let $S' \subseteq V(G \square K_{1,p})$. Then S' is an $F(G \square K_{1,p})$-set if and only if for each j $(0 \leq j \leq p)$, there exists a set $S'_j \subseteq V(G^{(v_j)})$ such that $S' = \cup_{j=0}^{p} S'_j$ and $S_j = p_G(S'_j)$ satisfying the following conditions:*

(i) S_j is a 2-packing in G, for each $j \in \{0, 1, \ldots, p\}$.

(ii) $(N[S_0] \times \{v_j\}) \cap S'_j = \emptyset$, for all $j \in \{1, 2, \ldots, p\}$ and $S_i \cap S_j = \emptyset$, for $i, j \in \{1, 2, \ldots, p\}$ and $i \neq j$.

(iii) $\sum_{j=0}^{p} I(S'_j)$ is maximum of all sets $S'_j \subseteq V(G^{(v_j)})$, for each j $(0 \leq j \leq p)$, such that $S' = \cup_{j=0}^{p} S'_j$.

Proof. Suppose that S' is an $F(G \square K_{1,p})$-set. Clearly, $S' = \cup_{j=0}^{p} S'_j$, where $S'_j \subseteq V(G^{(v_j)})$, for each j $(0 \leq j \leq p)$. Further by definition, each S'_j is a 2-packing in $G \square K_{1,p}$ and hence S_j is a 2-packing in G.

Moreover, if $S'_0 \neq \emptyset$, then for any $x \in (N[S_0] \times \{v_j\})$, $d(x, S'_0) \leq 2$, for each $j \in \{1, 2, \ldots, p\}$. Therefore, $x \notin S'_j$ and consequently, $(N[S_0] \times \{v_j\}) \cap S'_j = \emptyset$, for each $j \in \{1, 2, \ldots, p\}$. Now, suppose $u \in S_i \cap S_j$, for any $i, j \in \{1, 2, \ldots, p\}$ with $i \neq j$, then $(u, v_i) \in S'_i$ and $(u, v_j) \in S'_j$. Further, $d((u, v_i), (u, v_j)) \leq 2$ in $G \square K_{1,p}$, contradicting that S' is a 2-packing in $G \square K_{1,p}$. Hence, the sets S_j, for $1 \leq j \leq p$ are pairwise disjoint. Also, $I(S') = \sum_{j=0}^{p} I(S'_j)$ and is maximum, as S' is an $F(G \square K_{1,p})$-set.

Conversely, suppose that conditions (i), (ii) and (iii) hold for some subset S' of $V(G \square K_{1,p})$. Then, conditions (i) and (ii) together imply that S' is a 2-packing of $G\Box K_{1,p}$. Further, as $I(S') = \sum_{j=0}^{p} I(S'_j)$, condition (iii) guarantees that S' is an $F(G\Box K_{1,p})$-set. $\Box$

Theorem 5.3.3. *$G\Box K_{1,p} \in \mathscr{E}$ if and only if there exists a subset S' of $V(G\Box K_{1,p})$ such that the following conditions hold:*

(i) $p_G(S' \cap V(G^{(v_0)}))$ is a 2-packing in G.

(ii) If $S_0 = p_G(S' \cap V(G^{(v_0)}))$ and $G^* \cong < V(G) - N[S_0] >$, then $V(G^*)$ can be partitioned into p sets, say, $S_1, S_2, \ldots, S_p$ such that each S_j is an EDS of G^*.

(iii) For every vertex $v \in N(S_0)$ and for each j $(1 \leq j \leq p)$, $|N(v) \cap S_j| = 1$.

Proof. Suppose that there exists a subset S' of $V(G\Box K_{1,p})$ satisfying conditions (i), (ii) and (iii). Since $S_1, S_2, \ldots, S_p$ are pairwise disjoint efficient dominating sets of G^*, forming a partition of $V(G^*)$, it follows that $|V(G^*)| = p\gamma(G^*)$. For each j, $(0 \leq j \leq p)$, let $S'_j = S_j \times \{v_j\}$. For each $j\,(1 \leq j \leq p)$, as S_j is an EDS of G^*, S'_j is a 2-packing of $G\Box K_{1,p}$ and it follows from condition (i) that S'_0 is also a 2-packing of $G\Box K_{1,p}$. Further, $S' = \cup_{j=0}^{p} S'_j$.

Claim: S' is an EDS of $G\Box K_{1,p}$

Let $j \in \{1, 2, \ldots, p\}$. Then, S'_j dominates $G^{*(v_j)}$ and also copies of the vertices of S'_j in the layer $G^{*(v_0)}$. That is, S'_j dominates $V(G^{*(v_j)}) \cup (S_j \times \{v_0\})$. In addition, it follows from condition (iii) that each S'_j dominates $N(S_0) \times \{v_j\}$, as well. This is true for each $j\,(1 \leq j \leq p)$. Further, S'_0 dominates $N[S'_0]$. Thus, $S' = \cup_{j=0}^{p} S'_j$ forms an EDS of $G\Box K_{1,p}$ and $\gamma(G\Box K_{1,p}) = |S_0| + p\gamma(G^*)$.

Conversely, let $G\Box K_{1,p} \in \mathscr{E}$ and S' be its EDS.

Claim: S' satisfies conditions (i) to (iii).

For each j, $0 \leq j \leq p$, define $S'_j = S' \cap V(G^{(v_j)})$ and $S_j = p_G(S'_j)$ so that $S' = \cup_{j=0}^{p} S'_j$. Further, as S' is an EDS of $G\Box K_{1,p}$, each S'_j $(0 \leq j \leq p)$ is a 2-packing of $G\Box K_{1,p}$ and hence each S_j $(0 \leq j \leq p)$ is a 2-packing of G. Further, for each $j\,(1 \leq j \leq p)$, S'_j dominates efficiently all vertices in the layer $G^{(v_j)}$ except $V(G^{(v_j)}) \cap N(S'_0)$. Consequently, each $S_j\,(1 \leq j \leq p)$ is an EDS of G^*, where $G^* \cong < V(G) - N[S_0] >$.

Claim 1: $\cup_{j=1}^{p} S_j = V(G^*)$

Clearly, $\cup_{j=1}^{p} S_j \subseteq V(G^*)$. Suppose that there exists a vertex $w \in V(G^*)$ and $w \notin S_j$, for all $j \,(1 \leq j \leq p)$. Then $(w, v_j) \notin S'$, but it is dominated by S'_j and hence the vertex (w, v_0) is left undominated by S', contradicting that S' is an EDS of $G\Box K_{1,p}$. Hence, $\cup_{j=1}^{p} S_j = V(G^*)$.

Claim 2: $S_i \cap S_j = \emptyset$*, for all* $i \neq j$*,* $1 \leq i, j \leq p$

Suppose that $u \in S_i \cap S_j$. Then, $(u, v_i) \in S'_i$ and $(u, v_j) \in S'_j$, which implies that both $(u, v_i), (u, v_j)$ are in S'. But, (u, v_i) and (u, v_j) are at distance two in $G\Box K_{1,p}$, contradicting that S' is an EDS of $G\Box K_{1,p}$. Hence, $S_i \cap S_j = \emptyset$, for all $i, j \in \{1, 2, \ldots, p\}$ and $i \neq j$.

Therefore, $\{S_j : 1 \leq j \leq p\}$ is a partition of $V(G^*)$ where each S_j is an EDS of G^*. Further, as G is connected, for each i $(1 \leq i \leq p)$ and for any $v \in N(S_0)$, $|N(v) \cap S_j| \geq 1$. Moreover, as S' is an EDS of $G\Box K_{1,p}$, $|N(v) \cap S_j| \ngeq 2$ and hence, condition (iii) follows. □

Remark 5.3.1. *If* $G\Box K_{1,p} \in \mathscr{E}$ *and* $S'_0 \neq \emptyset$*, it follows from condition (ii) of Theorem 5.3.3 that* $\{S_1, S_2, \ldots, S_p\}$ *forms a partition of* $V(G^*)$*, where* $G^* \cong < V(G) - N[S_0] >$*. Hence,* $V(G) = N[S_0] \cup S_1 \cup \cdots \cup S_p$ *(disjoint union). Figure 5.13 gives an illustration of the general structure of* G *for which* $G\Box K_{1,p} \in \mathscr{E}$ *and has an EDS say,* S' *such that* $S'_0 \neq \emptyset$*.*

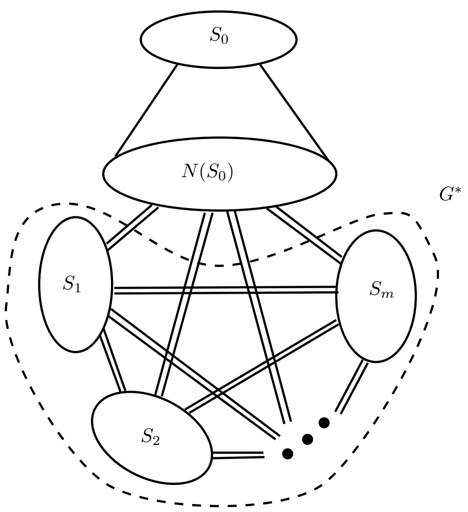


Figure 5.13: $V(G) = N[S_0] \cup S_1 \cup \cdots \cup S_p$ (disjoint union)

If $G \Box K_{1,p} \in \mathscr{E}$ and G has one of the structures shown in Figures 5.14 and 5.15, then G must also be efficiently dominatable. However, there may be other cases wherein both $G \Box K_{1,p}$ and G are efficiently dominatable. Few such cases are explored in Corollaries 5.3.3.1, 5.3.3.2 and 5.3.3.3. Precisely, the set $N(S_0)$ forms an EDS of G if the structure of G is as in Figure 5.14 and the set $(S_0 - \{u\}) \cup \{w\}$ forms an EDS of G if G has a structure similar to Figure 5.15. This fact is discussed in detail in Corollaries 5.3.3.1 and 5.3.3.2.

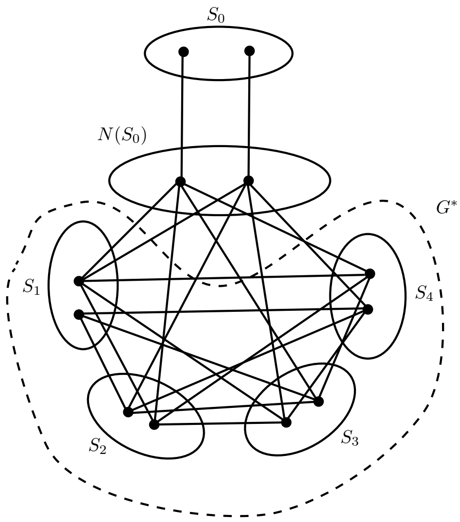


Figure 5.14: $G \in \mathscr{E}$ whenever $G \Box K_{1,p} \in \mathscr{E}$

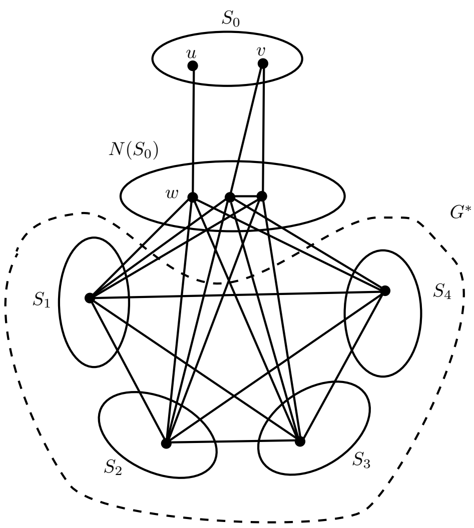


Figure 5.15: $G \in \mathscr{E}$ whenever $G \Box K_{1,p} \in \mathscr{E}$

Corollary 5.3.3.1. *Let G be connected, $G \in \mathscr{E}$ and $G \Box K_{1,p} \in \mathscr{E}$. If S' is an EDS of $G \Box K_{1,p}$ such that $S'_0 \neq \emptyset$, then the following conditions hold:*

(i) For any j $(0 \leq j \leq p)$, S_j is not an EDS of G.

(ii) $N(S_0)$ is an EDS of G if and only if $N(S_0)$ is a 2-packing of G and $|N(S_0)| = |S_j|$, for each $j \in \{0, 1, \dots, p\}$.

Proof. Let S be an EDS of G. As $G \Box K_{1,p} \in \mathscr{E}$ and $S'_0 \neq \emptyset$, it follows from condition (ii) of Theorem 5.3.3 that $\{S_1, S_2, \dots, S_p\}$ forms a partition of $V(G^*)$, where $G^* \cong < V(G) - N[S_0] >$ and hence $V(G) = N[S_0] \cup S_1 \cup \dots \cup S_p$.
Proof of (i):
Since $d_G(S_0, S_j) \geq 2$, for all $j \in \{1, 2, \dots, p\}$, it follows that S_0 cannot be an EDS of G. In addition, for any $j \in \{1, 2, \dots, p\}$, it follows from conditions (ii) and (iii) of Theorem 5.3.3 that each S_j efficiently dominates $V(G^*) \cup N(S_0)$, but does not dominate S_0. Hence, S_j cannot be an EDS of G, for all j $(0 \leq j \leq p)$.
Proof of (ii):
Suppose that $N(S_0)$ is an EDS of G. Then, clearly $N(S_0)$ is a 2-packing of G. Also, G is connected and hence, $|N(S_0)| = |S_0|$.
Claim: $|N(S_0)| = |S_j|$, for all $j \in \{1, 2, \dots, p\}$
It follows from condition (iii) of Theorem 5.3.3 that, $|N(S_0)| \leq |S_j|$, for all $j \in \{1, 2, \dots, p\}$. Suppose $|N(S_0)| < |S_j|$, for any j $(1 \leq j \leq p)$, then there exists a vertex $u \in S_j$, which is not adjacent to any vertex in $N(S_0)$, contradicting that $N(S_0)$ is an EDS of G.
Conversely, suppose that $N(S_0)$ is a 2-packing of G and $|N(S_0)| = |S_j|$, for all $j \in \{0, 1, \dots, p\}$. Then, clearly for all $u \in N[S_0]$, $|N[u] \cap N(S_0)| = 1$ and hence $N(S_0)$ efficiently dominates $N[S_0]$. Further, it follows from condition (iii) of Theorem 5.3.3 that for all $j \in \{1, 2, \dots, p\}$ and for every $u \in V(S_j)$, $|N(u) \cap N(S_0)| = 1$ and hence $N(S_0)$ efficiently dominates $\cup_{j=1}^{p} S_j$. Hence, $N(S_0)$ is an EDS of G. □

It is noted that if $G^* \cong K_p$, then $|S_j| = 1$ and $|N(S_0)| \geq |S_j|$, for all $j \in \{1, 2, \dots, p\}$. Corollary 5.3.3.1 states that if $G \in \mathscr{E}$ and $|N(S_0)| = |S_j|$, for all $j \in \{0, 1, \dots, p\}$, then $N(S_0)$ is an EDS of G. On the other hand, if $|N(S_0)| \neq |S_j|$, then G may or may not be efficiently dominatable. In particular, if $G \in \mathscr{E}$, then $N(S_0)$ cannot be an EDS of G. In Corollary 5.3.3.2, the necessary and sufficient conditions are determined for a graph G to be efficiently dominatable, whenever $|N(S_0)| \neq |S_j|$, for any $j \in \{0, 1, \ldots, p\}$.

Corollary 5.3.3.2. *Let $G \Box K_{1,p} \in \mathscr{E}$ and S' be an EDS of $G \Box K_{1,p}$ such that $S'_0 \neq \emptyset$. Suppose that $G^* \cong K_p$, where $G^* \cong < V(G) - N[S_0] >$. Then, $G \in \mathscr{E}$ if and only if G has a pendant vertex, say u, such that $u \in S_0$ and $d_G(u, v) > 3$, for all $v \in S_0 - \{u\}$.*

Proof. As $G \Box K_{1,p} \in \mathscr{E}$, it follows from Theorem 5.3.3 that S_j is an EDS of G^*, for all $j \in \{1, 2, \ldots, p\}$. In addition, it follows from condition (iii) of Theorem 5.3.3 that each vertex in $N(S_0)$ is adjacent to every vertex in $\cup_{j=1}^{p} S_j$. Thus, if $G^* \cong K_p$, then $|S_j| = 1$, for $j \in \{1, 2, \ldots, p\}$.

Let $G \in \mathscr{E}$ and S be an EDS of G. It follows from Corollary 5.3.3.1 that $S \neq S_j$. Also, $S \not\subseteq S_j$, for all $j \in \{1, 2, \ldots, p\}$. Thus, $S \subset N[S_0]$. Since, each vertex in $N(S_0)$ is adjacent to every vertex in $\cup_{j=1}^{p} S_j$, it follows that $|N(S_0) \cap S| = 1$. Let $w \in |N(S_0) \cap S|$.

Claim: $d_G(w, w') \geq 2$*, for all* $w' \in N(S_0) - \{w\}$

Suppose that, $d_G(w, w') = 1$, for some $w' \in N(S_0) - \{w\}$. Then, since $w \in S$, the vertex $N(w') \cap S_0$ is not dominated efficiently, contradicting that $G \in \mathscr{E}$. Thus, for all $w' \in N(S_0) - \{w\}$, $d_G(w, w') \geq 2$.

Hence, it follows that $d_G(w, v) \geq 3$, for all $v \in S_0$. Let $u \in N(w) \cap S_0$. Then, it follows that $d_G(u, v) \geq 4$, for all $v \in S_0 - \{u\}$.

Claim: $\deg_G(u) = 1$

Suppose that $\deg_G(u) \geq 2$. Then, $d_G(w, x) = 2$, for all $x \in N(u) - \{w\}$. As $w \in S$, this is not possible. Thus, u is a pendant vertex in S_0.

Conversely, suppose that G has a pendant vertex, say u, such that $u \in S_0$ and $d_G(u, v) > 3$, for all $v \in S_0 - \{u\}$. If $w \in N(u)$, then $d_G(w, v) \geq 3$, for all $v \in S_0 - \{u\}$. Then, the set $(S_0 - \{u\}) \cup \{w\}$ forms an EDS of G. □

Corollary 5.3.3.3. *$G \Box K_{1,p} \in \mathscr{E}$ and it has an EDS, say S' such that $S'_0 = \emptyset$ if and only if G has p pairwise disjoint efficient dominating sets. Moreover, $p =$*

$\dfrac{|V(G)|}{\gamma(G)}$.

Remark 5.3.2.

1. *If $G\square K_{1,p} \in \mathscr{E}$ and has an EDS, say S' such that $S'_0 = \emptyset$, then it follows from Corollary 5.3.3.3 that G has p pairwise disjoint efficient dominating sets, $S_j\,(1 \leq j \leq p)$. Hence, $\{S'_j : 1 \leq j \leq p\}$ can be chosen to efficiently dominate $G\square K_{1,p}$ in $p!$ ways. Therefore, there are $p!$ distinct efficient dominating sets and p pairwise disjoint efficient dominating sets in $G\square K_{1,p}$.*

2. *If $G\square K_{1,p} \in \mathscr{E}$ and has an EDS, say S' such that $S'_0 = \emptyset$, then it follows from Theorem 3.1.19 that G must be an $(p-1)$-regular efficiently dominatable graph, but not conversely.*

5.3.1 An Exact Exponential time Algorithm to find an $F(G\square K_{1,p})$-set

As already discussed, the problem of deciding whether or not, a graph G has an EDS is $\mathcal{NP}$-complete. The same is the case for the product $G\square K_{1,p}$. However, it is evident from the existing literature that designing efficient exact exponential algorithms is one of the well-adopted methods to solve most of the $\mathcal{NP}$-complete problems. So in this section, an attempt is made to solve the efficient domination problem for the product $G\square K_{1,p}$ using an exact exponential time algorithm, namely "*ED_ StarCProd*". To the best of our knowledge, this is the first of this kind which provides an exact exponential solution for the problem in the case of $G\square K_{1,p}$, whenever G is arbitrary.

The algorithm presented in this section, namely ED_ StarCProd, verifies whether the product $G\square K_{1,p}$ is efficiently dominatable or not and in case, the product is identified not to be efficiently dominatable, the algorithm computes the value of $F(G\square K_{1,p})$; Finally, it returns an EDS, if it exists or an F-set for the product and the value of $F(G\square K_{1,p})$, otherwise.

Except for the vertex set of $K_{1,p}$ as a part of its input, the proposed algorithm completely uses the sets (2-packings) generated from G and the structure of G rather than those of the product $G\Box K_{1,p}$. Thereby, the time complexity is reduced substantially compared to the traditional exhaustive search techniques. The algorithm begins by enumerating all 2-packings of G. Lemma 5.3.4 given below gives an upper bound on the total number of 2-packings of G enumerated in Step (1) of *ED_StarCProd*.

In general, an $F(G\Box K_{1,p})$-set may or may not intersect with $V(G^{(v_0)})$. That is, it may or may not include vertices of the form (u, v_0), for any $u \in V(G)$. Based on this, initially, among the various 2-packings of $G\Box K_{1,p}$ which do not intersect with $V(G^{(v_0)})$, the one having maximum influence is generated by using the subroutine "*M2P_StarCProd1*". In case, the influence of the 2-packing so generated is equal to $n(1+p)$, *M2P_StarCProd1* itself returns an EDS of the product, concluding that $G\Box K_{1,p}$ is efficiently dominatable.

On the other hand, if the influence of the set returned by *M2P_StarCProd1* is less than $n(1+p)$, then the main algorithm (*ED_StarCProd*) proceeds further. The other maximal 2-packings of $G\Box K_{1,p}$, which intersect with $V(G^{(v_0)})$ are also enumerated. Finally, among all these and the 2-packing returned by *M2P_StarCProd1*, the one with maximum influence is returned as the required $F(G\Box K_{1,p})$-set.

Before proceeding further to *ED_StarCProd*(G, n, p) (Algorithm 2), the significant steps required to analyse its time complexity are discussed below:
(1) Enumerating all 2-packings of G and
(2) The subroutine - *M2P_StarCProd1*$(G,\ n,\ p,\ \mathscr{P}, |\mathscr{P}|)$.

Lemma 5.3.4. *(Junosza-Szaniawski and Rzażewski, 2012) The maximum number of 2-packings in a connected graph on n vertices does not exceed $\mathcal{O}(1.5399\ldots^n)$. Moreover, all 2-packings in a connected graph on n vertices can be generated in time $\mathcal{O}^*(1.5399\ldots^n)$.*

Remark 5.3.3. *It is shown by K.J-Szaniawski and Paweł Rzażewski in Junosza-Szaniawski and Rzażewski (2012) that the maximum number of 2-packings in a connected graph is between $\Omega(1.4970\ldots^n)$ and $\mathcal{O}(1.5399\ldots^n)$. In Lemma 5.3.4, the authors claim that the number of 2-packings in G does not exceed $\mathcal{O}(1.5399\ldots^n)$.*

And, all the local operations involved in the process (that is, finding a spanning tree, finding the longest path in a tree, deleting vertices, checking if a set is a 2*-packing etc.) may be performed in polynomial time. Hence, the total computational complexity of the algorithm is* $\mathcal{O}^*(1.5399\ldots^n)$. *Precisely, if* G *is a graph on* n *vertices and* m *edges, then finding a spanning tree takes* $\mathcal{O}(n+m)$ *steps, finding the longest path in a tree can be done in linear time (Club et al., 2002; Uehara and Uno, 2007), deletion of the vertices can be done in* $\mathcal{O}(n)$ *steps, checking if a set is a* 2*-packing can be done in* $\mathcal{O}(n)$ *steps (by using an appropriate data structure, like Hashing technique). Hence, all* 2*-packings of a connected graph on* n *vertices can be generated in* $\mathcal{O}(n^2 l)$ *time, where* l *is the number of* 2*-packings of* G *and* $l \leq (1.5399\ldots)^n$.

An Overview of M2P_StarCProd1:

The main objective of *M2P_StarCProd1* is to generate a 2-packing of $G \square K_{1,p}$, say S'', having maximum influence among all those 2-packings of the product, which do not include (u, v_0), for any $u \in V(G)$. That is, to generate a 2-packing S'' of the product such that $S'' \cap S'_0 = \emptyset$ and has maximum influence among all such 2-packings. This is accomplished by generating a collection (of size at most p) of mutually disjoint 2-packings of G such that the total influence (the sum of influence of all the elements in the collection) is maximum among all such collections. To determine such a collection, one of the brute-force techniques is to generate all 2-packings of G, say $P_1, P_2, \ldots, P_l$; then for each P_i $(1 \leq i \leq l)$, all distinct collections of mutually disjoint 2-packings of G containing P_i can be generated, which in turn helps in generating all collections of mutually disjoint 2-packings of G; finally, the one with maximum total influence is picked up. But this procedure is not efficient in terms of complexity. Hence, with the intention of reducing the complexity, in *M2P_StarCProd1*, initially, all 2-packings of G are enumerated. Then, these 2-packings are sorted in the nonincreasing order of their influence in G. The sets with same influence are taken in the nonincreasing order of their cardinality. Then, for each i, where $1 \leq i \leq l$, a collection of mutually disjoint 2-packings of G containing P_i having maximum total influence is determined among all such collections containing P_i. Next, for each of the above newly generated collection, the elements in the collection are further sorted, in the nonincreasing order of their influence in G. In the event that a collection includes more than p elements, only the first p elements are retained after sorting. Finally, the required collection (of size at most p) whose total influence is the maximum compared to the others is determined.

Lemma 5.3.5. *Given the collection of all* 2*-packings of a connected graph G of order n, M2P_StarCProd1 generates a maximal* 2*-packing of $G\Box K_{1,p}$, say S'', which does not intersect with $V(G^{(v_0)})$ in $\mathcal{O}^*(c^n)$ time, where* $I(S'') = \max\{I(P') :$ *P' is a* 2*-packing of $G\Box K_{1,p}$; $P' \cap V(G^{(v_0)}) = \emptyset\}$ and* $5.0221 \cdots \leq c \leq 5.6230\ldots$.

Proof.
Correctness of M2P_StarCProd1: Let $\mathscr{P} = \{P_1, P_2, \ldots, P_l\}$ be the given collection of all 2-packings of G. *M2P_StarCProd1* starts by sorting $\mathscr{P}$ in the nonincreasing order of the influence (in G) of the 2-packings included in $\mathscr{P}$. To break a tie, if any, the sets are taken in the nonincreasing order of their cardinalities. Let $\mathscr{P}' = \{P'_1, P'_2, \ldots, P'_l\}$ be the sorted list. A series of steps is performed to generate a collection of mutually disjoint 2-packings of G by choosing an element from $\mathscr{P}'$. This step is carried out for all the elements in $\mathscr{P}'$.
Initially, starting with $P'_1 \in \mathscr{P}'$, a collection $\mathscr{S}_1$ of mutually disjoint 2-packings of G and containing P'_1 is generated by comparing P'_1 with the other elements in $\mathscr{P}'$. First, set $S_1 = P'_1$ and include it in $\mathscr{S}_1$. To choose the next element to include in $\mathscr{S}_1$, it is required to pick up the next packing of maximum influence as well as disjoint with S_1. So, if $P'_2 \cap S_1 = \emptyset$, then let $S_2 = P'_2$ and include it in $\mathscr{S}_1$. If not, proceed checking P'_3 and so on. For the third and subsequent choice of elements to include in $\mathscr{S}_1$, it is required to compare the next candidate P'_i with all $S'_i s$ included earlier in $\mathscr{S}_1$. In this way, a collection of mutually disjoint 2-packings containing P'_1, namely $\mathscr{S}_1$ is generated. At each stage, as the elements are chosen in the order they appear in the sort list (based on influence), it is evident that the generated collection has maximum total influence compared to all those disjoint collections containing P'_1.

Next, the elements in $\mathscr{P}'\backslash\mathscr{S}_1$ are considered in the order they appear in $\mathscr{P}'$. The above process is continued by starting with the first element appearing in the list $\mathscr{P}'\backslash\mathscr{S}_1$ and generate the collection $\mathscr{S}_2$ and so on.

Claim: The Collections $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_q$ are mutually distinct.

Proof of Claim: Once the collection $\mathscr{S}_1$ is determined, it is clear that all elements present in $\mathscr{S}_1$ are mutually disjoint from each other. Hence, repeating the step by choosing a 2-packing already in $\mathscr{S}_1$ may result either in a duplication of collections or a collection having lesser total influence than $\mathscr{S}_1$. Hence, to generate a distinct collection $\mathscr{S}_2$, the process is continued by choosing the first 2-packing in $\mathscr{P}'\backslash\mathscr{S}_1$. Similarly, the collection $\mathscr{S}_3$ is generated by choosing the first member of $\mathscr{P}'\backslash(\mathscr{S}_1 \cup \mathscr{S}_2)$ and so on. Hence, the collections $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_q$ are mutually distinct.

Next, for $1 \leq i \leq q$, the elements in each $\mathscr{S}_i$ are sorted in the nonincreasing order of their influence. Sets with same influence are taken in the nonincreasing order of their cardinality in the sorted list so that the influence is maximum in $G\Box K_{1,p}$. Finally, excluding $G^{(v_0)}$, as there are only p rows (or p copies of G) in $G\Box K_{1,p}$, at most k elements, where $k \leq p$, are retained in each $\mathscr{S}_i$.

Thus, for each $i\,(1 \leq i \leq q)$, if $\mathscr{S}'_i = \{S'_1, S'_2, \ldots, S'_k\}$ is the sorted list of 2-packings, then the set $S''_i = (S'_1 \times \{v_1\}) \cup (S'_2 \times \{v_2\}) \cup \cdots \cup (S'_k \times \{v_k\})$ will be the corresponding 2-packing of $G\Box K_{1,p}$ not containing (u, v_0), for any $u \in V(G)$. Then, among these S''_i's, the one with maximum influence is the required 2-packing of maximum influence in $G\Box K_{1,p}$ not intersecting with $V(G^{(v_0)})$.

Time Complexity of M2P_ StarCProd1*:* In Algorithm 1, the sorting done in Step 2 requires $\mathcal{O}(l \log l)$ time. Steps 14 - 19 take at most n^2 time. The **while** loop in Step 13 executes at most l times. Hence, the innermost **while** loop in Steps 13 - 20 takes at most ln^2 steps. The **while** loop in Step 11 is executed at most l times. Hence, Steps 11 - 24 take at most l^2n^2 time. Next, the **while** loop in Step 6 executes at most l times and Step 26 is executed q times, where $q \leq l$ and each execution takes $l \log l$ times. The sets S''_q in Step 28 and $I(S''_q)$ computed in Step 29 are used to generate a 2-packing, say S'', of $G\Box K_{1,p}$ in Step 38 and this takes $\mathcal{O}(p)$ time (since $m \leq p$). Thus, Steps 5 - 35 take at most $l(l^2n^2 + l^2 \log l + p)$

steps. As **while** loop in Step 4 executes at most l times, Steps 4 - 36 take at most $l(l(l^2n^2 + l^2 \log l + p)) = l^4n^2 + l^4 \log l + pl^2$ steps. Thus, *M2P_StarCProd1* takes $\mathcal{O}(l^4n^2 + l^4 \log l + pl^2) = \mathcal{O}(l^2(l^2n^2 + p))$ steps (using Lemma 5.3.4).

Now, suppose $p \leq n$, then clearly, $\mathcal{O}(l^4n^2 + l^2p) = \mathcal{O}(l^4n^2)$. On the other hand, if $p > n$, then $p = n + k$, for some $k > 0$. Therefore, as $l^2p = l^2(n + k) < l^4n^2$, $\mathcal{O}(l^4n^2 + l^2p) = \mathcal{O}(l^4n^2)$. Thus, in either case, it can be observed that the time complexity for *M2P_StarCProd1* is $\mathcal{O}(l^4n^2) = \mathcal{O}^*(l^4)$. Or precisely, it follows from Remark 5.3.3 that *M2P_StarCProd1* takes $\mathcal{O}^*(c^n)$ time, where $5.0221 \cdots \leq c \leq 5.6230 \ldots$. □

Theorem 5.3.6. *For any connected graph $G = (V, E)$, the algorithm ED_StarCProd(G, n, p) finds an EDS of $G\Box K_{1,p}$ or an $F(G\Box K_{1,p})$-set in $\mathcal{O}^*(c^n)$ time, where* $5.6230257 \cdots \leq c \leq 8.658897 \ldots$.

Proof. The correctness of the algorithm follows from Theorems 5.3.2 and Lemma 5.3.5. Next, it will be shown that *ED_StarCProd* computes an $F(G\Box K_{1,p})$-set (or an EDS of $G\Box K_{1,p}$) in $\mathcal{O}(l^3(l^2n^2 + p))$ steps, where l is the number of 2-packings of G.

In *ED_StarCProd*(G, n, p), Step 1 generates all 2-packings of G in $\mathcal{O}(n^2l)$ steps, where l is the number of 2-packings of G (refer to Remark 5.3.3). As discussed earlier, an $F(G\Box K_{1,p})$-set may or may not include elements from $V(G^{(v_0)})$. Based on this, the algorithm *ED_StarCProd* involves two major sequence of steps, executed based on the validity of the 'if' statement in Step 3. Initially, Step 2 calls *M2P_StarCProd1*$(G, n, p, \mathscr{P})$ to find a maximal 2-packing, say, S'' of $G\Box K_{1,p}$ which does not contain (u, v_0), for any $u \in V(G)$ and having maximum influence among all such 2-packings of the product. Upon checking whether $I(S'') = n(1+p)$ (that is, if S'' is an EDS of $G\Box K_{1,p}$) in Step 3, the algorithm *ED_StarCProd* either terminates by returning S'' (as an EDS of the product) and its influence or proceeds further. If it terminates at Step 7, then the total complexity of *ED_StarCProd* will be $\mathcal{O}(n^2l + l^2(l^2n^2 + p)) = \mathcal{O}^*(c^n)$, where $5.0221 \cdots \leq c \leq 5.6230 \ldots$ (refer to Lemma 5.3.5).

Algorithm 1: *M2P_StarCProd1*$(G, n, p, \mathscr{P}, |\mathscr{P}|)$

Input: A connected graph G of order n, $V(K_{1,p}) = \{v_0, v_1, \ldots, v_p\}$ $(p \geq 1)$, $\mathscr{P}$ - Set of all 2-packings of G and $|\mathscr{P}|$

Output: A 2-packing of $G \square K_{1,p}$ not containing (u, v_0), for any $u \in V(G)$ and having maximum influence among all such 2-packings of the product

```
1  Let $|\mathscr{P}| = l$ and $\mathscr{P} = \{P_1, P_2, \ldots, P_l\}$
2  Sort $\mathscr{P}$ in the nonincreasing order of influence of $P_i's$. Sets with same influence are taken
   in the nonincreasing order of their cardinality in the sorted list. Let
   $\mathscr{P}' = \{P'_1, P'_2, \ldots, P'_l\}$ be the sorted list of 2-packings.
3  $q = 0$; $r = 1$; $k = 0$; $\mathscr{T} = \emptyset$
4  while $r \leq l$ do
5      if $k \leq l$ then
6          while $P'_r \notin \mathscr{T}$ do
7              $q++$
8              $t = 1$; $S_t = P'_r$; $I(S_t) = \sum_{x \in S_t}(1 + deg_G(x))$
9              $\mathscr{T} = \mathscr{T} \cup \{P'_r\}$; $k++$
10             $j = 1$
11             while $j \leq l$ do
12                 $i = 1$
13                 while $i \leq t$ do
14                     if $P'_j \cap S_i \neq \emptyset$ then
15                         j++; goto Step 11
16                     end
17                     else
18                         $i++$
19                     end
20                 end
21                 $t++$; $S_t = P'_j$; $I(S_t) = \sum_{x \in S_t}(1 + deg_G(x))$
22                 $\mathscr{T} = \mathscr{T} \cup \{P'_j\}$; $k++$
23                 $j++$
24             end
25             $\mathscr{S}_q = \{S_1, S_2, \ldots, S_t\}$
26             Sort $\mathscr{S}_q$ in the nonincreasing order of influence of $S'_i s$. To break a tie, if any,
               take the sets in the nonincreasing order of their cardinalities. In the sorted
               collection $\mathscr{S}_q$, retain only the first $p$ elements, in case it includes more than
               $p$ sets.
27             $\mathscr{S}'_q = \{S'_1, S'_2, \ldots, S'_m\}_{m \leq p}$ be the sorted collection got in Step 26.
28             $S''_q = \cup_{i=1}^{m}(S'_i \times \{v_i\})$
29             $I(S''_q) = \sum_{i=1}^{m}(I(S'_i) + |S'_i|)$
30         end
31         $r++$; goto Step 4
32     end
33     else
34         goto Step 37
35     end
36 end
37 $I_{max} = \max\{I(S''_1), I(S''_2), \ldots, I(S''_q)\}$
38 $S'' = S''_q$ such that $I(S''_q) = I_{max}$
39 $I(S'') = I_{S''_q}$
40 return $S''$ and $I(S'')$
```

Algorithm 2: *ED_StarCProd*(G, n, p)

Input: A connected graph G of order n, $V(K_{1,p}) = \{v_0, v_1, \ldots, v_p\}$ $(p \geq 1)$
Output: $F(G \square K_{1,p})$ and an $F(G \square K_{1,p})$-set
// If $F(G \square K_{1,p} = n(1+p))$, then the F-set returned is an EDS of $G \square K_{1,p}$.
1 Generate all 2-packings of G. Let $\mathscr{P}$ be the set of all 2-packings of G.
2 Call *M2P_StarCProd1*(G, n, p, $\mathscr{P}$, $|\mathscr{P}|$)
3 **if** $I(S'') == n(1+p)$ **then**
4   print "$G \square K_{1,p}$ is efficiently dominatable and S'' is an EDS of $G \square K_{1,p}$."
5     $F(G \square K_{1,p}) = I(S'')$
6   **return** S'', $F(G \square K_{1,p})$
7 **end**
8 **else**
9   **for** $i = 1$ *to* l **do**
10         $S_0 = P_i$
11         $G^* \cong < V(G) - N[S_0] >$
12   Generate all 2-packings of G^*. Let $\mathscr{P}^*$ be the set of all 2-packings of G^*.
13   Call *M2P_StarCProd1*(G^*, $|V(G^*)|$, p, $\mathscr{P}^*$, $|\mathscr{P}^*|$)
14         $P_i'' = S'' \cup (S_0 \times \{v_0\})$
15         $I(P_i'') = I(S'') + \sum_{v \in S_0} (\deg_G(v) + 1) + p|S_0|$
16   **end**
17 **end**
18 $\mathscr{S} = \{S'', P_1'', P_2'', \ldots, P_l''\}$; $I_{max} = \max\{I(S) : S \in \mathscr{S}\}$
19 Let S' be the set in $\mathscr{S}$ whose influence is equal to I_{max}.
20 $F(G \square K_{1,p}) = I_{max}$
21 **if** $F(G \square K_{1,p}) = n(1+p)$ **then**
22   print "$G \square K_{1,p}$ is efficiently dominatable and S' is an EDS of $G \square K_{1,p}$."
23 **end**
24 **else**
25   print "$G \square K_{1,p}$ is not efficiently dominatable and S' is an $F(G \square K_{1,p})$-set."
26 **end**
27 **return** S', $F(G \square K_{1,p})$

On the other hand, if the test condition in Step 3 fails, then the algorithm proceeds further. For each $P_i \in \mathscr{P}$ $(1 \leq i \leq l)$, Step 13 calls the subroutine *M2P_StarCProd1* for G^*, where G^* is the graph induced by $V(G) - N[P_i]$. Every call of Step 13 takes $\mathcal{O}(l_i^4 n^2)$-steps, where l_i is the number of 2-packings of $< V(G) - N[P_i] >$. Thus, the **for** loop in Steps 9 - 16 takes $n^2(\sum_{i=1}^{l} l_i^4)$ steps. That is, the **for** loop in Steps 9 - 16 runs in $\mathcal{O}(l^5 n^2)$ time, since $l_i \leq l$, for each i $(1 \leq i \leq l)$.

The maximum influence computed in Step 18 takes $\mathcal{O}((l+1)\log(l+1))$ time and the remaining steps take constant time. Hence, the total time complexity to execute *ED_StarCProd* is $\mathcal{O}(n^2l+l^5n^2+(l+1)log(l+1)) = \mathcal{O}(l^5n^2) = \mathcal{O}^*(l^5)$. Or precisely, it takes $\mathcal{O}^*(c^n)$ time, where $7.5181\dots \leq c \leq 8.6589\dots$ (Using Remark 5.3.3). $\square$

Remark 5.3.4. *If ED_StarCProd(G,n,p) ends at Step 7, then the the overall complexity will be reduced by a factor of l. That is, the algorithm takes $\mathcal{O}^*(l^4)$ time, where l is the number of 2-packings of G. Otherwise, it takes $\mathcal{O}^*(l^5)$ time. Thus, the problem of finding an $F(G\square K_{1,p})$-set has time complexity $\Omega(l^4n^2)$ and $\mathcal{O}(l^5n^2)$.*

5.4 Efficient Domination in the cartesian Product $G\square K_p$

In this section, the notion of efficient domination is discussed for the cartesian product of complete graphs with other graphs in terms of their factors. A necessary and sufficient condition is obtained for the product $G\square K_p$ to be efficiently dominatable. Given a subset of $V(G\square K_p)$, a characterization is obtained for the existence of an $F(G\square K_p)$-set and finally an algorithm is presented to find an $F(G\square K_p)$-set. It is known that $G\square K_1 \cong G$ and hence, the product is efficiently dominatable if and only if $G \in \mathscr{E}$. Hence, it is assumed throughout that in the product $G\square K_p$, the factor G is connected, $G \ncong K_1$ and $p \geq 2$.
Throughout this section, the following notations are used:
Let $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(K_p) = \{v_1, v_2, \dots, v_p\}$. Then, $|V(G\square K_p)| = np$.
For any $(u_i, v_j) \in V(G\square K_p)$, $\deg_{G\square K_p}(u_i, v_j) = \deg_G(u_i) + (p-1)$, for $1 \leq i \leq n$ and $1 \leq j \leq p$.

Notation 5.4.1.

- *$|V(G)| = n$, $|V(K_p)| = p$*
- *For any set $S' \subseteq V(G\square K_p)$, S denotes $p_G(S')$*
- *For $1 \leq j \leq p$, $S'_j = V(G^{(v_j)}) \cap S'$ and $S_j = p_G(S'_j)$*

Fact 5.4.1. *If S' is an $F(G\square K_p)$-set and $S = p_G(S')$, then the following properties hold:*

1. *For any $i \in \{1, 2, \ldots, n\}$, $|V(K_p^{(u_i)}) \cap S'| \leq 1$ and hence, $|S'| \leq n$. Moreover, $|S'| = |S|$.*

2. *If S' is an $F(G\Box K_p)$-set, then S is independent in G.*

3. *$F(G\Box K_p) = I_G(S) + (p-1)|S|$*

   Proof. For any $(u_i, v_j) \in V(G\Box K_p)$, where $1 \leq i \leq n$ and $1 \leq j \leq p$, $\deg_{G\Box K_p}(u_i, v_j) = \deg_G(u_i) + (p-1)$. Further, as $|S'| = |S|$, $\sum_{(u_i,v_j)\in S'} \deg_{G\Box K_p}(u_i, v_j) = \sum_{u_i \in S}[\deg_G(u_i)+p-1]$, for all $(u_i, v_j) \in V(G\Box K_p)$. This implies that

$$\begin{aligned} F(G\Box K_p) &= \sum_{(u_i,v_j)\in S'} [\deg_{G\Box K_p}(u_i, v_j) + 1] \\ &= \sum_{u_i \in S} \deg_G(u_i) + p|S| \end{aligned}$$

   Equivalently, $F(G\Box K_p) = I_G(S) + (p-1)|S|$. □

Proposition 5.4.1. *Let G be a connected graph of order n, where $n \geq 2$. If $G\Box K_p \in \mathscr{E}$, then $p \leq n - \delta(G)$.*

Proof. Let $G\Box K_p \in \mathscr{E}$ and S' be an EDS of $G\Box K_p$. Without loss of generality, let $(u_1, v_1) \in S'$. Then, (u_1, v_1) dominates $V(K_p^{(u_1)}) \cup (N_G[u_1] \times v_1)$. Without loss of generality, let $u_2 \in N_G(u_1)$. Then, $V(K_p^{(u_2)}) \cap S' = \emptyset$. Hence, to efficiently dominate each of the $(p-1)$ vertices in $V(K_p^{(u_2)}) - \{u_2, v_1\}$, $(p-1)$ distinct vertices are required, one from each set $(V(G) - N[u_1]) \times v_j$, $(1 \leq j \leq p)$. Hence, $p - 1 \leq |V(G) - N[u_1]|$. That is, $p \leq n - \deg_G(u_1)$. Since u_1 is arbitrary, $p \leq n - \delta(G)$. □

Proposition 5.4.2. *Let $\mathscr{S} = \{S \subseteq V(G) : S$ is independent in G and $|S| \leq n - \frac{1}{p}\sum_{u_i \in S} \deg_G(u_i)\}$. If S' is an $F(G\Box K_p)$-set and $S = p_G(S')$, then the following conditions hold:*

(i) $S \in \mathscr{S}$.

(ii) $I_G(S) + |S|(p-1) = \max_{T \in \mathscr{S}}\{I_G(T) + |T|(p-1)\}$.

In particular, $|S'| \leq \alpha(G)$*, where* $\alpha(G)$ *is the independence number of* G*.*

Proof. Let S' be an $F(G\Box K_p)$-set and $S = p_G(S')$. Let $|V(G)| = n$. Then, it follows from Fact 5.4.1(2) that S is an independent set in G. Since S' is an $F(G\Box K_p)$-set,

$$I_{G\Box K_p}(S') \leq pn \tag{5.23}$$

But, using Fact 5.4.1(3),

$$\begin{aligned} I_{G\Box K_p}(S') &= I_G(S) + |S|(p-1) \\ &= \sum_{u_i \in S} (\deg_G(u_i) + 1) + |S|(p-1) \\ &= \sum_{u_i \in S} \deg_G(u_i) + p|S| \end{aligned} \tag{5.24}$$

Therefore, from (5.23) and (5.24), $|S| \leq n - \frac{1}{p}\sum_{u_i \in S} \deg_G(u_i)$. Hence, $S \in \mathscr{S}$. Also, as S' is an $F(G\Box K_p)$-set, it follows by definition that $I_{G\Box K_p}(S')$ is maximum among the influences of all 2-packings of $G\Box K_p$.
To prove condition (ii), it is required to show that $I_G(S) + |S|(p-1) \geq I_G(T) + |T|(p-1)$, for all $T \in \mathscr{S}$. Suppose to the contrary that there exists a set $T \in \mathscr{S}$ such that $I_G(T) + |T|(p-1) > I_G(S) + |S|(p-1)$. Then, $I_{G\Box K_p}(T') > I_{G\Box K_p}(S')$, where T' is the 2-packing in $G\Box K_p$ such that $T = p_G(T')$. This contradicts our hypothesis. Hence, condition (ii) holds.
Further, as S is independent in G, $|S| \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G. Therefore, $|S'| = |S|$ implies that $|S'| \leq \alpha(G)$. □

In general, if $S' = \cup_{j=1}^{p} S'_j$, where $S'_j \subseteq V(G^{(v_j)})$ and $S_j = p_G(S'_j)$, for $1 \leq j \leq p$, then it is observed that S' is a 2-packing in $G\Box K_p$ if and only if S'_j is a 2-packing of $G^{(v_j)}$ if and only if $p_G(S'_j)(= S_j)$ is a 2-packing of G, for each $j \in \{1, 2, \ldots, p\}$. Also, $I(S') = \sum_{j=1}^{p} I(S'_j)$. With these facts, the following result gives a necessary and sufficient condition for an arbitrary subset of $V(G\Box K_p)$ to be an $F(G\Box K_p)$-set.

Theorem 5.4.3. *Let* $S' \subseteq V(G\Box K_p)$*. Then,* S' *is an* $F(G\Box K_p)$*-set if and only if there exist sets* $S'_j \subseteq V(G^{(v_j)})$ $(1 \leq j \leq p)$ *such that* $S' = \cup_{j=1}^{p} S'_j$*, where one or* *more S'_j's are possibly empty and for each $j \in \{1, 2, \ldots, p\}$ such that $S'_j \neq \emptyset$ the following conditions hold:*

(i) $p_G(S'_j)$ *is a 2-packing in* G.

(ii) For any given k, l, *where* $k \neq l$ *and* $1 \leq k, l \leq p$, $S'_l \cap (N[p_G(S'_k)] \times \{v_j\}) = \emptyset$.

(iii) $\sum_{j=1}^{p} I(S'_j)$ *is the maximum among all sets* $T'_j \subseteq V(G^{(v_j)})$ $(1 \leq j \leq p)$ *such that* $S' = \cup_{j=1}^{p} T'_j$.

Proof. Let S' be an $F(G\square K_p)$-set. Define for each j $(1 \leq j \leq p)$, $S'_j = S' \cap V(G^{(v_j)})$. Clearly, one or more $S'_j s$ are possibly empty and $S'_j \subseteq V(G^{(v_j)})$, for all j $(1 \leq j \leq p)$. Further, $S' = \cup_{j=1}^{p} S'_j$.

For all $S'_j \neq \emptyset$, since S' is a 2-packing of $G\square K_p$, S'_j is a 2-packing of $G^{(v_j)}$ and hence, $p_G(S'_j)$ is a 2-packing in G. Thus, condition (i) holds.

For each $j \in \{1, 2, \ldots, p\}$ define $S_j = p_G(S'_j)$ and $S = p_G(S')$. Let $k, l \in \{1, 2, \ldots, p\}$ such that $k \neq l$ and $j \in \{1, 2, \ldots, p\}$ such that $S'_j \neq \emptyset$. Then, for each $x \in (N[S_k] \times \{v_j\})$, $d_{G\square K_p}(x, S'_j) \leq 2$ and hence, $x \notin S'_j$. Or equivalently,

$$(N[S_k] \times \{v_j\}) \cap S'_j = \emptyset \tag{5.25}$$

Further, as $(N[S_k] \times \{v_j\}) \subseteq V(G^{(v_j)})$,

$$(N[S_k] \times \{v_j\}) \cap S'_l = \emptyset \tag{5.26}$$

for all $k \neq l \quad (1 \leq k, l \leq p)$.

As (5.25) and (5.26) are true for all $k \neq l \quad (1 \leq k, l \leq p)$ and any arbitrary j $(1 \leq j \leq p)$ for which $S'_j \neq \emptyset$, condition (ii) holds. Further, as S' is an $F(G\square K_p)$-set, $I(S') = \sum_{j=1}^{p} I(S'_j)$ is maximum (Clearly, $I(S'_i) = 0$ whenever $S'_i = \emptyset$). That is, $\sum_{j=1}^{p} I(S'_j) = \max\{\sum_{j=1}^{p} I(T'_j) : T'_j \subseteq V(G^{(v_j)})$ and $\cup_{j=1}^{p} T'_j = S'\}$ and hence condition (iii) holds.

Conversely, suppose that conditions (i), (ii) and (iii) hold for any subset S' of $V(G\square K_p)$. Then, conditions (i) and (ii) together imply that S' is a 2-packing of $G\square K_p$. Further, as $I(S') = \sum_{j=1}^{p} I(S'_j)$, condition (iii) guarantees that S' is an $F(G\square K_p)$-set. $\square$

The following theorem gives a necessary and sufficient condition for $G\Box K_p$ to be efficiently dominatable.

Theorem 5.4.4. *Let G be a connected graph. $G\Box K_p \in \mathscr{E}$ if and only if there exists a collection $\mathcal{P}$ of p mutually disjoint equal sized subsets of $V(G\Box K_p)$ such that*

(i) $p_G(K) \cap p_G(T) = \emptyset$, for all $K, T \in \mathcal{P}$.

(ii) $\cup_{T\in\mathcal{P}} p_G(T)$ is a maximal independent set of G.

(iii) If $S = \cup_{T\in\mathcal{P}} p_G(T)$ and $u \in V - S$, then $|N_G(u) \cap p_G(T)| = 1$, for every $T \in \mathcal{P}$.

Proof. Let $G\Box K_p \in \mathscr{E}$ and S' be an EDS of $G\Box K_p$. For each $j \in \{1, 2, \ldots, p\}$, let $S'_j = V(G^{(v_j)}) \cap S'$. Then, clearly $S'_j \subset V(G\Box K_p)$. Define $\mathcal{P} = \{S'_j\}_{1\leq j\leq p}$.
S' consists of p subsets of $V(G\Box K_p)$. Let $\mathcal{P}$ be the collection of p such subsets. Let $S = \cup_{T\in\mathcal{P}} p_G(T)$. Then, it follows from Proposition 5.4.2 that S is independent in G. Since $G\Box K_p \in \mathscr{E}$, for any $u \in V - S$, $|N_G(u) \cap S| = p$. Furthermore, it follows that the independent set S is maximal. For, if there exist $w \in V(G)$ such that $S \cup \{w\}$ is independent in G, then for every $x \in N_G(w)$, $x \in V - (S \cup \{w\})$ and $|N_G(w) \cap (S - \{w\})| = 0$, a contradiction. As S' is a 2-packing of $G\Box K_p$, it follows that have $|T \cap (N_G(u) \times \{v_j\})| = 1$, for $j \in \{1, 2, \ldots, p\}$, for every $u \in V - S$ and $T \in \mathcal{P}$. In other words, $|N_G(u) \cap p_G(T)| = 1$, for every $u \in V - S$ and $T \in \mathcal{P}$. Also, as for every $u \in V - S$, $|N_G(u) \cap S| = p$, it follows that the elements in $\mathcal{P}$ are mutually disjoint and $|p_G(K)| = |p_G(T)|$, for all $K, T \in \mathcal{P}$.
Conversely, let $\mathcal{P}$ be a collection of p mutually disjoint equal sized subsets of $V(G\Box K_p)$ such that conditions (i) and (ii) hold. It follows from conditions (i) and (ii) that $|N_G(u) \cap S| = p$. Since $S(= \cup_{T\in\mathcal{P}} p_G(T))$ is a maximal independent set of G, we have $|T \cap (N(u) \times \{v_j\})| = 1$, for $j \in \{1, 2, \ldots, p\}$, which inturn implies that each $T \in \mathcal{P}$ forms a 2-packing of $G\Box K_p$ and hence $S' = \cup_{T\in\mathcal{P}} T$ in turn forms a 2-packing of $G\Box K_p$. As for every $u \in V - S$, $|N(u) \cap S| = p$, $S' = \cup_{T\in\mathcal{P}} T$ efficiently dominates $V(G\Box K_p)$. Hence, $G\Box K_p \in \mathscr{E}$. □

5.4.1 An Exact Exponential time Algorithm to identify an $F(G\Box K_p)$-set

Following the discussions given in Section 5.3.1, it is known that the problem of deciding whether or not a graph G has an EDS is $\mathcal{NP}$-complete and so also for the product $G\Box K_p$. Therefore, in this section, an exact exponential time algorithm is proposed to compute the exact value of $F(G\Box K_p)$ and thereby, to determine whether or not the product $G\Box K_p$ is efficiently dominatable.

Given a connected graph G of order n and knowing the value of p (the order of the complete graph in the product $G\Box K_p$), the proposed algorithm "*ED_ CompCProd(G,n,p)*" (refer to Algorithm 4) computes $F(G\Box K_p)$. Based on the value of $F(G\Box K_p)$, it is determined whether the product is efficiently dominatable. The algorithm generates an $F(G\Box K_p)$-set simply by using the independent sets of G rather than directly searching for subsets of $G\Box K_p$. This helps in considerably reducing the time complexity compared to the traditional exhaustive search techniques.

Based on the results discussed in Proposition 5.4.2 and Theorems 5.4.3 and 5.4.4, given a connected graph G of order n, the proposed algorithm "*ED_ CompCProd(G,n,p)*" generates an $F(G\Box K_p)$-set using the following procedure:

1. Find all distinct independent sets, say, $I_1, I_2, \ldots, I_k$ of G.

2. Among these independent sets, identify those sets which satisfy the condition $|I_i| \leq n - \frac{1}{p}(\sum_{u\in I_i} \deg(u))$.

3. For those independent sets identified in Step 2, partition each independent set into 2-packings of G. Suppose I_j is an independent set satisfying the inequality in Step 2, then I_j is partitioned into 2-packings, say S_1, S_2, $\ldots$, S_t of G. Then, placing each of these 2-packings of G in distinct rows of the product $G\Box K_p$ results in a 2-packing of the product $G\Box K_p$. Repeating this process for each independent set identified in Step 2 results in different 2-packings of the product. Upon comparing the influences of all the

2-packings of $G\Box K_p$ so obtained, the one with maximum influence is returned as an $F(G\Box K_p)$-set. Based on the value of $F(G\Box K_p)$, it is decided whether or not $G\Box K_p$ is efficiently dominatable. It is guaranteed by Proposition 5.4.2 and Theorem 5.4.3 that an $F(G\Box K_p)$-set must be one among the sets generated as above and hence, it is sufficient to compare the influences of these sets rather than all 2-packings of $G\Box K_p$. This again helps in significantly reducing the complexity of the algorithm.

The above procedure involves two major steps, which significantly influence the complexity of the proposed algorithm: (1) Generating all independent sets of G and (2) the procedure "*M2P_ CompCProd*" (used as a subroutine in the proposed algorithm).

It is shown in (Kirschenhofer et al., 1983) that if G is a connected graph of order n and k is the number of independent sets in G, then $1 + n \leq k \leq 2^{n-1} + 1$. An outline of the subroutine "*M2P_ CompCProd*" is discussed below:

An Overview of M2P_ CompCProd:

It is known that if P is a 2-packing of a graph H, then $P \times \{v_j\}$, for some j $(1 \leq j \leq p)$, forms a 2-packing in the product $H\Box K_p$ and $I_{H\Box K_p}(P \times \{v_j\}) = I_H(P) + |P|(p-1)$.

Given any connected graph H and an independent set S of H, the main objective of *M2P_ CompCProd* is to partition S into 2-packings of H, say $\mathscr{S} = \{S'_1, S'_2, \ldots, S'_m\}_{m\leq p}$ in such a way that the $\cup_{i=1}^{m}(S'_i \times \{v_i\})$ has maximum influence among the influences of all 2-packings of $H\Box K_p$ generated using such partition of S into 2-packings. To determine such a collection, the algorithm takes an independent set of H as input. Among all those 2-packings of H, a collection of 2-packings $\mathcal{P}$ of H is generated such that $P \subseteq S$ $(P \in \mathcal{P})$. Using this collection $\mathcal{P}$, a partition of S is identified. This is done by sorting these 2-packings in the nonincreasing order of their influence in H. The sets with same influence are taken in the nonincreasing order of their cardinality. Let $\mathcal{P} = \{P_1, P_2, \ldots, P_{l'}\}$. For a given i $(1 \leq i \leq l')$, there may be one or more collections of mutually disjoint 2-packings of H containing P_i. Among all such collections containing P_i, the one having maximum influence (the sum of influence of all the elements in the collection), say $\mathscr{S}_q$ is determined (refer to Step 25). Next, for each of the above newly generated collections $\mathscr{S}_q$, the elements in the collection are further sorted in the nonincreasing order of their influence in H. In the event that a collection includes more than p elements, only the first p elements are retained after sorting. Finally, the required collection (of size at most p) whose total influence in the corresponding product $H \Box K_p$ is maximum among all such collections generated using the above procedure is determined (refer to Step 37).

Lemma 5.4.5 is proved by using a similar discussion as in Lemma 5.3.5 and is stated as below.

Lemma 5.4.5. *Given the collection of all 2-packings of a connected graph H of order n', M2P_ CompCProd generates a 2-packing of $H \Box K_p$, say S'', in $\mathcal{O}(l^2(l^2n'^2 + p))$ steps, where l is the number of 2-packings of H.*

Theorem 5.4.6. *For any connected graph $G = (V, E)$ of order n, the algorithm ED_ CompC_ prod(G,n,p) identifies an EDS of $G \Box K_p$ or an $F(G \Box K_p)$-set in $\mathcal{O}(kl^2(l^2n^2 + p))$ steps, where k and l are respectively the number of independent sets and 2-packings of G.*

Proof. The correctness of the algorithm follows from Proposition 5.4.2, Theorem 5.4.3 and Lemma 5.4.5.

In the main algorithm *ED_ CompC_ prod(G,n,p)* (Algorithm 4), Step 1 generates all independent sets of G in $\mathcal{O}(nk)$ steps, where k is the number of independent sets of G (Lawler et al., 1980). The **for** loop in Steps 3-10 generates a collection $\mathscr{S} = \{S'_i : 1 \leq i \leq k\}$ of pairwise disjoint 2-packings of the independent sets S_j's, where $\mathscr{S}$ is of size at most k. Steps 4-6 takes constant time. For each S_j in Step 6, Step 8 calls the subroutine *M2P_ CompCProd*$(H, S, p, \mathscr{P}, |\mathscr{P}|)$ for the subgraph $H =< N[S_j] >$ which partitions S_j into 2-packings such that its corresponding influence in $H \Box K_p$ is maximum among all 2-packings of $H \Box K_p$ generated using such partitions of S_j into 2-packings. Every call of Step 8 takes $\mathcal{O}(l'^2(l'^2n'^2 + p))$ steps, where $n' = |V(< N[S_j] >)|$ and l' is the number of 2-packings of $< N[S_j] >$

Algorithm 3: *M2P_ CompCProd*$(H,\, S,\, p,\, \mathscr{P}, |\mathscr{P}|)$

Input: A connected graph H of order n', an independent set S of H, $V(K_p) = \{v_1, v_2, \ldots, v_p\}$ $(p \geq 1)$, $\mathscr{P} = \{P : P$ is a 2-packing of H and $P \subseteq S\}$ and $|\mathscr{P}|$

Output: A partition of S into 2-packings $\mathscr{S} = \{S'_1, S'_2, \ldots, S'_m\}_{m \leq p}$ of H such that $\cup_{i=1}^{m}(S'_i \times \{v_i\})$ has maximum influence among the influences of all those 2-packings in $H \square K_p$ obtained by using any such partition of S.

1 Let $|\mathscr{P}| = l'$ and $\mathscr{P} = \{P_1, P_2, \ldots, P_{l'}\}$
2 Sort $\mathscr{P}$ in the nonincreasing order of influence of $P'_i s$. Sets with same influence are taken in the nonincreasing order of their cardinality in the sorted list. Let $\mathscr{P}' = \{P'_1, P'_2, \ldots, P'_{l'}\}$ be the sorted list of 2-packings.
3 $q = 0$; $r = 1$; $k = 0$; $\mathscr{T} = \emptyset$
4 **while** $r \leq l'$ **do**
5   **if** $k \leq l'$ **then**
6   **while** $P'_r \notin \mathscr{T}$ **do**
7             $q++$
8             $t = 1$; $S_t = P'_r$; $I(S_t) = \sum_{x \in S_t}(1 + deg_H(x))$
9             $\mathscr{T} = \mathscr{T} \cup \{P'_r\}$; $k++$
10             $j = 1$
11   **while** $j \leq l'$ **do**
12                 $i = 1$
13   **while** $i \leq t$ **do**
14   **if** $P'_j \cap S_i \neq \emptyset$ **then**
15   j++; goto Step 11
16   **end**
17   **else**
18                         $i++$
19   **end**
20   **end**
21                 $t++$; $S_t = P'_j$; $I(S_t) = \sum_{x \in S_t}(1 + deg_H(x))$
22                 $\mathscr{T} = \mathscr{T} \cup \{P'_j\}$; $k++$
23                 $j++$
24   **end**
25             $\mathscr{S}_q = \{S_1, S_2, \ldots, S_t\}$
26   Sort $\mathscr{S}_q$ in the nonincreasing order of influence of $S'_i s$. To break a tie, if any, take the sets in the nonincreasing order of their cardinalities. In the sorted collection $\mathscr{S}_q$, retain only the first p elements, in case it includes more than p sets.
27             $\mathscr{S}'_q = \{S'_1, S'_2, \ldots, S'_m\}_{m \leq p}$ be the sorted collection got in Step 26.
28             $S''_q = \cup_{i=1}^{m}(S'_i \times \{v_i\})$
29             $I(S''_q) = \sum_{i=1}^{m}(I(S'_i) + |S'_i|(p-1))$
30   **end**
31         $r++$; goto Step 4
32   **end**
33   **else**
34   goto Step 37
35   **end**
36 **end**
37 $I_{max} = \max\{I(S''_1), I(S''_2), \ldots, I(S''_q)\}$
38 $S'' = S''_q$ such that $I(S''_q) = I_{max}$
39 $I(S'') = I(S''_q)$
40 **return** S'' and $I(S'')$

Algorithm 4: *ED_ CompCProd(G,n,p))*

Input: A connected graph G of order n, $V(K_p) = \{v_1, v_2, \ldots, v_p\}$
Output: $F(G\Box K_p)$ and an $F(G\Box K_p)$-set

1 Generate all independent sets $I_1, I_2, \ldots, I_k$ of G.
2 $j = 0$
3 **for** $i = 1$ *to* k **do**
4   **if** $|I_i| \leq n - \frac{1}{p}(\sum_{u\in I_i} \deg(u))$ **then**
5         $j = j + 1$
6         $S_j = I_i$
7   Generate all 2-packings P of $< N[S_j] >$ such that $P \subseteq S_j$. Let $\mathscr{P}_j$ be the collection of all such 2-packings.
8   S''_j =*M2P_ CompCProd*($< N[S_j] >$, S_j, p, $\mathscr{P}_j, |\mathscr{P}_j|$)
9   **end**
10 **end**
11 $S_{max} = S''_i$ such that $I(S''_i) = \max\{I(S''_1), I(S''_2), \ldots, I(S''_j)\}$ and $F(G\Box K_p) = I(S''_i)$
12 **if** $F(G\Box K_p) = np$ **then**
13   print "$G\Box K_p$ is efficiently dominatable and S'' is an EDS of $G\Box K_p$."
14 **end**
15 **else**
16   print "$G\Box K_p$ is not efficiently dominatable and S'' is an $F(G\Box K_p)$-set."
17 **end**
18 **return** S'' and $F(G\Box K_p)$

such that each 2-packing is a subset of S_j. As $n' \leq n$ and $l' \leq l$, where l is the number of 2-packings of G, Steps 3-10 takes $\mathcal{O}(kl^2(l^2n^2 + p))$ steps. The collection $\mathscr{S} = \{S'_i : 1 \leq i \leq k\}$ of pairwise disjoint 2-packings of S_j's, generated at the end of Step 10, is used in Step 11 to generate a 2-packing, say S_{max} of $G\Box K_p$, which takes $\mathcal{O}(n \log n)$ time. The remaining steps are executed in constant time. Hence, the total time complexity to execute Algorithm *ED_ CompC_ prod(G,n,p)* (Algorithm 4) is at most $kl^2(l^2n^2 + p) = kl^4n^2 + kl^2p$. □

5.4.2 Some special classes of graphs G for which $G\Box K_p \in \mathscr{E}$

In this section, the existence of some special classes of graphs G for which $G\Box K_p \in \mathscr{E}$ are discussed.

1. Let $G \cong K_{1,p}$, where $V(G) = \{u_0, u_1, u_2, \ldots, u_p\}$, $V(K_p) = \{v_1, v_2, \ldots, v_p\}$ and u_0 be the central vertex. Then, the set $S' = \{(u_1, v_1), (u_2, v_2), \ldots, (u_p, v_p)\}$

forms an EDS of $G\Box K_p$.

2. A special class of graph $G \cong T^{(l)}$ is defined as follows:

   $T^{(l)}$ is a rooted tree whose root, say r, is of degree l and all the vertices at an even distance from the root r are also of degree l. Equivalently, all the vertices at an even level from the root (including the root) are of degree l. $T^{(l)}\Box K_p \in \mathscr{E}$ if and only if $l = p$. If S' is any EDS of $G\Box K_p$, then $p_G(S')$ ={All the vertices at an odd distance from the root r in G}.

3. Let $G \cong K_{p,n}$ be a complete bipartite graph with partite sets V_1 and V_2, where $|V_1| = p$ and $|V_2| = n$. For any set $S' \subseteq V(G\Box K_p)$, if $p_G(S') = V_1$, then S' is an EDS of $G\Box K_p$. Similarly, if $p_G(S') = V_2$, then S' is an EDS of $G\Box K_n$.

5.5 Efficient Domination in the cartesian Product $\Box_{i=1}^{l} K_{n_i}$

Hamming graphs, a special class of graphs, is the cartesian product of complete graphs. Some known results in the existence of perfect Hamming error correcting codes can be found in (Bannai, 1977; Hamming, 1950). In this section, the results discussed in Section 5.4 are extended to identify some efficiently dominatable graphs among the Hamming graphs.

Let $G \cong \Box_{i=1}^{l} K_{n_i} = K_{n_1}\Box K_{n_2}\Box \ldots \Box K_{n_l}$, for positive integers $l, n_1, n_2, \ldots, n_l$. Then, G is a regular graph of degree $(n_1 - 1) + (n_2 - 1) + \cdots + (n_l - 1) = (n_1 + n_2 + \cdots + n_l) - l$. For ease of reference, (i, j) is used to represent (u_i, v_j).

For positive integers $l, n_1, n_2, \ldots, n_l$, let $G \cong K_{n_1}\Box K_{n_2}\Box \ldots \Box K_{n_l}$, where $n_1 \geq n_2 \geq \cdots \geq n_l$. Let $G' \cong K_{n_1}\Box K_{n_2}$ be called as a block. For ease of representation, the edges in the block (with respect to the product of two complete graphs) are drawn in dotted lines, where each row and each column induces a complete graph (For an example, refer to Figure 5.16).

Fact 5.5.1. *For positive integers $l, n_1, n_2, \ldots, n_l$, let $G \cong K_{n_1}\Box K_{n_2}\Box \ldots \Box K_{n_l}$, where $n_1 \geq n_2 \geq \cdots \geq n_l$. Then, there are $(n_3 \times n_4 \times \cdots \times n_l)$ blocks G' in the product graph G.*

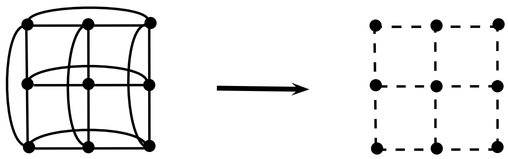


Figure 5.16: The Block representing $K_3\Box K_3$

Theorem 5.5.1. *For positive integers $l, n_1, n_2, \ldots, n_l$, let $G \cong K_{n_1}\Box K_{n_2}\Box \ldots \Box K_{n_l}$, where $n_1 \geq n_2 \geq \cdots \geq n_l$ and S be a maximum independent set of G. Then, the following conditions hold:*

(i) $|S| = n_2 \times n_3 \times \cdots \times n_l$.

(ii) For every $u \in V(G) - S$, $|N_G(u) \cap S| \leq l$. Equality holds if and only if $n_1 = n_2 = \cdots = n_l$.

Proof. (i) Let $G' \cong K_{n_1}\Box K_{n_2}$ be considered as a block. Then, by Fact 5.5.1, G contains $(n_3\times n_4\times\cdots\times n_l)$ blocks of G'. Let $V(G') = \{(i, j) : 1 \leq i \leq n_1,\ 1 \leq j \leq n_2\}$. Then, G' is a diameter two graph, in which at most one vertex from each row or each column forms an independent set of G'. Since $n_1 \geq n_2$, exactly one vertex from each n_2 rows or n_2 columns forms an independent set and hence, there will be $(n_1 - n_2)$ columns whose vertices do not belong to any independent set of G'. As there are $(n_3 \times n_4 \times \cdots \times n_l)$ blocks G' in G, a set of n_2 vertices (one vertex from each row) from each block G' can be chosen to be in any independent set S of G. The vertex so chosen from each block follows a permutation order so that S is independent. For instance, if $\{(i, i) : 1 \leq i \leq n_2\}$ forms an independent set for one block, then $\{(i, i + 1) : 1 \leq i \leq n_1,\ i + 1 \equiv 0 \pmod{n_1}\}$, $\{(i + 1, i) : 1 \leq i \leq n_2,\ i + 1 \equiv 0 \pmod{n_2}\}$ forms the independent set for the other blocks. It can be noted that as exactly one vertex from each row and each column forms an independent set, the set S so obtained is the best maximum possible. Thus, any maximum independent set S of G can be generated by choosing n_2 vertices from each block and hence, $|S| = n_2 \times n_3 \times \cdots \times n_l$.
(ii) Consider, $G \cong \Box_{i=1}^{l} K_{n_i} = \Box_{i=1}^{l-1} K_{n_i}\Box K_{n_l} \cong G^*\Box K_{n_l}$ (say). If S is an independent set of G, then for each $u \in V(G) - S$, u is adjacent to at most two vertices

from the same block and to at most $l-2$ vertices from the remaining blocks. Thus, $|N_G(u) \cap S| \leq l$ and the equality holds if and only if $n_1 = n_2 = \cdots = n_l$. $\square$

Proposition 5.5.2. *Let $G \cong K_{n_1} \square K_{n_2} \square \ldots \square K_{n_l}$, where $n_1 \geq n_2 \geq \cdots \geq n_l$. If S' is an $F(G)$-set, then $|S'| \leq n_3 \times n_4 \times \cdots \times n_l$.*

Proof. Let S' be any $F(G)$-set. If $G' \cong K_{n_1} \square K_{n_2}$ is considered as a block, then G contains $(n_3 \times n_4 \times \cdots \times n_l)$ blocks of G'. Since each G' is a diameter two graph, at most one vertex from each block can be included in S' and thus, $|S'| \leq n_3 \times n_4 \times \cdots \times n_l$. $\square$

For any positive integers p and l, let $G \cong \square_{i=1}^{l} K_p = K_p \square K_p \square \ldots \square K_p$ (l times). Then, $|V(G)| = p^l$ and G is a regular graph of degree $(p-1) + (p-1) + \cdots + (p-1)$ (l times)$= l(p-1)$.

Theorem 5.5.3. *Let $G \cong \square_{i=1}^{l} K_{n_i}$. If $n_1 = n_2 = \cdots = n_l$ and $l = p+1$, then $G \in \mathscr{E}$. In particular, $\gamma(G) = p^{l-2}$.*

Proof. Let $G \cong \square_{i=1}^{l} K_{n_i} = \square_{i=1}^{l-1} K_{n_i} \square K_{n_l} \cong G^* \square K_{n_l}$ (say). Let S' be a maximum independent set of G^*. Then, by Theorem 5.5.1, $|S'| \leq n_2 \times n_3 \times \cdots \times n_{l-1}$ and for every $u \in V(G) - S'$, $|N_G(u) \cap S'| \leq l-1$. Let $n_1 = n_2 = \cdots = n_l$ and $l = p+1$. Then, $G \cong \square_1^l K_p = \square_1^{l-1} K_p \square K_p \cong G^* \square K_p$ (say). Let S' be a maximum independent set of G^*. Then, by Theorem 5.5.1, $|S'| = p^{l-2}$ and for every $u \in V(G) - S'$, $|N_G(u) \cap S'| = l-1$. It follows from the discussion in Theorem 5.4.4 that if for every $u \in V(G) - S'$, $|N_G(u) \cap S'| = p$, then $G \cong G^* \square K_p \in \mathscr{E}$. Since $l-1 = p$, $G \cong \square_1^l K_p \in \mathscr{E}$. In particular, $\gamma(G) = |S'| = p^{l-2}$. $\square$

Remark 5.5.1. *From the Theorem 5.5.3 it follows that,*

a) For $p = 1$, $K_1 \square K_1 = \square_1^2 K_1 \in \mathscr{E}$ and $\gamma(\square_1^2 K_1) = 1$.

b) For $p = 2$, $K_2 \square K_2 \square K_2 = \square_1^3 K_2 \in \mathscr{E}$ and $\gamma(\square_1^3 K_2) = 2$.

c) For $p = 3$, $K_3 \square K_3 \square K_3 \square K_3 = \square_1^4 K_3 \in \mathscr{E}$ and $\gamma(\square_1^4 K_3) = 3^2 = 9$.

d) For $p = 4$, $K_4 \square K_4 \square K_4 \square K_4 \square K_4 = \square_1^5 K_4 \in \mathscr{E}$ and $\gamma(\square_1^5 K_4) = 4^3 = 64$.

In general, $\square_1^{p+1} K_p \in \mathscr{E}$ and $\gamma(\square_1^{p+1} K_p) = p^{p-1}$.

Example 5.5.1. *Let $G \cong \Box_1^4 K_3 = K_3 \Box K_3 \Box K_3 \Box K_3 \cong G^* \Box K_3$, where $G^* \cong K_3 \Box K_3 \Box K_3$. If S is an independent set of G^*, then $|S| = 9$ (refer to Figure 5.17). Here, $G \in \mathscr{E}$ and $\gamma(G) = 9$ (refer to Figure 5.18). (In Figures 5.17 and 5.18, each block (in dotted lines) represents $K_3 \Box K_3$. For ease of visualization, only a few set of edges are shown in Figure 5.18).*

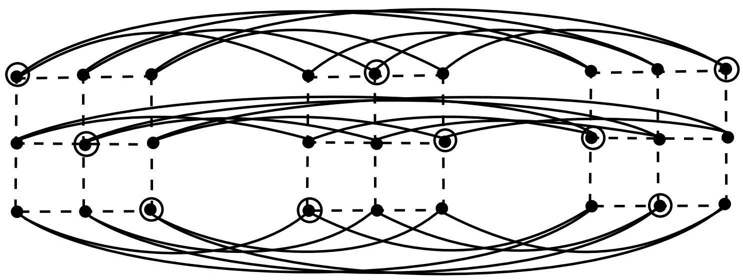


Figure 5.17: An Independent set of $K_3 \Box K_3 \Box K_3$ (Encircled vertices)

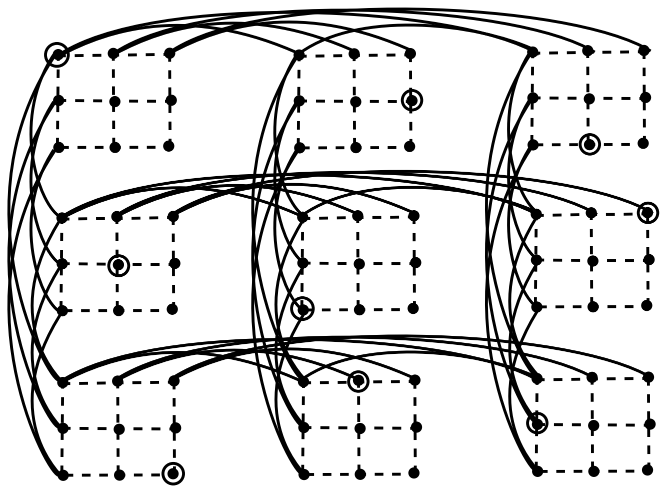


Figure 5.18: An Efficient dominating set of $K_3 \Box K_3 \Box K_3 \Box K_3$ (Encircled vertices)

Remark 5.5.2. *The converse of Theorem 5.5.3 is not true. For example, $\Box_1^7 K_2 \in \mathscr{E}$. In this case, $l = 7$ and $p = 2$, but $l \neq p + 1$.*

Conclusion

This chapter deals with the concept of efficient domination in the cartesian product of graphs. Initially, the notion of efficient domination is discussed for the product $G\Box H$, when G and H are isomorphic to one of the graphs: P_n, C_n, K_n and $K_{1,n}$. The conditions are identified under which these products are efficiently dominatable or otherwise; the exact values of their respective efficient domination numbers are evaluated. Further, the efficiently dominatable products $G\Box K_{1,p}$ and $G\Box K_p$ are characterized in terms of their factors . Furthermore, two exact-exponential time algorithms are proposed for identifying when the products $G\Box K_{1,p}$ and $G\Box K_p$ are efficiently dominatable or not. Finally, the study is extended to efficiently dominatable Hamming graphs.

Chapter 6

Summary and Conclusion

For a graph $G = (V, E)$, a subset S of V is a *dominating set*, if each vertex in V is either in S or has a neighbor in S. The size of a minimum dominating set is called the *domination number of* G, denoted by $\gamma(G)$. A set S is an *efficient dominating set* (EDS) of G if each vertex u in V is either in S or has **exactly one** neighbor in S (inclusive of u). In general, not all graphs possess an EDS; a graph which possesses an EDS is said to be *efficiently dominatable*. Hence, the general interest is to find a subset of V which dominates the maximum number of vertices such that each vertex is dominated exactly once. This maximum number is referred to as the *efficient domination number* of G, denoted by $F(G)$.

In this thesis, the notation $\mathscr{E}$ is used to denote the class of efficiently dominatable graphs. Thus, $G \in \mathscr{E}$ if and only if G has an efficient dominating set (EDS). If $G \in \mathscr{E}$, then any EDS of G has its cardinality equal to the domination number of G, denoted by $\gamma(G)$ (Bange et al., 1988). The structural properties of a graph G having a given domination number, say $\gamma(G) = k$, have been well studied in the literature. But, the properties of an efficiently dominatable graph G with $\gamma(G) = k$ need not be the same for a graph G, where $G \notin \mathscr{E}$, but with $\gamma(G) = k$. This necessitates an independent study of the class of efficiently dominatable graphs.

6.1 Summary

Based on the research gap identified in the literature and motivated by the applications of the notion of efficient domination, this research work focuses on three aspects: (1) Study on efficient domination in general graphs (2) Critical aspects of efficient domination and (3) Efficient domination in the cartesian product of graphs. The results discussed on these three aspects are categorized into three chapters and some of the significant contributions in these chapters are summarized as below:

Chapter 3: In this chapter, the focus is on exploring the notion of efficient domination in arbitrary graphs and trees. Some significant contributions to this chapter are summarized as follows:

- Given any positive integer k, the existence of efficiently dominatable graphs having domination number k is discussed together with a procedure for the construction of such graphs.

- Some improved bounds are obtained for the domination number of an efficiently dominatable graph.

- The properties of graphs possessing pairwise disjoint efficient dominating sets are discussed.

- For $r \geq 1$, G is an r-regular graph containing $(r+1)$ pairwise disjoint efficient dominating sets if and only if $V(G)$ can be partitioned into $(r+1)$ independent sets S_i (for $i = 1$ to $r+1$), each of cardinality $\frac{|V(G)|}{r+1}$, such that each vertex $u \in S_i$ has a unique neighbor in S_j, for every $i \neq j$.

- As an attempt to explore the applications of such r-regular structures, a discussion is included which guarantees that these structures possess an inbuilt simultaneous solution to the problems related to topology control, fault-tolerance, efficient routing, channel assignment in ad hoc as well as sensor networks.

- The properties of efficiently dominatable trees and those of trees which are not efficiently dominatable are studied based on the existence/non-existence and nature of support vertices.

- If $S(T)$ denotes the support vertices of a tree T, then it is shown that, for any tree T with $S(T) = \emptyset$, $\left\lceil \frac{n+2}{4} \right\rceil \leq \gamma(T) \leq \left\lfloor \frac{n}{2} \right\rfloor$.

- Some efficiently dominatable trees are also identified based on the distance between any pair of distinct leaf nodes. That is, if $T \in \mathscr{L}$, where $\mathscr{L}$ denotes the family of trees in which for any pair of distinct leaf nodes x and y, $d(x, y) \equiv c \pmod 3$, where $c \in \{0, 1, 2\}$, then $T \in \mathscr{E}$.

- Efficiently dominatable trees of diameter upto five are characterized.

- Efficient domination in some special classes of graphs, namely, join, one-point union and corona of graphs are also discussed.

Chapter 4: This chapter is devoted to the study of the critical aspects in efficiently dominatable graphs. On that line, the study on changing and unchanging efficient domination in graphs is initiated with respect to vertex criticality (vertex removal), edge criticality (edge removal and addition).

In general, on removing a vertex u from G, $\gamma(G - u)$ is either same as $\gamma(G)$ or lesser or greater than that of G. Interest is shown on studying the properties of such vertices whose removal leaves $\gamma(G)$ unaltered, those which decrease or increase $\gamma(G)$. Some of the significant results obtained on these topics are listed below:

Vertex Removal:

- Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G - u \in \mathscr{E}$. Then, u is γ-critical if and only if u is in every EDS of G.

- Let $G \in \mathscr{E}$ and $u \in V(G)$ such that $G - u \in \mathscr{E}$. Then, the following conditions are equivalent:

  (i) u is γ-critical.

  (ii) u is in every EDS of G.

  (iii) $|N_G(u) \cap S_u| \neq 1$, for every EDS S_u of $G - u$.

- Let $G \in \mathscr{G}_{-v}$ and $|V(G)| = n$. Then, the following properties hold:

(i) $n - \gamma(G) \leq |V^0| \leq n$

(ii) $0 \leq |V^+| \leq \gamma(G)$

(iii) $0 \leq |V^-| \leq \gamma(G)$

- Let $G \in \mathscr{G}_{-v}$. Then, $G \in CVR_{\mathscr{E}}$ if and only if $G \cong mK_1$, for $m \geq 1$.

- Let $G \in \mathscr{G}_{-v}$. Then, $|V^0| = n - \gamma(G)$ if and only if G has a unique EDS.

- Let $G \in \mathscr{G}_{-v}$ such that G is connected and $\gamma(G) \leq 2$. Then either $V(G) = V^0$ or $V(G) = V^0 \cup V^-$ or $V(G) = V^0 \cup V^+$.

- Let $G \in \mathscr{G}_{-v}$ such that G is connected and $\gamma(G) \geq 3$. Then, for any $u \in V^+$ and $v \in V^-$, $d_G(u, v) \geq 4$.

- Let G be a graph of order n, where $n \geq 2$. Then, $G \in UVR_{\mathscr{E}}$ if and only if G has k efficient dominating sets $S_1, S_2, \ldots, S_k$ $(k \geq 2)$ such that $\cap_{i=1}^{k} S_i = \emptyset$.

Edge Removal:

- It is shown that for any edge $e = uv$ in G, if S_e is an EDS of $G - e$ and S is an EDS of G containing either u or v, then it is always possible to relate S and S_e. A procedure is also proposed to construction of an EDS of $G - e$, knowing an EDS of G containing either u or v and this helps in comparing $\gamma(G)$ and $\gamma(G - e)$ easily.

- Let $e \in E(G)$ and $e = uv$. If there exists an EDS S of G such that $u \notin S$ and $v \notin S$, then $e \in ER^0$.

- Let $e \in E(G)$ and $e = uv$. Suppose that G has an EDS containing u. Then, $e \in ER^0$ if and only if v is not in any EDS of $G - e$.

- Let $e \in E(G)$ and $e = uv$. Suppose that G has an EDS containing u. Then, $e \in ER^0$ if and only if v is not in any EDS of $G - u$.

- It is defined that a graph G satisfies the property **P**, if for every pair of vertices $u, v \in V(G)$, there exists an EDS of G not containing both u and

v. Using this, it is shown that $G \in UER_E$ if and only if one of the following holds:

(i) Graph G satisfies Property **P**.

(ii) If S is an EDS of G and $e = uv \in E(G)$ such that one of its end vertices, say $u \in S$, then for every EDS S_u of $G - u$, either $N_G(u) \cap S_u = \emptyset$ or $N_G(u) \cap S_u$ is not unique.

- For any graph G, $G \in CER_E$ if and only if $G \cong K_{1,n}$.

- For any tree T, $T \in UER_E$ if and only if V^- forms an EDS of T.

Edge Addition:

- Let $G \in \mathscr{E}$ and $e \in E(\overline{G})$, where $e = uv$. If G has an EDS containing both u and v and if S' is an EDS of $G + e$, then $|S' - (N_G[u] \cup N_G[v])| = \gamma(G) - 2$.

- Let $G \in \mathscr{E}$ and $e \in E(\overline{G})$, where $e = uv$. If either, both u and v belong to an EDS of G, or both do not belong to an EDS of G, then, $e \in EA^0$ if and only if $G + e$ has an EDS not containing both u and v.

- Let $G \in \mathscr{E}$ and $e \in E(\overline{G})$, where $e = uv$. If S is any EDS of G such that $u \in S$ and $v \notin S$, then $e \in EA^0$ if and only if $G + e$ also has an EDS, say S', such that $v \notin S'$.

- If $G \in \mathscr{E}$, then $G \in CEA_E$ if and only $G \cong mK_1$, for $m \geq 1$.

- Let $G \in \mathscr{E}$ and $V^+ \neq \emptyset$. Then, $G \in UEA_E$ if and only if $\gamma(G) = 1$.

- If $\gamma(G) \geq 2$ and $G \in UEA_E$, then $V^+ = \emptyset$ and $V^- = \emptyset$. Equivalently, $V(G) = V^0$.

- Let $G \in \mathscr{E}$. If G satisfies property **P**, then $G \in UEA_E$.

- Let $G \in \mathscr{E}$ and $\gamma(G) \geq 2$. If $S = V^+$, then $G \notin \mathscr{G}_{+e}$.

- All the categories of classes arising from the notion changing/unchanging efficient domination with respect to vertex removal, edge removal and edge addition are related and represented through a Venn diagram.

Chapter 5: This chapter deals with the concept of efficient domination in the cartesian product graphs. Some of the properties of the product are discussed in terms of its factors. Mainly, the class of efficiently dominatable product graphs $G\Box K_{1,p}$ and $G\Box K_p$, for an arbitrary graph G, are characterized. As the problem of deciding whether a graph G is efficiently dominatable is $\mathcal{NP}$-complete and so also, for the above two products, exact exponential algorithms are presented to identify an $\mathrm{F}(G\Box K_{1,p})$-set and an $\mathrm{F}(G\Box K_p)$-set in the respective products and thereby, to decide whether the products are efficiently dominatable. Finally, the result is extended to identify efficiently dominatable graphs among the product of complete graphs (Hamming graphs). The following are some significant contributions in this chapter:

1. Efficient domination number of Cartesian Product of some well known graphs are obtained.

2. For any nonempty subset S' of $V(G\Box H)$, $I_{G\Box H}(S') \geq I_G(S_1)+I_H(S_2)-|S'|$, where $S_1 = p_G(S')$ and $S_2 = p_H(S')$. The equality holds if and only if $|S'| = |S_1| = |S_2|$.

3. If $G\Box H \in \mathscr{E}$, where G and H are graphs of order n and p respectively, then $\gamma(G\Box H) \leq \min\{p \times \rho(G), n \times \rho(H)\}$.

Efficient domination in Cartesian product $G\Box K_{1,p}$

- Let G be a graph of order n, where $n \geq 2$. If $G\Box K_{1,p} \in \mathscr{E}$ and S' is its EDS, then either $p \leq \delta(G)+1$ or $p \leq n-\Delta'(G)-1$, where $\Delta'(G) = \max\{\deg(u) : u \in p_G(S'_0)\}$.

- Let $S' \subseteq V(G\Box K_{1,p})$. Then S' is an $F(G\Box K_{1,p})$-set if and only if for each j $(0 \leq j \leq p)$, there exists a set $S'_j \subseteq V(G^{(v_j)})$ such that $S' = \cup_{j=0}^{p} S'_j$ and $S_j = p_G(S'_j)$ satisfying the following conditions:

(i) S_j is a 2-packing in G, for each $j \in \{0, 1, \ldots, p\}$.

(ii) $(N[S_0] \times \{v_j\}) \cap S'_j = \emptyset$, for all $j \in \{1, 2, \ldots, p\}$ and $S_i \cap S_j = \emptyset$, for $i, j \in \{1, 2, \ldots, p\}$ and $i \neq j$.

(iii) $\sum_{j=0}^{p} I(S'_j)$ is maximum of all sets $S'_j \subseteq V(G^{(v_j)})$, for each j $(0 \leq j \leq p)$, such that $S' = \cup_{j=0}^{p} S'_j$.

- $G\Box K_{1,p} \in \mathscr{E}$ if and only if there exists a subset S' of $V(G\Box K_{1,p})$ such that the following conditions hold:

(i) $p_G(S' \cap V(G^{(v_0)}))$ is a 2-packing in G.

(ii) If $S_0 = p_G(S' \cap V(G^{(v_0)}))$ and $G^* \cong < V(G) - N[S_0] >$, then $V(G^*)$ can be partitioned into p sets, say, $S_1, S_2, \ldots, S_p$ such that each S_j is an EDS of G^*.

(iii) For every vertex $v \in N(S_0)$ and for each j $(1 \leq j \leq p)$, $|N(v) \cap S_j| = 1$.

- For any connected graph $G = (V, E)$, the algorithm *ED_StarCProd*(G, n, p) finds an EDS of $G\Box K_{1,p}$ or an $F(G\Box K_{1,p})$-set in $\mathcal{O}^*(c^n)$ time, where $5.6230257 \cdots \leq c \leq 8.658897 \ldots$.

Efficient domination in Cartesian product $G\Box K_p$

- Let G be a connected graph of order n, where $n \geq 2$. If $G\Box K_p \in \mathscr{E}$, then $p \leq n - \delta(G)$.

- Let $\mathscr{S} = \{S \subseteq V(G) : S \text{ is independent in } G \text{ and } |S| \leq n - \frac{1}{p}\sum_{u_i \in S} \deg_G(u_i)\}$. If S' is an $F(G\Box K_p)$-set and $S = p_G(S')$, then the following conditions hold:

(i) $S \in \mathscr{S}$.

(ii) $I_G(S) + |S|(p-1) = \max_{T \in \mathscr{S}}\{I_G(T) + |T|(p-1)\}$.

In particular, $|S'| \leq \alpha(G)$, where $\alpha(G)$ is the independence number of G.

- Let $S' \subseteq V(G\Box K_p)$. Then, S' is an $F(G\Box K_p)$-set if and only if there exist sets $S'_j \subseteq V(G^{(v_j)})$ $(1 \leq j \leq p)$ such that $S' = \cup_{j=1}^{p} S'_j$, where one or more

S'_j's are possibly empty and for each $j \in \{1, 2, \ldots, p\}$ such that $S'_j \neq \emptyset$ the following conditions hold:

(i) $p_G(S'_j)$ is a 2-packing in G.

(ii) For any given k, l, where $k \neq l$ and $1 \leq k, l \leq p$, $S'_l \cap (N[p_G(S'_k)] \times \{v_j\}) = \emptyset$.

(iii) $\sum_{j=1}^{p} I(S'_j)$ is the maximum among all sets $T'_j \subseteq V(G^{(v_j)})$ $(1 \leq j \leq p)$ such that $S' = \cup_{j=1}^{p} T'_j$.

- Let G be a connected graph. $G \Box K_p \in \mathscr{E}$ if and only if there exists a collection $\mathcal{P}$ of p mutually disjoint equal sized subsets of $V(G \Box K_p)$ such that

  (i) $p_G(K) \cap p_G(T) = \emptyset$, for all $K, T \in \mathcal{P}$.

  (ii) $\cup_{T \in \mathcal{P}} p_G(T)$ is a maximal independent set of G.

  (iii) If $S = \cup_{T \in \mathcal{P}} p_G(T)$ and $u \in V - S$, then $|N_G(u) \cap p_G(T)| = 1$, for every $T \in \mathcal{P}$.

- For any connected graph $G = (V, E)$ of order n, the algorithm *ED_ CompC_ prod(G,n,p)* identifies an EDS of $G \Box K_p$ or an $F(G \Box K_p)$-set in $\mathcal{O}(kl^2(l^2n^2 + p))$ steps, where k and l are respectively the number of independent sets and 2-packings of G.

Efficient domination in Cartesian product $\Box_{i=1}^{l} K_{n_i}$

- For positive integers $l, n_1, n_2, \ldots, n_l$, let $G \cong K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_l}$, where $n_1 \geq n_2 \geq \cdots \geq n_l$ and S be a maximum independent set of G. Then, the following conditions hold:

  (i) $|S| = n_2 \times n_3 \times \cdots \times n_l$.

  (ii) For every $u \in V(G) - S$, $|N_G(u) \cap S| \leq l$. Equality holds if and only if $n_1 = n_2 = \cdots = n_l$.

- Let $G \cong K_{n_1} \Box K_{n_2} \Box \ldots \Box K_{n_l}$, where $n_1 \geq n_2 \geq \cdots \geq n_l$. If S' is an $F(G)$-set, then $|S'| \leq n_3 \times n_4 \times \cdots \times n_l$.

- Let $G \cong \Box_{i=1}^{l} K_{n_i}$. If $n_1 = n_2 = \cdots = n_l$ and $l = p + 1$, then $G \in \mathscr{E}$. In particular, $\gamma(G) = p^{l-2}$.

6.2 Conclusion

The problems studied in this thesis are motivated by the applications of efficient domination in coding theory (Biggs, 1973; Hammond and Smith, 1975), resource allocation in distributed/parallel computing (Livingston and Stout, 1988, 1990; Van Wieren et al., 1993; Milanič, 2013), communication in sensor and ad hoc networks etc. (Yu and Chong, 2003, 2005; Janakiraman and Thilak, 2011; Thilak, 2013).

Based on the results and discussions in this thesis, it is justified that even though every efficient dominating set is also a minimum dominating set and all efficient dominating sets have the same cardinality, namely, the domination number of the graph, the properties possessed by an efficiently dominatable graph differ considerably from those possessed by a graph which is not efficiently dominatable. By revisiting some of the existing results related to the concept of criticality and exploring some new properties of critical vertices and critical edges, it is noted that the properties of such elements differ significantly when restricted to the class of efficiently dominatable graphs (refer to Tables 4.1, 4.2 and 4.3).

Further, the structure of cartesian product of graphs is one of the widely used multi-dimensional architectures in distributed computing systems and is also one of the commonly used topologies for ad hoc, sensor and vehicular networks. Thus, the problem studied in this thesis will facilitate the problems related to the design of efficient resource management protocols in distributed computing. Further, an efficient dominating set possesses three significant properties, namely, domination, independence and 2-packing, which makes it unique among other domination parameters and makes it suitable for the design of energy efficient and interference free communication protocols in ad hoc and sensor networks. From a graph theoretic perspective, the two exact exponential algorithms proposed in this thesis will help in the solving the decision version of the efficient domination problem, at least for the two products under consideration.

6.3 Scope for future work

The concept of efficient domination in graphs is explored to some extent in some special class of graphs, both from theoretical and algorithmic perspectives. Attempts can be made to improve further, the bounds on domination number of an efficiently dominatable graph G, by imposing additional constraints on G, or focusing on some special significant classes of graphs. To the best of our knowledge, a strong characterization for a graph to be efficiently dominatable or otherwise, is yet to be obtained. The properties of efficiently dominatable graphs can still be explored to a great extent.

It is known that the decision version of the efficient domination problem is $\mathcal{NP}$-complete for an arbitrary graph and even in case of some special classes of graphs. To the best of our knowledge, an efficient approximation or an exponential time algorithm is yet to be proposed for an arbitrary graph.

In this thesis, some properties of efficiently dominatable trees are discussed and efficiently dominatable trees upto diameter five have been characterized. However, the properties of efficiently dominatable trees of arbitrary diameter, are yet to be explored. Extending the ideas discussed in this thesis, or exploring some other better procedures, will be helpful in characterizing trees with diameter d, for $d \geq 6$. Thus, with respect to trees, the following problems is worth exploring:

- Characterize efficiently dominatable trees of an arbitrary diameter.

Further, among all the products, cartesian product of graphs is of special interest from both graph theoretic as well as application perspective, as it is one of the widely used multi-dimensional architectures in distributed computing. To the best of our knowledge, there exist very limited results concerning the concept of efficient domination in the cartesian product of two or more arbitrary graphs. On that line, the thesis deals with the results on the notion of efficient domination in the cartesian products having $K_{1,p}$ or K_p as one of the factors. The study on similar lines for products of two or more arbitrary graphs will be of special interest and significance. Thus, the following problems will be interesting to deal with:

- For arbitrary graphs G and H, obtain some properties/bounds on efficient domination number for the product $G \Box H$.
- Study the concept of efficient domination in the cartesian product of graphs having trees and/or other special classes of graphs, as factors.
- Explore the notion of efficient domination in other interesting graph products.

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LIST OF PUBLICATIONS/ CONFERENCE PAPERS

1. A Senthil Thilak, Sujatha V Shet and S.S. Kamath (2020), *Changing and unchanging efficient domination in graphs with respect to edge addition,* Mathematics in Engineering, Science and Aerospace, 11(1): 201–213.

2. A Senthil Thilak, Sujatha V Shet and S.S. Kamath (2021),*On graphs with pairwise disjoint efficient dominating sets and efficient domination in trees in terms of support vertices*, Advances and Applications in Discrete Mathematics, 26(1):83-108.

3. A Senthil Thilak, S.S. Kamath and Sujatha V Shet, *Efficient domination in Cartesian product of graphs*, In: Proceedings of the International Workshop and Conference on Analysis and Applied Mathematics, IWCAAM'16, pp. 179 - 194.

4. Sujatha V Shet, A Senthil Thilak and S.S. Kamath, *Efficient Domination in Trees up to diameter five using support vertices*, *International Conference on Advances in Mathematical Sciences 2017*, VIT, Vellore (Abstract only).

5. A Senthil Thilak, Sujatha V Shet and S.S. Kamath, *Changing and Unchanging Efficient Domination with respect to Vertex Criticality*, 83^{rd} *Annual Conference of Indian Mathematical Society- An International Meet 2017*, S V University, Tirupati (Abstract only).

6. A Senthil Thilak, Sujatha V Shet and S.S. Kamath, *Graphs having pairwise disjoint Efficient Dominating sets and their Applications to Fault-Tolerant*

communication in Wireless Sensor Networks, *International Conference on Discrete Mathematics and Data Sciences-2018*, SASTRA University, Thanjavur (Abstract only).

7. A Senthil Thilak, Sujatha V Shet and S.S. Kamath, *Changing and Unchanging Efficient Domination with respect to Edge removal*, *International Conference on Emerging Trends in Graph Theory-2019*, CHRIST (Deemed to be University), Bangalore (Abstract only).

8. A Senthil Thilak, Sujatha V Shet and S.S. Kamath, *Changing and Unchanging Efficient Domination with respect to Edge Removal.* (Extended version submitted to ICETGT2019 conference proceedings, Christ University)

9. A Senthil Thilak, Sujatha V Shet and S.S. Kamath, *Efficient domination in Cartesian product of an arbitrary graph G with* $K_{1,p}$. (To be communicated)

10. A Senthil Thilak, Sujatha V Shet and S.S. Kamath, *Changing and Unchanging Efficient Domination with respect to Vertex Criticality.* (To be communicated)

11. A Senthil Thilak, Sujatha V Shet and S.S. Kamath, *On Efficient domination in Cartesian product of graphs with* K_p *as one of its factors.* (To be communicated)

BIO DATA

Name : Sujatha V Shet

Email Id : sujashet@gmail.com

Date of Birth : 24-07-1976

Permanent Address : "ViswaKiran", Ambagilu,
Santhekatte post, Udupi-576105.
Karnataka.

EDUCATIONAL QUALIFICATIONS:

Degree	Year of Passing	Institute
B.Sc. (Mathematics)	1996	Seethalakshmi Ramaswamy College, Tiruchirapalli, Bharathidasan University, Tamilnadu.
M.Sc. (Mathematics)	1998	Seethalakshmi Ramaswamy College, Tiruchirapalli, Bharathidasan University, Tamilnadu.
B.Ed	2000	Sri Sathya Sai Institute of Higher Learning, Anantapur, Andhrapradesh.
M.Phil. (Mathematics)	2003	Madurai Kamaraj University, Madurai, Tamilnadu.