FRACTIONAL REGULARIZATION METHODS FOR ILL-POSED PROBLEMS IN HILBERT SCALES

Thesis

Submitted in partial fulfillment of the requirements for the degree of

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by

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JUNE 2022

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DECLARATION

By the Ph.D. Research Scholar

I hereby **declare** that the research thesis entitled **"FRACTIONAL REGULARIZATION METHODS FOR ILL-POSED PROBLEMS IN HILBERT SCALES"** which is being submitted to the **National Institute of Technology Karnataka, Surathkal** in partial fulfillment of the requirements for the award of the Degree of **Doctor of Philosophy** in **Department of Mathematical and Computational Sciences** is a **bonafide report of the research work carried out by me**. The material contained in this research thesis has not been submitted to any University or Institution for the award of any degree.

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CERTIFICATE

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ABSTRACT

Regularization methods are widely used for solving ill-posed problems. In this thesis we consider the fractional regularization methods for solving equations of the form $T(x) = y$ where $T : X \longrightarrow Y$ is a linear operator between the Hilbert spaces X and Y. In practical applications, we mostly have only the noisy data y^δ such that $\|y - y^\delta\| \leq \delta$. Throughout our study, we work in the setting of Hilbert scales as it improves the order of convergence. We study the Fractional Tikhonov Regularization method in Hilbert scales and for selecting the regularization parameter the adaptive choice method introduced by Pereverzev and Schock (2005) is used. Also we introduce a new parameter choice strategy. While performing numerical calculations, it is always easier to work in a finite dimensional setting than in an infinite dimensional one. For this reason we study the finite dimensional realization of the fractional Tikhonov and the fractional Lavrentiev regularization methods in Hilbert scales. We also study the analogous of the discrepancy principle considered in George and Nair (1993) for Fractional Lavrentiev method.

Keywords: ***Ill-posed problem; Fractional Tikhonov regularization; Hilbert scales; Parameter choice strategy; Adaptive parameter choice; Lavrentiev regularization; Finite dimensional realization; Discrepancy principle.***

Mathematics Subject Classification: 47A52, 65R10, 65J10, 47H09, 49J30

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Chapter 1

INTRODUCTION

An inverse problem determines the causes of a desired or observed result. Inverse problems is a very useful and active field of research in applied sciences. Many inverse problems have their mathematical formulation as an operator equation of the form

$$T(x) = y\,, \tag{1.0.1}$$

where $T : D(T) \subseteq X \longrightarrow Y$ is a linear or nonlinear operator between suitable normed linear spaces X and Y. Here y is the observation and x is the sought for solution.

1.1 ILL-POSED PROBLEMS

Inverse problems most often do not fulfill Hadamard's postulates of well-posedness. French mathematician Hadamard (1953), formulated the following conditions of well-posedness of mathematical problems. The problem of solving the operator equation (1.0.1) is said to be well-posed if the following three conditions are fulfilled:

(1.) **Existence**: For each $y \in Y$, there is a solution $x \in X$ of (1.0.1) ;

(2.) **Uniqueness**: The solution x is unique ;

(3.) **Stability**: The dependence of x upon T is continuous .

Problems that are not well-posed in the sense of Hadamard [Hadamard (1953)] are termed ill-posed.

A typical example would be the Fredholm integral equation of the first kind, with $T : L^2[a,b] \longrightarrow L^2[a,b]$ defined by

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt = y(s), \;\; a \leq s \leq b \; ,$$

where $k(.,.)$ is a non-degenerate square integrable function and $y(.)$ is a known data. The Fredholm integral equations of the first kind appear in many problems with practical applications. Hereafter in our study, we consider the operator T to be a bounded linear operator.

Next we provide two examples where Fredholm integral equations appear.

Example 2.1(A Gravitation problem)(Groetsch (2007))

Suppose mass is distributed on a circular ring of radius 1/2 centered on the origin with density $f = f(\theta)$, where θ is a polar angle. Let $g(\varphi)$ be the centrally directed component of gravitational force at the points on the concentric circle of radius 1, where φ is polar angle. Then the relation between density and centrally directed force is given by

$$g(\varphi) = \int_0^{2\pi} k(\phi,\theta)f(\theta)d\theta \; ,$$

let r be distance between a mass element at position θ on the inner ring and a point on the outer ring located at polar angle φ. Then by law of cosines we have

$$r^2 = 5/4 - cos(\varphi - \theta).$$

Similarly, by law of cosines, the angle ψ between the centrally directed vector emanating from the attracted point on the unit circle and the vector from the attracted point and the gravitating element $f(\theta)d\theta$ on the inner circle satisfies

$$cos\psi = \frac{1 - \frac{1}{2}cos(\varphi - \theta)}{r}$$

and hence the total centrally directed force on a point at polar angle at φ on the outer circle is

$$g(\varphi) = \gamma \int_0^{2\pi} \frac{2 - cos(\varphi - \theta)}{(5 - 4cos(\varphi - \theta))^{3/2}} f(\theta)d\theta \; ,$$

where γ is the universal gravity constant. The inverse problem of determining the interior mass distribution f from observation of the force g on the outer ring is thus formulated as an integral equation of the first kind.

Example 2.2(Steady State Heat Distributions)(Groetsch (2007))

Consider the problem of determining the temperature flux (cause) on the left edge of a semi-infinite strip from observation of the temperature on that face (effect) when the temperature in the strip is at steady state. The problem may be stated mathematically as follows. Let

$$\Omega = \{(x, y) : 0 < x, 0 < y < \pi\}$$

and suppose $u = u(x, y)$ is a function defined on the closure of Ω and satisfying

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ in \ \Omega$$

and

$$u(x, 0) = u(x, \pi) = 0 \ \ for \ \ x > 0 \, .$$

Suppose we wish to determine the temperature flux

$$f(y) = \frac{\partial u}{\partial x}(0, y), \ \ 0 < y < \pi$$

given the temperature distribution $g(y) = u(0, y)$.

Elementary separation of variables techniques lead to a representation of the form

$$u(x, y) = \sum_{n=1}^{\infty} a_n e^{-nx} sin(ny).$$

Proceeding formally, we find that

$$f(y) = \sum_{n=1}^{\infty} (-na_n) sin(ny)$$

and hence

$$a_n = -\frac{2}{n\pi} \int_0^{\pi} f(\xi) sin(n\xi) d\xi$$

while

$$g(y) = \sum_{n=1}^{\infty} a_n sin(ny)$$

$$= -\sum_{n=1}^{\infty} \frac{2}{n\pi} \int_0^{\pi} f(\xi) sin(n\xi) d\xi sin(ny)$$

$$= \int_0^{\pi} k(y, \xi) f(\xi) d\xi \, ,$$

where

$$k(y,\xi) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}sin(ny)sin(n\xi)\,.$$

Again the inverse problem is modeled by an integral equation of the first kind.

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt, \;\; a \leq s \leq b$$

where $k(s,t) = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}sin(ns)sin(nt)$, $x(t) = \sum_{n=1}^{\infty}(-na_n)sin(nt)$ and $a = 0$,$b = \pi$.

1.2 BEST APPROXIMATE SOLUTION AND GENERALIZED INVERSE

In equation (1.0.1) if $y \in R(T)$ then we say that y is attainable and condition (a) of well-posedness holds true. However, in real world problems this need not happen, but one would want to find some approximate solution to (1.0.1). Hence we look at the following definition

DEFINITION 1.2.1. *(Engl et al. (1996)) Let $T : X \longrightarrow Y$ be a bounded linear operator.*

(i) $x \in X$ is called least-squares solution of equation (1.0.1) if

$$\|Tx - y\| = inf\{\|Tz - y\| | z \in X\}. \tag{1.2.1}$$

(ii) $x \in X$ is called best-approximate solution of $Tx = y$ if x is a least-squares solution of $Tx = y$ and

$$\|x\| = inf\{\|z\| \mid z \text{ is least-squares solution of } Tx = y\} \tag{1.2.2}$$

holds.

DEFINITION 1.2.2. *Let $T \in B(X,Y)$. The operator $T^\dagger : D(T^\dagger) \subseteq Y \longrightarrow X$, where $D(T^\dagger) = R(T) + R(T)^\perp$, defined by $T^\dagger y = \hat{x}$, where $\hat{x}$ is the least-squares solution of minimal norm of the equation $Tx = y$, is called the generalized inverse of T.*

THEOREM 1.2.3. *(Nair (2009))*
Let X and Y be Hilbert spaces, X_0 be a subspace of X and $T : X_0 \longrightarrow Y$ be a closed linear operator. Then

(i) $T^\dagger$ is a closed linear operator, and

(ii) $T^\dagger$ is continuous if and only if $R(T)$ is a closed subspace of Y .

So, if $R(T)$ is not closed, then finding a generalized inverse of T is also ill-posed. In this case one has to consider, regularization methods for solving (1.0.1).

1.3 REGULARIZATION METHOD

In equation (1.0.1) if $y \in R(T)$ then we say that y is attainable and if not then we can find the best approximation to the solution x using the generalized inverse $T^\dagger$ (Engl et al. (1996)). However if $R(T)$ is not closed, then finding a generalized inverse $T^\dagger$ of T is also ill-posed. In this case one has to consider, regularization methods for solving (1.0.1). Regularization is the approximation of an ill-posed problem by a family of neighbouring well-posed problems. Since (1.0.1) is ill-posed in general, the strong convergence and stability of approximate solutions can be attained only by exercising some regularization procedure. In practice, most of the time the exact data is not known and the available data is y^δ(called the noisy data) with

$$\|y - y^\delta\| \leq \delta\,. \tag{1.3.3}$$

We therefore look for some approximation, say x_α^δ which depends continuously on the (noisy) data y^δ with the property that as the noise level δ decreases to zero and if α is chosen appropriately, then x_α^δ tends to $\hat{x}$.

DEFINITION 1.3.1. *(Engl et al. (1996)) Let $T : X \longrightarrow Y$ be a bounded linear operator between the Hilbert spaces X and Y and $\alpha_0 \in (0, +\infty)$. For every $\alpha \in (0, \alpha_0)$, let $R_\alpha : Y \longrightarrow X$ be a continuous operator. The family $\{R_\alpha\}$ is called a regularization or a regularization operator(for $T^\dagger$), if, for all $y \in D(T^\dagger)$, there* *exists a parameter choice rule* $\alpha = \alpha(\delta, y^\delta)$ *such that*

$$\lim_{\delta \longrightarrow 0} \sup\{\|R_{\alpha(\delta,y^\delta)}y^\delta - T^\dagger y\| | y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0 \tag{1.3.4}$$

holds. Here, $\alpha : \mathbb{R}^+ \times Y \longrightarrow (0, \alpha_0)$ *is such that*

$$\lim_{\delta \longrightarrow 0} \sup\{\alpha(\delta, y^\delta) | y^\delta \in Y, \|y - y^\delta\| \leq \delta\} = 0. \tag{1.3.5}$$

For a specific $y \in D(T^\dagger)$*, a pair* (R_α, α) *is called a (convergent) regularization method (for solving* $Tx = y$*) if (1.3.4) and (1.3.5) hold.*

A regularization method therefore consists of a regularization operator and a parameter choice rule. Since this parameter choice rule is chosen depending on δ and y^δ, α in Definition (1.3.1) can be defined as a function

$$\alpha : \{(\delta, y^\delta) | \delta > 0, \|y - y^\delta\| \leq \delta\} \longrightarrow (0, \alpha_0).$$

DEFINITION 1.3.2. *(Engl et al. (1996)) Let* α *be a parameter choice rule according to definition (1.3.1). If* α *does not depend on* y^δ*, but only on* δ*, then we call* α *an a-priori parameter choice rule and write* $\alpha = \alpha(\delta)$*. Otherwise, we call* α *an a-posteriori parameter choice rule.*

The most widely used regularization methods for (1.0.1) are:

1. When T is a bounded linear operator, Tikhonov regularization method in which the solution x_α^δ of the equation

$$(T^*T + \alpha I)x = T^*y^\delta \tag{1.3.6}$$

   is taken as the approximate solution of (1.0.1)(Groetsch (1984)). In other words, x_α^δ is taken as the minimizer of the functional $J_\alpha(x)$ with

$$J_\alpha(x) = \|Tx - y^\delta\|^2 + \alpha\|x\|^2. \tag{1.3.7}$$

2. If T is a positive, self-adjoint operator and $X = Y$, then one considers Lavrentiev regularization method, in which the solution x_α^δ of the equation

$$(T + \alpha I)x = y^\delta \tag{1.3.8}$$

is taken as an approximate for $\hat{x}$(Hochstenbach et al. (2015)). Note that in this case x_α^δ is the minimizer of the functional $J_\alpha(x)$ with

$$J_\alpha(x) = \langle Tx, x\rangle - 2\left\langle y^\delta, x\right\rangle + \alpha\|x\|^2.$$

It is known that if $\hat{x} \in R(T^*T)$ and $\|y - y^\delta\| \le \delta$ for some $\delta > 0$, the convergence rate $\|\hat{x} - x_\alpha^\delta\| = O(\delta^{2/3})$ is optimal(George and Nair (1994), Engl (1983), Groetsch (1984)) and it is usually attained by an a-priori choice $\alpha = O(\delta^{2/3})$ or by an a-posteriori choice. Morozov(Morozov (1966)) suggests to choose α such that

$$\|Tx_\alpha^\delta - y^\delta\| = \delta$$

and the Arcangelis (Arcangeli (1966))method suggests

$$\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta}{\sqrt{\alpha}}.$$

Another type of discrepancy principle is suggested by Schock in which α is chosen such that

$$\|Tx_\alpha^\delta - y^\delta\| = \frac{\delta^p}{\alpha^q},\ p > 0, q > 0.$$

1.3.1 SOURCE CONDITIONS AND ORDER OPTIMALITY

DEFINITION 1.3.3. *(Nair (2009)) A priori assumptions on the unknown solution $\hat{x}$ $(orT^\dagger y)$', are called source conditions.*

For example, suppose T is a compact operator and $\{(\sigma_n, u_n, v_n) : n \in N\}$ be a singular system of T. Then, from the representations

$$Tx = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n\rangle v_n,\ x \in X,$$

$$T^*Tx = \sum_{n=1}^{\infty} \sigma_n^2 \langle x, u_n\rangle u_n,\ x \in X,$$

we have

$$\hat{x} \in R(T^*) \Longleftrightarrow \sum_{n=1}^{\infty} \frac{|\langle \hat{x}, u_n\rangle|^2}{\sigma_n^2} < \infty\,, \tag{1.3.9}$$

$$\hat{x} \in R(T^*T) \Longleftrightarrow \sum_{n=1}^{\infty} \frac{|\langle \hat{x}, u_n \rangle|^2}{\sigma_n^4} < \infty. \tag{1.3.10}$$

We observe that the source conditions (1.3.9) and (1.3.10) are special cases of the condition

$$\sum_{n=1}^{\infty} \frac{|\langle \hat{x}, u_n \rangle|^2}{\sigma_n^{4\nu}} < \infty$$

for $\nu > 0$. This condition is known as a Holder-type source condition.

For a particular regularization method (R_α, α), its rate of convergence helps us determine its effectiveness. Let

$$x_\alpha = R_\alpha y. \tag{1.3.11}$$

Consider the rate with which

$$\|x_\alpha - \hat{x}\| \longrightarrow 0 \text{ as } \alpha \longrightarrow 0, \tag{1.3.12}$$

where x_α is defined by (1.3.11) and $\hat{x} = T^\dagger y$, or the rate with which

$$\|x_\alpha^\delta - \hat{x}\| \longrightarrow 0 \text{ as } \alpha \longrightarrow 0 \tag{1.3.13}$$

where $x_\alpha^\delta = R_\alpha y^\delta$ and (1.3.3) holds. The rate $\|x_\alpha - \hat{x}\|$ depends on the parameter choice rule (α) and that of $\|x_\alpha^\delta - \hat{x}\|$ depends on the regularization operator. However

$$\|x_\alpha^\delta - \hat{x}\| \leq \|x_\alpha^\delta - x_\alpha\| + \|x_\alpha - \hat{x}\|, \tag{1.3.14}$$

so both the rates are connected. A regularization method R for equation (1.0.1) is said to be of order optimal with respect to the source condition that $\hat{x} \in M_{\nu,\rho} := \{x = (T^*T)^\nu u : 0 < \nu \leq 1, \|u\| \leq \rho\}$ if there exists a constant c_0 independent of δ and ρ such that

$$\|\hat{x} - Ry^\delta\| \leq c_0 \rho^{1/(2\nu+1)} \delta^{2\nu/(2\nu+1)} \tag{1.3.15}$$

whenever $y^\delta \in Y$ and $\|y - y^\delta\| \leq \delta$. In order to improve convergence rates in (1.3.15), many authors studied (George and Nair (1997), George et al. (2013), Mahale and Dadsena (2018), Natterer (1984), Tautenhahn (1993), Argyros et al. (2017), Shobha et al. (2014), Argyros et al. (2014), Shobha and George (2014), George and Kanagaraj (2019), George (2008), George and Nair (2004), George and Nair (2003)) regularization methods in the setting of Hilbert scales.

1.4 HILBERT SCALES

In order to improve convergence rates we study regularization methods in the setting of Hilbert scales.

DEFINITION 1.4.1. *(Mahale and Nair (2007)) A family* $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ *of Hilbert spaces is called a Hilbert scale if it satisfies the following conditions:*

- *For* $s < t$, $\mathcal{X}_t \subseteq \mathcal{X}_s$ *and* $\mathcal{X}_t$ *is a dense subset of* $\mathcal{X}_s$.

- *As Hilbert spaces, the above inclusion is a continuous embedding, i.e. there exists* $c_{s,t} > 0$ *such that*

$$\|x\|_s \leq c_{s,t}\|x\|_t \text{ for all } x \in \mathcal{X}_t.$$

Let $L : D(L) \subset \mathcal{X} \longrightarrow \mathcal{X}$ be a strictly positive definite, unbounded, densely defined, self-adjoint operator. That is, L satisfies;

$$\langle Lx, x\rangle > 0,$$

$D(L)$ is dense in $\mathcal{X}$ and

$$\|Lx\| \geq \|x\|,\ x \in D(L).$$

Let $\mathcal{X}_t$ be the completion of $D := \cap_{k=0}^{\infty} D(L^k)$ with respect to the norm $\|x\|_t = \|L^t x\|$, (here and below $\|.\|$ denote the norm in $\mathcal{X}$) induced by the inner product

$$\langle u, v\rangle_t = \langle L^t u, L^t v\rangle,\ \ u, v \in D.$$

Then, $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ (cf. George and Nair (1997)) satisfies the Definition 1.4.1 (George and Nair (1997),George and Nair (2003),Tautenhahn (2002),Vasin and George (2014)). In this study, we consider the Hilbert scale $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$. Note that the Hilbert scale generated by L connects $\mathcal{X}$ with $\mathcal{X}_s$ through the relation $\|x\|_s = \|x\|_{\mathcal{X}_s} = \|L^s x\|$ (Egger and Hofmann (2018), see also Kreĭn and Petunin (1966)[Page 145]).

1.4.1 INTERPOLATION INEQUALITY IN HILBERT SCALES

Let $\{H_s\}_{s\in\mathbb{R}}$ be a Hilbert scale. We already know that for $r < s < t$, $H_t \subseteq H_s \subseteq H_r$ and there exists constants $c_{r,s}$ and $c_{s,t}$ such that

$$c_{s,t}\|x\|_t \leq \|x\|_s \leq c_{r,s}\|x\|_r \ \ \forall x \in H_t.$$

Another inequality which holds in Hilbert scales is

$$\|x\|_s \leq \|x\|_r^{1-\lambda}\|x\|_t^{\lambda}, \ \ \forall x \in H_t \tag{1.4.16}$$

whenever $r < s < t$, where $\lambda = \frac{t-s}{t-r}$. The inequality in (1.4.16) is called the interpolation inequality on $\{H_s\}_{s\in\mathbb{R}}$.

It is known that the (Klann and Ramlau (2008)) Tikhonov and Lavrentiev regularization methods oversmoothen the solution $\hat{x}$, i.e., sharp or fine features of the reconstructed function are lost. In order to overcome this, the fractional Tikhonov and fractional Lavrentiev methods were studied.

1.5 WEIGHTED OR FRACTIONAL REGULARIZATION METHODS

Klann and Ramlau (2008) first introduced the fractional or weighted Tikhonov regularization method. Following this, another approach also referred to as fractional Tikhonov regularization method was investigated by Hochstenbach and Reichel (2011). By replacing the norm in the first term on the right side of equation (1.3.7) by the weighted seminorm

$$\|y\|_W = \|W^{1/2}y\|_Y$$

with $W = (TT^*)^{(\beta-1)/2}$ for some parameter $0 \leq \beta \leq 1$, where W is the Moore-Penrose inverse of TT^* when $\beta < 1$ we can derive the method in Hochstenbach and Reichel (2011).

In fractional Tikhonov regularization method, $x^\delta_{\alpha,\beta}$ the minimizer of the functional $J_{\alpha,\beta}(x)$,

$$J_{\alpha,\beta}(x) = \|Tx - y^\delta\|_\beta^2 + \alpha\|x\|^2, \ x \in X, \ \alpha > 0 \tag{1.5.1}$$

is taken as an approximation for $\hat{x}$. Note that $x^\delta_{\alpha,\beta}$ satisfies the equation

$$((T^*T)^{(\beta+1)/2} + \alpha I)x = (T^*T)^{(\beta-1)/2}T^*y^\delta.$$

It is known (Kanagaraj and George (2019)) that $x^\delta_{\alpha,\beta}$ reduces the oversmoothing.

In fractional Lavrentiev regularization method in the setting of Hilbert space, the minimizer $x^\delta_{\alpha,\beta}$ of the functional

$$J^\beta_\alpha(x) = \langle Tx, x\rangle - 2\langle y^\delta, x\rangle + \alpha\langle T^\beta x, x\rangle, \quad \alpha > 0 \tag{1.5.2}$$

is taken as an approximation for $\hat{x}$. Here T is a positive self-adjoint operator and $X = Y$.

In the equation (1.3.7), $\alpha > 0$ is the regularization parameter and $\|x\|^2$ is the penalty term. The penalty term in Tikhonov regularization(Lavrentiev regularization) oversmoothens the solution. We propose to study the fractional Tikhonov regularization method and fractional Lavrentiev regularization method which reduces oversmoothing; in the setting of Hilbert scales.

Gerth et al. (2015) compared the convergence properties of fractional Tikhonov regularization method (two different approaches)(Klann and Ramlau (2008), Hochstenbach and Reichel (2011)) with the results published by some other authors(Louis (1989), Mathé and Tautenhahn (2011)) and showed that in both their methods when α is chosen according to the discrepancy principle

$$\|Tx^\delta_\alpha - y^\delta\| = \tau\delta \tag{1.5.3}$$

for some $\tau > 1$ it gives the optimal convergence rate. But the main drawback of this method is that it does not provide optimal order $O(\delta^{\frac{2\nu}{2\nu+1}})$ under the assumption $\hat{x} \in R((T^*T)^\upsilon)$, for all $0 < \upsilon < 1$.

Reddy (2018) considered the Engl (1987) type discrepancy principle for choosing the regularization parameter α for weighted Tikhonov regularization method. Reddy (2018) considered the following discrepancy principles

(a) $G(\alpha, y^\delta) := \|\alpha((T^*T)^{\frac{\beta+1}{2}} + \alpha I)^{-1}(T^*T)^{\frac{\beta-1}{2}}T^*y^\delta\|^2 = \tau_1\frac{\delta^p}{\alpha^q}, \tau_1 > 0$

(b) $G_1(\alpha, y^\delta) := \|(T^*T)x^\delta_\alpha - T^*y^\delta\|^2 = \frac{\delta^p}{\alpha^q}\ p > 0, q > 0, \alpha > 0.$

Recently, Kanagaraj and George (2019), considered Schock-type discrepancy principle, namely

$$\|Tx_{\alpha,\beta}^{\delta} - y^{\delta}\| = \frac{\delta^p}{\alpha^q}$$

for choosing the regularization parameter α and compared the numerical results with the numerical results obtained using Morozovs discrepancy principle (1.5.3). In the rest of the chapters, we have extended the above mentioned work in the setting of Hilbert scales.

1.6 RESEARCH OBJECTIVES

Our central aim in this thesis is to study fractional regularization methods in the setting of Hilbert scales. The overall objectives can be summarized as follows:

1. To study the fractional Tikhonov regularization method in the setting of Hilbert scales.

2. To study the finite dimensional realization of fractional Tikhonov and fractional Lavrentiev regularization methods in Hilbert scales.

3. To introduce a new parameter choice strategy for the above methods and also compare it with the adaptive parameter choice strategy.

1.7 OUTLINE OF THE THESIS

The rest of the thesis is structured as follows.

In Chapter 2, we consider the fractional Tikhonov regularization method in Hilbert scales wherein we take the solution $x_{\alpha,\beta}^{s,\delta}$ of the operator equation

$$((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})x_{\alpha,\beta}^{s,\delta} = (T^*T)^{\frac{\beta}{2}} y^{\delta}$$

where $L : D(L) \subset X \longrightarrow X$ is a strictly positive definite, unbounded, densely defined, self-adjoint operator as an approximation for $\hat{x}$ and obtain the optimal order error estimate. We further introduce a new parameter choice strategy to choose the regularization parameter, i.e. α is chosen such that

$$\|\alpha^2(L^{-s}(T^*T)^{\frac{1+\beta}{2}}L^{-s}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s}=c\delta,$$

where $c>0$ is some constant. The adaptive paramether choice strategy is also used for selecting the regularization parameter and we observe that we get the optimal order error estimates in both cases. This is further examined through numerical examples.

In Chapter 3, we study the finite dimensional extention of the fractional Tikhonov regularization method in Hilbert scales. The parameter choice strategy discussed in Chapter 2 is further modified to suit the finite dimensional setting. We manage to prove that this method also gives us the optimal order error estimate while also making it easier to perform numerical calculations.

Chapter 4 deals with fractional Lavrentiev regularization method in the setting of Hilbert scales and its finite dimensional realization. We also study the analogous of the discrepancy principle considered by George and Nair in (George and Nair, 1993) for choosing the regularization parameter α. The efficiency of this method is verified through numerical experiments as well.

Chapter 5 gives the conclusion of the thesis and scope for future work.

Chapter 2

FRACTIONAL TIKHONOV REGULARIZATION METHOD IN HILBERT SCALES

2.1 INTRODUCTION

Finding a solution for the equation

$$Tx = y, \tag{2.1.1}$$

where $T : X \longrightarrow Y$ is a bounded linear operator between the Hilbert spaces X and Y is an important problem due to its wide applications. A typical example of T is, $T : L^2[a,b] \longrightarrow L^2[a,b]$ defined by

$$(Tx)(s) = \int_a^b k(s,t)x(t)dt, \;\; a \leq s \leq b.$$

where $k(s,t) \in L^2([a,b] \times [a,b])$. In practice, the available data is y^δ with

$$\|y - y^\delta\| \leq \delta. \tag{2.1.2}$$

Therefore, one has to deal with the equation

$$Tx = y^\delta$$

instead of (2.1.1). In Tikhonov regularization method (Groetsch (1977), Engl et al. (1996), Egger and Hofmann (2018)) the minimizer x_α^δ of the functional

$$J_\alpha(x) = \|Tx - y^\delta\|^2 + \alpha \, \|x\|^2, \tag{2.1.3}$$

is used as an approximation for the solution $\hat{x}$ (assumed to exist) of (2.1.1).

It is known that the solution of (2.1.3) over smoothens the solution $\hat{x}$ (Klann and Ramlau (2008)), to overcome this, fractional Tikhonov regularization method (Gerth et al. (2015), Hochstenbach and Reichel (2011), Klann and Ramlau (2008), Morigi et al. (2017)) was studied. In fractional Tikhonov regularization method, the minimizer $x_{\alpha,\beta}^{\delta}$ of the functional

$$J_{\alpha}^{\beta}(x) = \|Tx - y^{\delta}\|_{\beta}^{2} + \alpha \|x\|^{2}, \tag{2.1.4}$$

is taken as an approximation for $\hat{x}$. Here $\|x\|_{\beta} = \|(T\,T^{*})^{(\beta-1)/4}x\|$ for some parameter $0 \leq \beta \leq 1$ (see Gerth et al. (2015); Hochstenbach and Reichel (2011)).

Morigi et al. (2017), modified (2.1.4) by replacing the L^2-norm in (2.1.4) by $TV-$norm, i.e., the minimizer of the functional

$$J_{\alpha}^{\beta}(x) = \|Tx - y^{\delta}\|_{\beta}^{2} + \alpha \|x\|_{TV}^{2},$$

was considered as an approximation for $\hat{x}$. Klann and Ramlau (2008) considered

$$x_{\alpha,\gamma}^{\delta} = (T^{*}T + \alpha I)^{-\gamma}(T^{*}T)^{\gamma-1}T^{*}y^{\delta}$$

for some $\gamma > \frac{1}{2}$ as an approximation for $\hat{x}$ and refereed the scheme as fractional Tikhonov method. Note that for $\gamma = 1$, $x_{\alpha,\gamma}^{\delta}$ reduces to the minimizer of (2.1.4) and for $\gamma \neq 1, x_{\alpha,\gamma}^{\delta}$ is not a minimizer of functionals of the form $J_{\alpha}(x)$.

Note that in (2.1.4), $\alpha > 0$ is the regularization parameter and $\|x\|^2$ is the penalty term and the L^2 norm in the penalty over smooths, the regularized solution. Further, note that the minimizer of the functional $J_{\alpha}^{\beta}(x)$ in (2.1.4) can also be derived from the minimization problem

$$\min_{x \in X} \{\|Tx - y^{\delta}\|^{2} + \alpha\|(TT^{*})^{\frac{1-\beta}{4}}x\|^{2}\}$$

for $0 \leq \beta \leq 1$. The penalty term $\|(TT^{*})^{\frac{1-\beta}{4}}x\|^{2}$ reduces the over smoothing. The over smoothing of the Tikhonov regularization, was perhaps noticed first by Natterer (1984).

Observe that the minimizer of (2.1.4) satisfies the equation

$$((T^{*}T)^{1+\gamma} + \alpha I)x_{\alpha,\beta}^{\delta} = (T^{*}T)^{\gamma}T^{*}y^{\delta}, \tag{2.1.5}$$

where $\gamma = \frac{\beta-1}{2} < 0$ (Reddy (2018)).

It is observed that the fractional regularization method reduces the over smoothing occurring in the Tikhonov regularization, but the order of convergence is pessimistic. In order to overcome this, one can study Fractional Tikhonov Regularization Method(FTRM) in the setting of Hilbert scales, so as to obtain a better convergence rate as well as to reduce the over smoothing.

The goals of this Chapter are (1) to study the fractional Tikhonov regularization method in the setting of Hilbert scales and (2) to study a new discrepancy principle for FTRM in the setting of Hilbert scales.

The rest of the Chapter is organized as follows: Preliminaries are given in Section 2.2, the method and its convergence analysis are given in Section 2.3. A comparison between Standard Tikhonov and Fractional Tikhonov regularization method in Hilbert scales are given in Section 2.4, error bounds are given in Section 2.5 and the numerical examples are given in Section 2.6.

2.2 PRELIMINARIES

Let $\{X_s\}_{s\in\mathbb{R}}$ (cf. George and Nair (1997)) be the Hilbert scale given in the introduction. We assume throughout this Chapter that, the operator T satisfies:

$$b_1\|x\|_{-a} \le \|Tx\| \le b_2\|x\|_{-a},\ x \in X \tag{2.2.1}$$

for some $a > 0, b_1 > 0$ and $b_2 > 0$.

Let $f(t) := \min\{b_1^t, b_2^t\},\ g(t) := \max\{b_1^t, b_2^t\},\ t \in \mathbb{R}$ and $|t| \le 1$.

With the above notation, we have the following Proposition.

PROPOSITION 2.2.1. *(cf. Natterer (1984)[Proposition 1]) Let T satisfy (2.2.1). Then, for $|\nu| \le 1$,*

$$f(\nu)\|x\|_{-\nu a} \le \|(T^*T)^{\nu/2}x\| \le g(\nu)\|x\|_{-\nu a},\ x \in D((T^*T)^{\nu/2}).$$

For $0 \le \beta \le 1$, let $F(t) := \min\{f(\frac{1+\beta}{2})^t, g(\frac{1+\beta}{2})^t\}$, $G(t) := \max\{f(\frac{1+\beta}{2})^t, g(\frac{1+\beta}{2})^t\}$.

PROPOSITION 2.2.2. *Let T be a bounded linear operator satisfying (2.2.1). Then, the following holds:*

$$F(\nu)\|x\|_{-\frac{\nu}{2}((1+\beta)a+2s)} \leq \|(L^{-s}(T^*T)^{\frac{1+\beta}{2}}L^{-s})^{\nu/2}x\| \leq G(\nu)\|x\|_{-\frac{\nu}{2}((1+\beta)a+2s)},$$

$x \in D((L^{-s}(T^*T)^{\frac{1+\beta}{2}}L^{-s})^{\nu/2}), s > 0, 0 \leq \beta \leq 1, |\nu| \leq 1.$

Proof. By Proposition 2.2.1, with $\nu = \frac{1+\beta}{2}$, we obtain

$$f(\frac{1+\beta}{2})\|x\|_{-(\frac{1+\beta}{2})a} \leq \|T^*T^{\frac{1+\beta}{4}}x\| \leq g(\frac{1+\beta}{2})\|x\|_{-(\frac{1+\beta}{2})a},\ x \in D((T^*T)^{\frac{1+\beta}{4}}).$$

Now, the proof follows by taking first, $x = L^{-s}x$ in the above equation and then applying Proposition 2.2.1 for the operator $(T^*T)^{\frac{1+\beta}{4}}L^{-s}$.

□

2.3 FRACTIONAL TIKHONOV REGULARIZATION IN HILBERT SCALES

In this Section, we introduce the fractional Tikhonov regularization method for approximately solving the ill-posed operator equation (2.1.1). Let $x^s_{\alpha,\beta}$ be the minimizer of the functional

$$J^s_{\alpha,\beta}(x) = \|Tx - y\|_\beta + \alpha\|x\|^2_s, \quad \alpha > 0, \tag{2.3.1}$$

where $0 \leq \beta \leq 1$. Then, $x^s_{\alpha,\beta}$ satisfies the equation

$$((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})x^s_{\alpha,\beta} = (T^*T)^{\frac{\beta}{2}}y. \tag{2.3.2}$$

Note that, for $\beta = 1, s = 0$, (2.3.2) is Tikhonov regularization of (2.1.1).

Let

$$A_{s,\beta} := L^{-s}(T^*T)^{\frac{1+\beta}{2}}L^{-s}.$$

Then

$$x^s_{\alpha,\beta} = L^{-s}(A_{s,\beta} + \alpha I)^{-1}L^{-s}(T^*T)^{\frac{\beta}{2}}y \tag{2.3.3}$$

and let

$$x^{s,\delta}_{\alpha,\beta} = L^{-s}(A_{s,\beta} + \alpha I)^{-1}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta. \tag{2.3.4}$$

Furthermore, by spectral properties of the self-adjoint operator $A_{s,\beta}, s > 0, \beta \in [0,1]$, we have

$$\|(A_{s,\beta} + \alpha I)^{-1} A_{s,\beta}^{\mu}\| \leq \alpha^{\mu-1}, \alpha > 0,\ 0 \leq \mu \leq 1. \tag{2.3.5}$$

Next, we present the error estimate for $\|x_{\alpha,\beta}^{s} - x_{\alpha,\beta}^{s,\delta}\|$ and $\|\hat{x} - x_{\alpha,\beta}^{s}\|$ using the above notation and propositions.

LEMMA 2.3.1. *Let $x_{\alpha,\beta}^{s}, x_{\alpha,\beta}^{s,\delta}$ be as in (2.3.3) and (2.3.4), respectively. Let the assumptions in Proposition 2.2.1 and Proposition 2.2.2 hold. Then, for $0 \leq \beta \leq 1$,*

$$\|x_{\alpha,\beta}^{s} - x_{\alpha,\beta}^{s,\delta}\| \leq \varphi(s,a,\beta) \alpha^{\frac{-a}{(1+\beta)a+2s}} \delta,$$

where $\varphi(s,a,\beta) := \frac{G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{F\left(\frac{2s}{(1+\beta)a+2s}\right) f(-\beta)}$.

Proof.

By (2.3.3) and (2.3.4), we have

$$\begin{aligned} \|x_{\alpha,\beta}^{s,\delta} - x_{\alpha,\beta}^{s}\| &= \|L^{-s}(A_{s,\beta} + \alpha I)^{-1} L^{-s} (T^*T)^{\frac{\beta}{2}} (y^\delta - y)\| \\ &= \|(A_{s,\beta} + \alpha I)^{-1} L^{-s} (T^*T)^{\frac{\beta}{2}} (y^\delta - y)\|_{-s}. \end{aligned}$$

Therefore, by Proposition 2.2.2, with $\nu = \frac{2s}{(1+\beta)a+2s}$, and $x = (A_{s,\beta} + \alpha I)^{-1} L^{-s} (T^*T)^{\frac{\beta}{2}} (y^\delta - y)$ and (2.3.5) we obtain in turn that

$$\begin{aligned} &\|x_{\alpha,\beta}^{s,\delta} - x_{\alpha,\beta}^{s}\| \\ \leq\ & \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \|A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} L^{-s} (T^*T)^{\frac{\beta}{2}} (y^\delta - y)\| \\ =\ & \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \\ & \times \|A_{s,\beta}^{\frac{2s+\beta a}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} L^{-s} (T^*T)^{\frac{\beta}{2}} (y^\delta - y)\| \qquad (2.3.6) \\ \leq\ & \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \|A_{s,\beta}^{\frac{2s+\beta a}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1}\| \|A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} L^{-s} (T^*T)^{\frac{\beta}{2}} (y^\delta - y)\| \\ \leq\ & \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \alpha^{\frac{-a}{(1+\beta)a+2s}} \|A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} L^{-s} (T^*T)^{\frac{\beta}{2}} (y^\delta - y)\|. \end{aligned}$$

So the Lemma is proved, if we prove

$$\|A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} L^{-s}(T^*T)^{\frac{\beta}{2}}(y^\delta - y)\| \leq \frac{G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{f(-\beta)}\delta. \tag{2.3.7}$$

But this can be seen as follows; by taking $\nu = \frac{-2(\beta a+s)}{(1+\beta)a+2s}$, and $x = L^{-s}(T^*T)^{\frac{\beta}{2}}(y^\delta - y)$ in Proposition 2.2.2, we obtain

$$\begin{aligned}
\|A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} L^{-s}(T^*T)^{\frac{\beta}{2}}(y^\delta - y)\| &\leq G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\|L^{-s}(T^*T)^{\frac{\beta}{2}}(y^\delta - y)\|_{s+\beta a} \\
&= G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\|(T^*T)^{\frac{\beta}{2}}(y^\delta - y)\|_{\beta a} \\
&\leq \frac{G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{f(-\beta)}\|y^\delta - y\| \\
&\leq \frac{G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{f(-\beta)}\delta.
\end{aligned}$$

The last but one step follows again from Proposition 2.2.1, by taking $\nu = -\beta$.

□

We use the following assumption, to obtain an error estimate for $\|\hat{x} - x_{\alpha,\beta}^s\|$.

ASSUMPTION 2.3.2. *There exists some* $E > 0, 0 < t \leq \frac{1+\beta}{2}a + 2s$ *such that* $\hat{x} \in M_{t,E} = \{x \in X : \|x\|_t \leq E\}$.

LEMMA 2.3.3. *Let* $x_{\alpha,\beta}^s$ *be as in (2.3.3), Assumption 2.3.2 holds,* $0 \leq \beta \leq 1$. *Further let the assumptions in Proposition 2.2.1 and Proposition 2.2.2 hold. Then*

$$\|\hat{x} - x_{\alpha,\beta}^s\| \leq \psi_1(s, a, \beta, t)\alpha^{\frac{t}{(1+\beta)a+2s}},$$

where $\psi_1(s, a, \beta, t) := \frac{G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{2s}{(1+\beta)a+2s}\right)}E.$

Proof. By (2.3.3) and Assumption 2.3.2, we have in turn that

$$\begin{aligned}
\hat{x} - x_{\alpha,\beta}^s &= \hat{x} - ((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T^*T)^{\frac{\beta}{2}}y \\
&= \alpha((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}L^{2s}\hat{x} \\
&= \alpha L^{-s}(A_{s,\beta} + \alpha I)^{-1}L^s\hat{x},
\end{aligned}$$

that is

$$\|\hat{x} - x^s_{\alpha,\beta}\| = \alpha\|(A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}\|_{-s}.$$

So, by Proposition 2.2.2(by taking $\nu = \frac{2s}{(1+\beta)a+2s}$) and (2.3.5), we have

$$\begin{aligned}
\|\hat{x} - x^s_{\alpha,\beta}\| &\leq \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \|\alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}\| \\
&\leq \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \|\alpha A_{s,\beta}^{\frac{t}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1}\| \|A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}} L^s \hat{x}\| \\
&\leq \frac{G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \alpha^{\frac{t}{(1+\beta)a+2s}} \|\hat{x}\|_t \\
&\leq \frac{G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \alpha^{\frac{t}{(1+\beta)a+2s}} E.
\end{aligned}$$

□

Combining the Lemma 2.3.1 and Lemma 2.3.3, we obtain the following theorem.

THEOREM 2.3.4. *Let $x^s_{\alpha,\beta}, x^{s,\delta}_{\alpha,\beta}$ be as in (2.3.3) and (2.3.4), respectively, Assumption 2.3.2 holds. Further let the assumptions in Proposition 2.2.1, Proposition 2.2.2, Lemma 2.3.1 and Lemma 2.3.3 hold. Then*

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta}\| \leq \varphi(s,a,\beta)\alpha^{\frac{-a}{(1+\beta)a+2s}}\delta + \psi_1(s,a,\beta,t)\alpha^{\frac{t}{(1+\beta)a+2s}}.$$

In particular, if $\alpha := \alpha(s,a,\beta,t) = c_0\delta^{\frac{(1+\beta)a+2s}{t+a}}$ for some $c_0 > 0$, then

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta}\| \leq \eta(s,a,\beta,t)\delta^{\frac{t}{t+a}},$$

where $\eta(s,a,\beta,t) = \max\{\varphi(s,a,\beta)c_0^{\frac{-a}{(1+\beta)a+2s}}, \psi_1(s,a,\beta,t)c_0^{\frac{t}{(1+\beta)a+2s}}\}$.

□

REMARK 2.3.5. *Note that under the Assumption 2.3.2, (2.2.1) and (2.1.2), the optimal order of error estimate for Tikhonov regularization in Hilbert scale is $O(\delta^{\frac{t}{t+a}})$, $0 < t \leq 2s + a$ (see Engl et al. (1996); George and Nair (1997);*

Goldenshluger and Pereverzev (2000); Egger and Hofmann (2018); Jin (2000); Lu et al. (2010); Mathé and Pereverzev (2003); Natterer (1984); Neubauer (1988, 1992, 2000)). We have obtained the optimal order $O(\delta^{\frac{t}{t+a}})$, *for* $0 < t \leq \frac{1+\beta}{2}a + 2s$. *Even though,* $\frac{1+\beta}{2}a + 2s \leq a + 2s$, *the advantage of our method is that, it avoids the over smoothing of the solution.*

2.4 STANDARD TIKHONOV METHOD VS FTRM IN HILBERT SCALES

In this Section, we compare the filter factors (Hochstenbach et al. (2015)) of Tikhonov regularization method and fractional Tikhonov regularization method in the Hilbert scales. Recall (Lu et al. (2010), Natterer (1984), Neubauer (1988), Neubauer (1992), Neubauer (2000), Tautenhahn (1996), Tautenhahn (1998)) the Tikhonov regularized solution for (2.1.1) in Hilbert scales is given by

$$x_\alpha^{s,\delta} = L^{-s}(L^{-s}T^*TL^{-s} + \alpha I)^{-1}L^{-s}T^*y^\delta. \tag{2.4.1}$$

So, using Proposition 2.2.2 with $\beta = 1$, we have

$$\begin{aligned} \|x_\alpha^{s,\delta}\| &\leq \frac{1}{F(\frac{s}{s+a})}\|(A^*A)^{\frac{s}{2(s+a)}}(A^*A + \alpha I)^{-1}A^*y^\delta\| \\ &\leq \frac{1}{F(\frac{s}{s+a})}\|(A^*A)^{\frac{2s+a}{2(s+a)}}(A^*A + \alpha I)^{-1}y^\delta\|, \end{aligned}$$

where $A = TL^{-s}$ and hence

$$\|x_\alpha^{s,\delta}\|^2 \leq \frac{1}{F(\frac{s}{s+a})^2}\int_0^{\|A^*A\|}\left(\frac{\lambda^{\frac{2s+a}{2(s+a)}}}{\lambda + \alpha}\right)^2 d\langle E_\lambda y^\delta, y^\delta\rangle, \tag{2.4.2}$$

where $\{E_\lambda : 0 \leq \lambda \leq \|A^*A\|\}$ is the spectral family of A^*A. Similarly by taking $y = 0$ in (2.3.6), we have

$$\begin{aligned} \|x_{\alpha,\beta}^{s,\delta}\| \leq\ & \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \\ & \times\|A_{s,\beta}^{\frac{2s+\beta a}{(1+\beta)a+2s}}(A_{s,\beta} + \alpha I)^{-1}A_{s,\beta}^{\frac{-(s+\beta a)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta\| \end{aligned} \tag{2.4.3}$$

and hence

$$\|x^{s,\delta}_{\alpha,\beta}\|^2 \;\leq\; \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)^2}\int_0^{\|A_{s,\beta}\|}\left(\frac{\lambda^{\frac{2s+\beta a}{(1+\beta)a+2s}}}{\lambda+\alpha}\right)^2 \tag{2.4.4}$$
$$\times d\langle F_\lambda A_{s,\beta}^{\frac{-(s+\beta a)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta, A_{s,\beta}^{\frac{-(s+\beta a)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta\rangle,$$

where $\{F_\lambda : 0 \leq \lambda \leq \|A_{s,\beta}\|\}$ is the spectral family of $A_{s,\beta}$. Further, note that

$$d\langle F_\lambda A_{s,\beta}^{\frac{-(s+\beta a)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta, A_{s,\beta}^{\frac{-(s+\beta a)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta\rangle$$
$$\leq \left(\frac{G\left(\frac{-2(s+\beta a)}{(1+\beta)a+2s}\right)}{f(-\beta)}\right)^2\|y^\delta\|^2.$$

Therefore, the quality of the approximate solution $x^{s,\delta}_\alpha$ and $x^{s,\delta}_{\alpha,\beta}$ are depending on the integrands in (2.4.2) and (2.4.4), respectively. Let $\varphi_1(t) := \frac{t^{\frac{2s+a}{2(s+a)}}}{t+\alpha}$ and $\varphi_2(t) := \frac{t^{\frac{2s+\beta a}{(1+\beta)a+2s}}}{t+\alpha}$. We call the functions φ_1 and φ_2 the filter factors (Hochstenbach et al. (2015), Klann and Ramlau (2008)) associated with the standard Tikhonov regularization method in Hilbert scales and fractional Tikhonov regularization method in Hilbert scales, respectively. Fig:2.1, displays the filter function $t \longrightarrow \varphi_1(t)$ for standard Tikhonov regularization method in Hilbert scales. The Fig:2.2 – 2.4, displays the filter function $t \longrightarrow \varphi_2(t)$ for fractional Tikhonov regularization method in Hilbert scales for $\beta = 0.5, 0.7, 0.9$, respectively.

Note that, when the desired solution $\hat{x}$ is smooth, one would like the filter functions to satisfy

$$\lim_{t\longrightarrow 0}\varphi_1(t) = 0 \text{ and } \lim_{t\longrightarrow 0}\varphi_2(t) = 0.$$

We observed that (see Fig: 2.1 – 2.4) the filter function $\varphi_2(t)$ is smoother than the filter function $\varphi_1(t)$ near 0. So we expect the computed solution obtained by fractional Tikhonov regularization method in Hilbert scales approximates the desired solution $\hat{x}$ better than the standard Tikhonov regularization method in Hilbert scales.

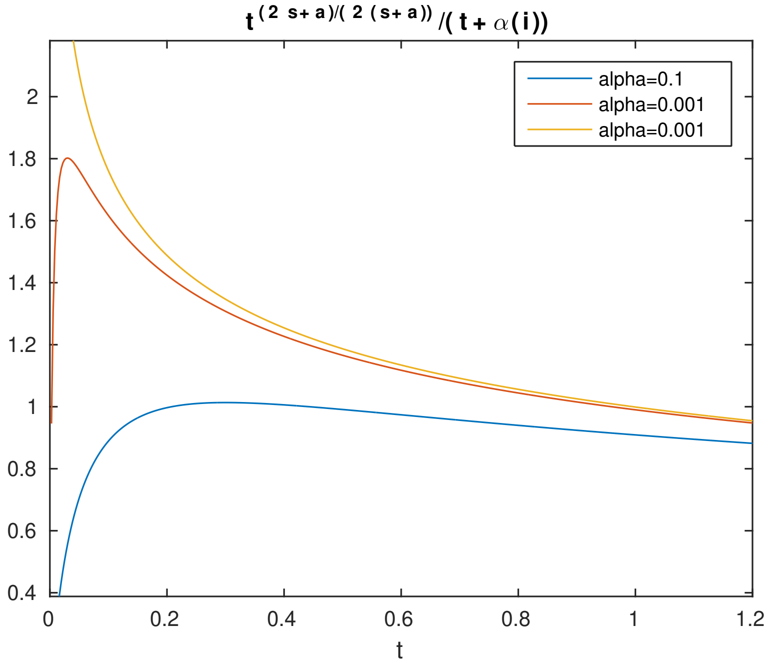

Figure 2.1: Filter function $\varphi_1(t)$ as a function of t.

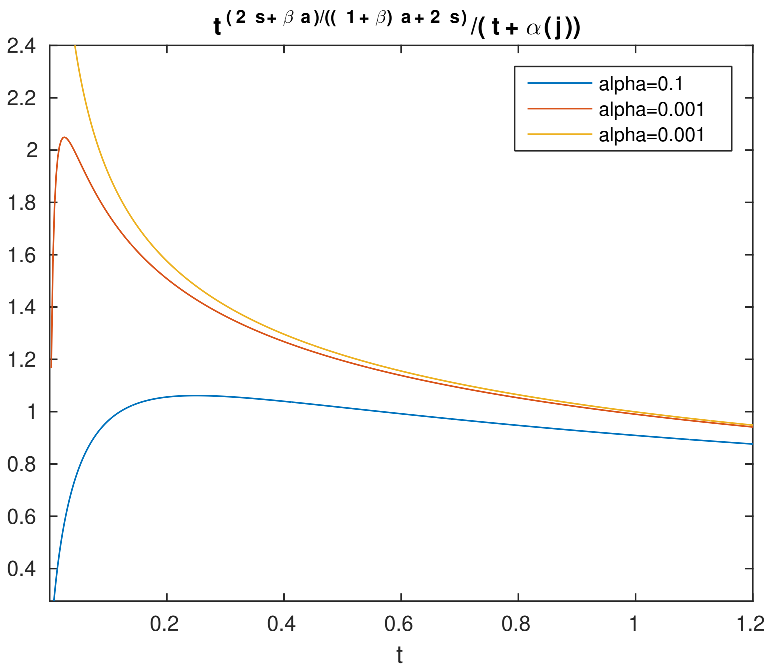

Figure 2.2: Filter function $\varphi_2(t)$ for $\beta = 0.5$.

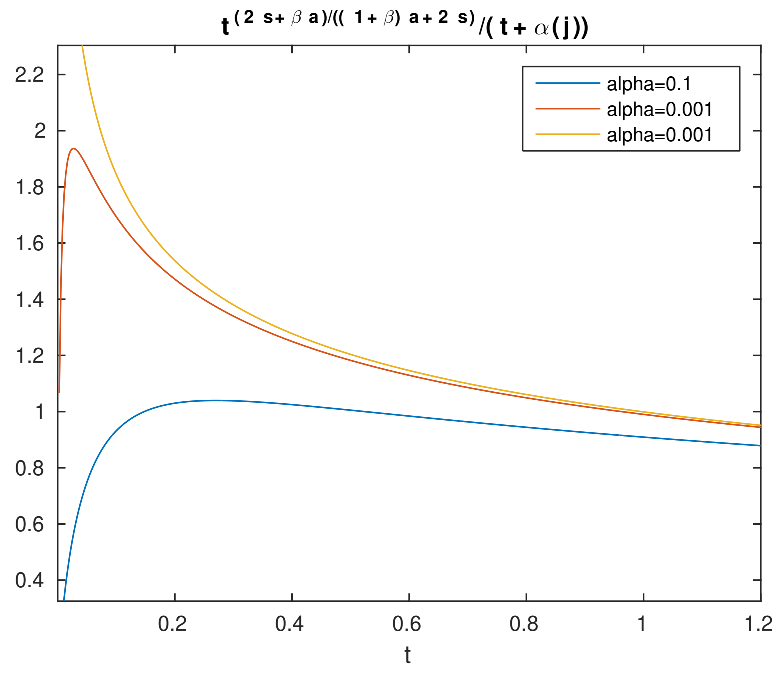

Figure 2.3: Filter function $\varphi_2(t)$ for $\beta = 0.7$.

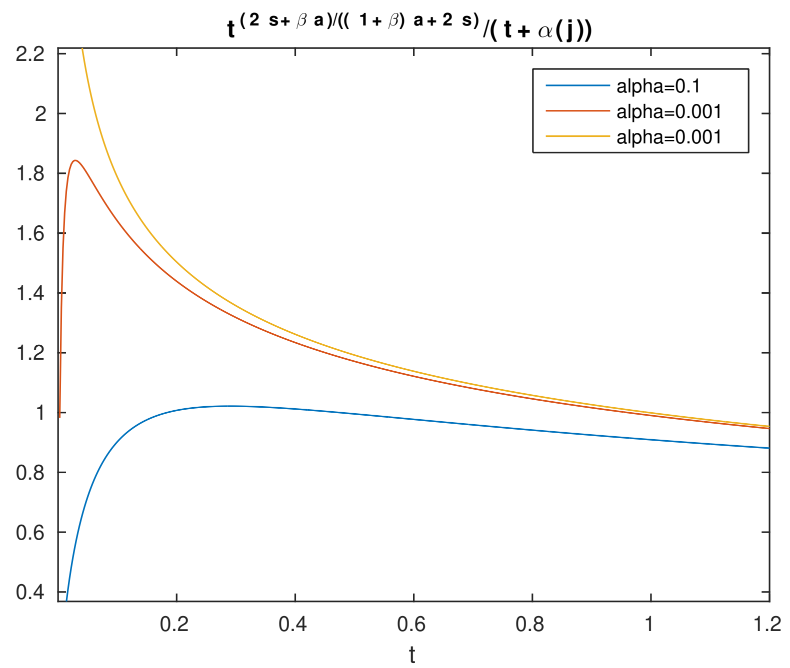

Figure 2.4: Filter function $\varphi_2(t)$ for $\beta = 0.9$.

2.5 DISCREPANCY PRINCIPLE

In this Section, we introduce a new discrepancy principle for choosing the regularization parameter α in FTRM. Let

$$\phi(\alpha, y^\delta) = \|\alpha^2 (A_{s,\beta} + \alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y^\delta\|_{\beta a+s} \ . \tag{2.5.1}$$

THEOREM 2.5.1. *For each non-zero y^δ, the function $\alpha \longrightarrow \phi(\alpha, y^\delta)$ for $\alpha > 0$, as defined in (2.5.1), is continuous and increasing. In addition*

$$\lim_{\alpha\to 0} \phi(\alpha, y^\delta) = 0, \lim_{\alpha\to\infty} \phi(\alpha, y^\delta) = \|(T^*T)^{\frac{\beta}{2}} y^\delta\|_{\beta a} \ . \tag{2.5.2}$$

Proof. Let $\{E_\lambda : 0 \leq \lambda \leq \|A_{s,\beta}\|\}$ be the spectral family of $A_{s,\beta}$. Then

$$\phi(\alpha, y^\delta)^2 \ = \ \int_0^{\|A_{s,\beta}\|} \left(\frac{\alpha}{\lambda+\alpha}\right)^4 d\langle E_\lambda L^{-s}(T^*T)^{\beta/2} y^\delta, L^{-s}(T^*T)^{\beta/2} y^\delta\rangle_{\beta a+s}.$$

Now since $\alpha \longrightarrow \left(\frac{\alpha}{\lambda+\alpha}\right)^4$ for $\lambda > 0$ is strictly increasing, $\lim_{\alpha\longrightarrow 0} \left(\frac{\alpha}{\lambda+\alpha}\right)^4 = 0$ and $\lim_{\alpha\longrightarrow\infty} \left(\frac{\alpha}{\lambda+\alpha}\right)^4 = 1$, by Dominated Convergence Theorem, we have

$$\lim_{\alpha\to 0} \phi(\alpha, y^\delta) = 0, \text{ and } \lim_{\alpha\to\infty} \phi(\alpha, y^\delta) = \|(T^*T)^{\frac{\beta}{2}} y^\delta\|_{\beta a} \ . \tag{2.5.3}$$

□

THEOREM 2.5.2. *Suppose (2.1.2) holds and*

$$\|(T^*T)^{\frac{\beta}{2}} y^\delta\|_{\beta a} \geq c\delta > 0 \tag{2.5.4}$$

for some $c > 0$. Then there exists a unique $\alpha = \alpha(\delta)$ satisfying

$$\phi(\alpha, y^\delta) = c\delta \tag{2.5.5}$$

Proof. Follows from Intermediate Value Theorem and Theorem 2.5.1.

□

REMARK 2.5.3. *Note that, by (2.1.2) and Proposition 2.2.2, we have*

$$\|y^\delta\| = \|(T^*T)^{\frac{-\beta}{2}} (T^*T)^{\frac{\beta}{2}} y^\delta\| \leq g(-\beta)\|(T^*T)^{\frac{\beta}{2}} y^\delta\|_{\beta a}.$$

Now, since $\|y^\delta\| \geq \|y\| - \delta$, *if*

$$\delta \leq \frac{\|y\|}{cg(-\beta)+1}$$

then δ *satisfies (2.5.4).*

LEMMA 2.5.4. *Suppose Assumption 2.3.2 holds and* $\alpha := \alpha(\delta) > 0$ *is the unique solution of (2.5.5) where* $c > \frac{G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)f(-\beta)}$. *Then,*

$$\alpha \geq c_{\beta,a,s}\delta^{\frac{(1+\beta)a+2s}{a+t}}, \tag{2.5.6}$$

where $c_{\beta,a,s} = \left[c - \frac{G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)f(-\beta)}\right] \Big/ \left[\frac{G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)E}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}\right]$.

Proof. Note that, from Proposition 2.2.1, Proposition 2.2.2 and (2.5.5), we have

$$\begin{aligned}
\phi(\alpha, y^\delta) &\leq \frac{1}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta}+\alpha I)^{-2}A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta\| \\
&\leq \frac{\alpha^2}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\|A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}(y^\delta - y)\| \\
&\quad + \frac{\alpha^2}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\|A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}T\hat{x}\| \\
&\leq \frac{G(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\|L^{-s}(T^*T)^{\frac{\beta}{2}}(y^\delta - y)\|_{\beta a+s} \\
&\quad + \frac{\alpha^2}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\|(A_{s,\beta}+\alpha I)^{-2}A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{(\beta+1)/2}L^{-s}L^s\hat{x}\| \\
&\leq \frac{G(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\|(T^*T)^{\frac{\beta}{2}}(y^\delta - y)\|_{\beta a} \\
&\quad + \frac{\alpha^2}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\|(A_{s,\beta}+\alpha I)^{-2}A_{s,\beta}^{\frac{a+t}{(1+\beta)a+2s}}A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}}L^s\hat{x}\| \\
&\leq \frac{G(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})f(-\beta)}\|y^\delta - y\| \\
&\quad + \frac{G(\frac{2(s-t)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta}+\alpha I)^{-2}A_{s,\beta}^{\frac{a+t}{(1+\beta)a+2s}}\|\|L^s\hat{x}\|_{t-s}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{G(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})f(-\beta)}\|y^\delta - y\| \\
&\quad + \frac{G(\frac{2(s-t)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta}+\alpha I)^{-1}A_{s,\beta}^{\frac{a+t}{2[(1+\beta)a+2s]}}\|^2\|\hat{x}\|_t \\
&\leq \frac{G(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})f(-\beta)}\delta + \frac{G(\frac{2(s-t)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^{\frac{a+t}{(1+\beta)a+2s}}E.
\end{aligned}
$$

Thus

$$
[c - \frac{G(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})f(-\beta)}]\delta \leq \frac{G(\frac{2(s-t)}{(1+\beta)a+2s})}{F(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^{\frac{a+t}{(1+\beta)a+2s}}E
$$

which implies that

$$
\alpha \geq c_{\beta,a,s}\delta^{\frac{(1+\beta)a+2s}{a+t}}. \tag{2.5.7}
$$

□

THEOREM 2.5.5. *Under the assumptions in Lemma 2.5.4,*

$$
\|\hat{x} - x_{\alpha,\beta}^{s,\delta}\| = O(\delta^{\frac{t}{t+a}}) \tag{2.5.8}
$$

Proof. As in the proof of Lemma 2.3.3, we have

$$
\hat{x} - x_{\alpha,\beta}^s = \alpha L^{-s}(A_{s,\beta}+\alpha I)^{-1}L^s\hat{x}. \tag{2.5.9}
$$

So by Proposition 2.2.2 we have

$$
\|\hat{x} - x_{\alpha,\beta}^s\| \leq \frac{1}{F(\frac{2s}{(1+\beta)a+2s})}\|\alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-1}L^s\hat{x}\|.
$$

We make use of the following moment inequality

$$
\|B^u z\| \leq \|B^v z\|^{u/v}\|z\|^{1-u/v}, 0 \leq u \leq v \tag{2.5.10}
$$

where B is a positive self-adjoint operator, to obtain an estimate for $\|\alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-1}L^s\hat{x}\|$.

Let $u = \frac{t}{a}, v = 1 + \frac{t}{a}, B = \alpha A_{s,\beta}^{\frac{a}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-1}$ and $z = \alpha^{1-\frac{t}{a}}(A_{s,\beta} + \alpha I)^{-1+\frac{t}{a}}A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}}L^s\hat{x}$.

Then $B^u z = \alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-1}L^s\hat{x}$. Also, from (2.5.10) we have

$$\begin{aligned}\|B^u z\| &\leq \|B^{1+u}z\|^{\frac{t}{t+a}}\|z\|^{\frac{a}{t+a}}\\ &= \|\alpha^2(A_{s,\beta}+\alpha I)^{-2}A_{s,\beta}^{\frac{s+a}{(1+\beta)a+2s}}L^s\hat{x}\|^{\frac{t}{t+a}}\\ &\quad \times\|\alpha^{1-\frac{t}{a}}(A_{s,\beta}+\alpha I)^{-1+\frac{t}{a}}A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}}L^s\hat{x}\|^{\frac{a}{t+a}}. \end{aligned} \tag{2.5.11}$$

From Proposition 2.2.1, Proposition 2.2.2 and Theorem 2.5.2, we have

$$\begin{aligned}\|B^{1+\frac{t}{a}}z\| &= \|\alpha^2(A_{s,\beta}+\alpha I)^{-2}A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}A_{s,\beta}L^s\hat{x}\|\\ &\leq G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\|\alpha^2(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}(T^*T)^{\frac{1}{2}}\hat{x}\|_{\beta a+s}\\ &= G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\|\alpha^2(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}(y-y^\delta+y^\delta)\|_{\beta a+s}\\ &\leq G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\left[\|L^{-s}(T^*T)^{\frac{\beta}{2}}(y-y^\delta)\|_{\beta a+s}+\phi(\alpha,y^\delta)\right]\\ &\leq G\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\left[\frac{1}{f(-\beta)}\|y-y^\delta\|+\phi(\alpha,y^\delta)\right]\\ &\leq \left[\frac{G(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{f(-\beta)}+G(\frac{-2(\beta a+s)}{(1+\beta)a+2s})c\right]\delta. \end{aligned}\tag{2.5.12}$$

Further, we have

$$\begin{aligned}\|z\| &\leq \|A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}}L^s\hat{x}\|\\ &\leq G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)\|L^s\hat{x}\|_{t-s}\\ &\leq G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)E, \end{aligned}\tag{2.5.13}$$

so by (2.5.11), (2.5.12) and (2.5.13), we have

$$\|\hat{x}-x_{\alpha,\beta}^s\| = O(\delta^{\frac{t}{t+a}}). \tag{2.5.14}$$

Notice that in (2.5.7) we have $\alpha \geq c_{\beta,a,s}\delta^{\frac{(1+\beta)a+2s}{a+t}}$ which implies that

$$\begin{aligned}\frac{1}{\alpha} &\leq \frac{1}{c_{\beta,a,s}\delta^{\frac{(1+\beta)a+2s}{a+t}}},\\ \frac{\delta}{\alpha^{\frac{a}{(1+\beta)a+2s}}} &\leq \frac{\delta}{c_{\beta,a,s}\delta^{\frac{a}{a+t}}}\\ &= \frac{1}{c_{\beta,a,s}}\delta^{\frac{t}{t+a}}. \end{aligned}\tag{2.5.15}$$

Therefore, by Lemma 2.3.1, (2.5.14) and (2.5.15), we have

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta}\| = O(\delta^{\frac{t}{t+a}}).$$

□

2.5.1 ADAPTIVE PARAMETER CHOICE STRATEGY AND ITS IMPLEMENTATION

By Theorem 2.3.4, one can write

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta}\| \leq C(\alpha^{\frac{-a}{(1+\beta)a+2s}}\delta + \alpha^{\frac{t}{(1+\beta)a+2s}}), \tag{2.5.16}$$

where

$$C = \max\{\varphi_(s, a, \beta), \psi_1(s, a, \beta, t)\}. \tag{2.5.17}$$

We, shall now briefly discuss the adaptive parameter choice strategy, introduced by Pereverzev and Schock (2005). This strategy involves the following steps.

Choose $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i\alpha_0$ where $\mu > 1$ and $\alpha_0 = (\delta)^{1+\beta+\frac{2s}{a}}$.

Let

$$l := \max\left\{i : \alpha_i^{\frac{t+a}{(1+\beta)a+2s}} \leq \delta\right\} < N \quad \text{and} \tag{2.5.18}$$

$$k := \max\left\{i : \|x^{s,\delta}_{\alpha_i,\beta} - x^{s,\delta}_{\alpha_j,\beta}\| \leq 4C\alpha_j^{\frac{-a}{(1+\beta)a+2s}}\delta\right\}, j = 0, 1, 2, \cdots, i-1. \tag{2.5.19}$$

THEOREM 2.5.6. *Assume that there exists $i \in \{0, 1, \cdots, N\}$ such that $\alpha_i^{\frac{t}{(1+\beta)a+2s}} \leq \frac{\delta}{\alpha_i^{\frac{a}{(1+\beta)a+2s}}}$ and let l and k be as in (2.5.18) and (2.5.19), respectively. If assumptions of Theorem 2.3.4 are fulfilled, then $l \leq k$ and*

$$\|\hat{x} - x^{s,\delta}_{\alpha_k,\beta}\| \leq 6C\mu(\delta)^{\frac{t}{t+a}},$$

where C is as in (2.5.17).

Proof. To prove $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \ldots N\}$, $\alpha_i^{\frac{t}{(1+\beta)a+2s}} \leq \frac{\delta}{\alpha_i^{\frac{a}{(1+\beta)a+2s}}} \Longrightarrow \|x^{s,\delta}_{\alpha_i,\beta} - x^{s,\delta}_{\alpha_j,\beta}\| \leq 4\,C\,\frac{\delta}{\alpha_j^{\frac{a}{(1+\beta)a+2s}}}$, $\forall\, j = 0, 1, 2, \ldots i-1$.

For $j < i$, we have

$$\begin{aligned}
\| x^{s,\delta}_{\alpha_i,\beta} - x^{s,\delta}_{\alpha_j,\beta} \| &\leq \| x^{s,\delta}_{\alpha_i,\beta} - \hat{x} \| + \| \hat{x} - x^{s,\delta}_{\alpha_j,\beta} \| \\
&\leq C \left(\alpha_i^{\frac{t}{(1+\beta)a+2s}} + \frac{\delta}{\alpha_i^{\frac{a}{(1+\beta)a+2s}}} \right) + C \left(\alpha_j^{\frac{t}{(1+\beta)a+2s}} + \frac{\delta}{\alpha_j^{\frac{a}{(1+\beta)a+2s}}} \right) \\
&\leq 2\,C\,\alpha_i^{\frac{t}{(1+\beta)a+2s}} + 2\,C\,\frac{\delta}{\alpha_j^{\frac{a}{(1+\beta)a+2s}}} \\
&\leq 4\,C\,\frac{\delta}{\alpha_j^{\frac{t}{(1+\beta)a+2s}}}.
\end{aligned}$$

Thus the relation $l \leq k$ is proved. Further note that

$$\| \hat{x} - x^{s,\delta}_{\alpha_k,\beta} \| \leq \| \hat{x} - x^{s,\delta}_{\alpha_l,\beta} \| + \| x^{s,\delta}_{\alpha_l,\beta} - x^{s,\delta}_{\alpha_k,\beta} \|$$

where,

$$\| \hat{x} - x^{s,\delta}_{\alpha_l,\beta} \| \leq C \left(\alpha_l^{\frac{t}{(1+\beta)a+2s}} + \frac{\delta}{\alpha_l^{\frac{a}{(1+\beta)a+2s}}} \right) \leq 2\,C\,\frac{\delta}{\alpha_l^{\frac{a}{(1+\beta)a+2s}}}.$$

Now since $l \leq k$, we have

$$\| x^{s,\delta}_{\alpha_k,\beta} - x^{s,\delta}_{\alpha_l,\beta} \| \leq 4\,C\,\frac{\delta}{\alpha_l^{\frac{a}{(1+\beta)a+2s}}}.$$

Hence

$$\| \hat{x} - x^{s,\delta}_{\alpha_k,\beta} \| \leq 6\,C\,\frac{\delta}{\alpha_l^{\frac{a}{(1+\beta)a+2s}}}.$$

Let $\alpha_\delta^{\frac{t+a}{(1+\beta)a+2s}} = \delta$. Then, $\alpha_l \leq \alpha_\delta \leq \alpha_{l+1}$, hence $\alpha_\delta^{\frac{t+a}{(1+\beta)a+2s}} = \delta \leq \alpha_{l+1}^{\frac{t+a}{(1+\beta)a+2s}} \leq \mu^{\frac{t+a}{(1+\beta)a+2s}} \alpha_l{}^{\frac{t+a}{(1+\beta)a+2s}}$, it follows that

$$\frac{\delta}{\alpha_\delta^{\frac{a}{(1+\beta)a+2s}}} \leq \frac{\delta}{\alpha_l^{\frac{a}{(1+\beta)a+2s}}} \leq \mu^{\frac{t+a}{(1+\beta)a+2s}} \alpha_l^{\frac{t}{(1+\beta)a+2s}} \leq \mu^{\frac{t+a}{(1+\beta)a+2s}} \alpha_\delta{}^{\frac{t}{(1+\beta)a+2s}} \leq \mu^{\frac{t+a}{(1+\beta)a+2s}} (\delta)^{\frac{t}{t+a}}.$$

$\square$

2.5.2 IMPLEMENTATION OF ADAPTIVE CHOICE RULE

The choice of the regularization parameter stated in Theorem 2.5.6 has the following steps:

- Choose $\alpha_0 > 0$ such that $(\delta_0)^{1+\beta+\frac{2s}{a}} < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0, i = 0, 1, 2, \cdots, N$.

Algorithm

1. Set $i = 0$.
2. Solve $x_i := x^{s,\delta}_{\alpha_i,\beta}$ by using the iteration (2.3.4).
3. If $\|x_i - x_j\| > 4C\frac{\delta}{\alpha_j}, j < i$, then take $k = i - 1$ and return x_k.
4. Else set $i = i + 1$ and go to 2.

2.6 NUMERICAL EXAMPLES

In this section we consider four examples in the Hilbert scales generated by the linear operator $L : H^2 \cap H^1_0[0, 1] \subset L^2[0, 1] \mapsto L^2[0, 1]$ by $Lx = -x''$. Note that L is densely defined, self-adjoint and positive definite (Jin (2000)) and the Hilbert scale $\{\mathcal{X}\}_s$ generated by L is given by

$$\mathcal{X}_s = \{x \in H^s[0, 1] : x^{(2l)}(0) = x^{(2l)}(1) = 0, l = 0, 1, \ldots [\frac{s}{2} - \frac{1}{4}]\} \qquad (2.6.1)$$

for any $s \in \mathbb{R}$, where $H^s[0, 1]$ is the usual Sobolev space and $\|x\|_s = \int_0^1 |x^{(s)}(t)|dt$ for all $s = 0, 1, 2, \ldots$. We have taken $s = a = 2$ in our computation.

The discrete version of the operator T in the first four examples are taken from the Regularization Toolbox by Hansen (2007). We use the Newton's method to solve the nonlinear equations (2.5.5) for α with different values β, δ. Relative error $E_{\alpha,\beta} := \left(\frac{\|\hat{x}-x^{s,\delta}_{\alpha,\beta}\|}{\|\hat{x}\|}\right)$, α and the size of the mesh n are presented in the tables for different values of β. We have introduced the random noise level $\delta = 0.01$ and 0.001 in the exact data.

EXAMPLE 2.6.1. *(Baker (1977)[Fox and Goodwin]) Let*

$$[Tx](s) := \int_0^1 (s^2 + t^2)^{\frac{1}{2}} x(t)dt = y(s), \ -\pi \le s \le \pi, \tag{2.6.2}$$

where $y(s) = \frac{1}{3}[(1+s^2)^{\frac{3}{2}} - s^3]$*. The solution* $\hat{x}$ *is given by* $\hat{x}(t) = t$*. Relative error and* α *values for different values of* β *and* δ *are given in Table 2.1. Exact data and noise data are given in Fig:2.5 and Fig:2.6 - 2.8 for various values of* β*.*

Table 2.1: Relative errors for Fox and Goodwin.

β		$n = 300$		$n = 500$		$n = 700$	
		$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$
	α	1.165426e-03	8.877524e-04	1.032442e-03	8.166164e-04	9.565734e-04	7.746511e-04
0.9	$E_{\alpha,\beta}$	7.048677e-02	6.424199e-02	6.436813e-02	6.656812e-02	7.832833e-02	6.682691e-02
	α	1.180655e-03	8.947242e-04	1.045896e-03	8.232822e-04	9.689167e-04	7.810231e-04
0.8	$E_{\alpha,\beta}$	6.549125e-02	5.877244e-02	5.884892e-02	6.102979e-02	7.109961e-02	6.101624e-02
	α	1.191953e-03	8.996208e-04	1.056228e-03	8.282967e-04	9.786773e-04	7.859925e-04
0.7	$E_{\alpha,\beta}$	6.325134e-02	5.426629e-02	5.522432e-02	5.663493e-02	6.459783e-02	5.631811e-02
	α	1.199959e-03	9.024373e-04	1.063518e-03	8.315022e-04	9.856851e-04	7.893331e-04
0.6	$E_{\alpha,\beta}$	6.500898e-02	5.043934e-02	5.349323e-02	5.323522e-02	5.861911e-02	5.255965e-02
	α	1.206698e-03	9.036926e-04	1.068838e-03	8.331773e-04	9.905773e-04	7.911876e-04
0.5	$E_{\alpha,\beta}$	7.286383e-02	4.690518e-02	5.361898e-02	5.064374e-02	5.287708e-02	4.952055e-02
	α	1.215665e-03	9.044719e-04	1.074223e-03	8.341338e-04	9.949262e-04	7.921900e-04
0.4	$E_{\alpha,\beta}$	9.056197e-02	4.321000e-02	5.635061e-02	4.870151e-02	4.723531e-02	4.701206e-02
	α	1.232036e-03	9.064131e-04	1.082293e-03	8.357359e-04	1.001165e-03	7.935109e-04
0.3	$E_{\alpha,\beta}$	1.255151e-01	3.882410e-02	6.653837e-02	4.737038e-02	4.278472e-02	4.499207e-02
	α	1.263494e-03	9.114670e-04	1.095076e-03	8.397204e-04	1.012156e-03	7.967197e-04
0.2	$E_{\alpha,\beta}$	1.956613e-01	3.300298e-02	1.019581e-01	4.683095e-02	4.796712e-02	4.363088e-02
	α	1.320953e-03	9.213254e-04	1.110233e-03	8.476533e-04	1.029293e-03	8.033514e-04
0.1	$E_{\alpha,\beta}$	3.742901e-01	3.434926e-02	2.367990e-01	5.041605e-02	1.548342e-01	4.505147e-02

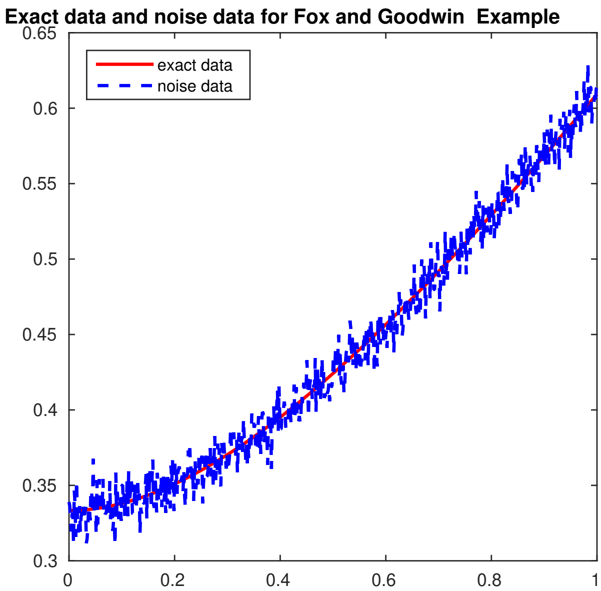

Figure 2.5: Exact data and noise data for $\delta = 0.01$ and $n = 700$ for Fox and Goodwin.

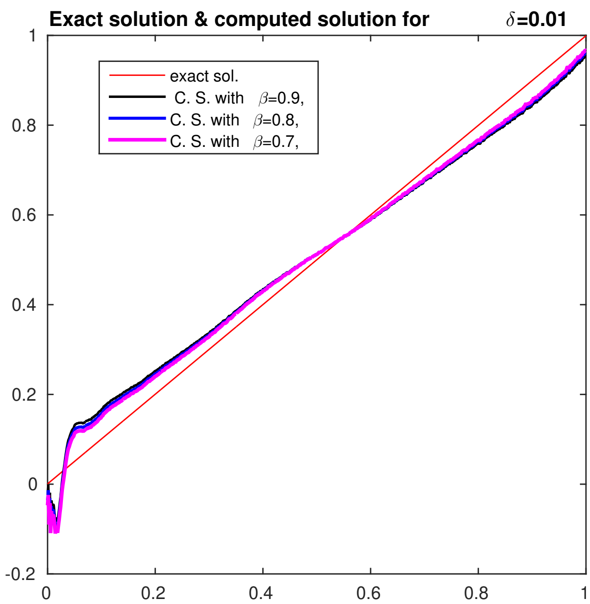

Figure 2.6: Solutions with $\delta = 0.01$ and $n = 700$ for Fox and Goodwin.

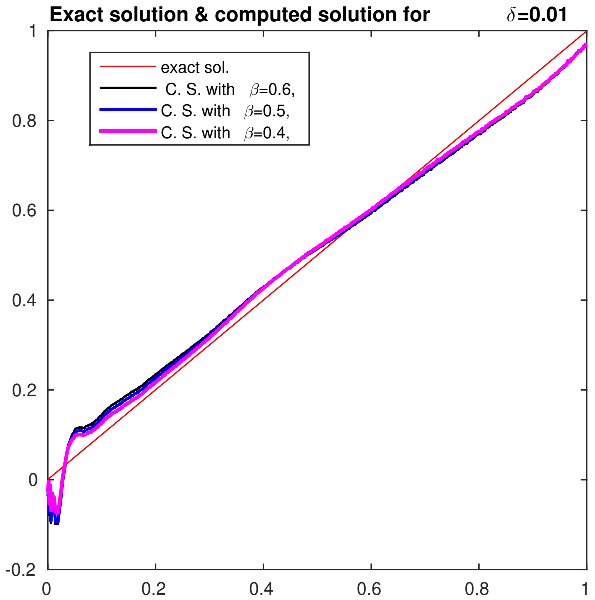

Figure 2.7: Solutions with $\delta = 0.01$ and $n = 700$ for Fox and Goodwin.

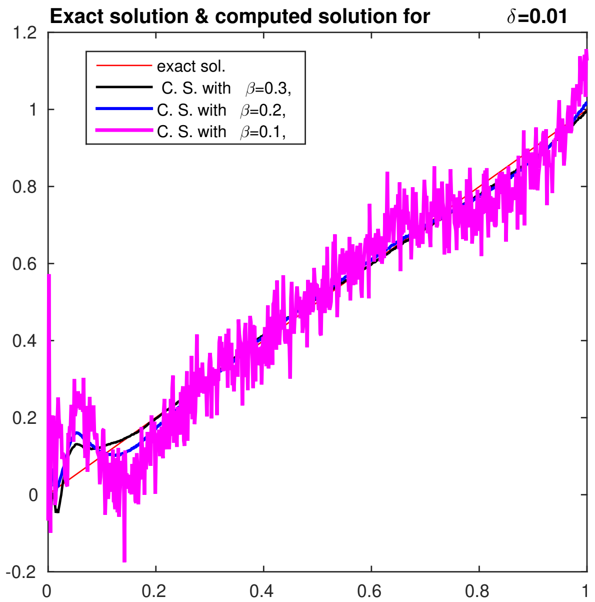

Figure 2.8: Solutions with $\delta = 0.01$ and $n = 700$ for Fox and Goodwin.

EXAMPLE 2.6.2. *(Carasso (1982)[Heat])*

Let

$$[Tx](s) := \int_0^1 k(s,t)x(t)dt = y(s), \ -\pi \leq s \leq \pi, \tag{2.6.3}$$

where, kernel is $K(s,t) = k(s-t)$ *with* $k(t) = \frac{t^{-\frac{3}{2}}}{2\sqrt{\pi}} e^{-\frac{1}{4t}}$.

An exact soltuion is constructed, and then the right-hand side y *is produced as* $y = Tx$. *Relative error and* α *values for different values of* β *and* δ *are given in Table 2.2. Fig:2.9 and Fig:2.10 - 2.12 displays the exact solution and computed solution for various values of* β.

Table 2.2: Relative errors for Heat example.

β		$n = 300$		$n = 500$		$n = 700$	
		$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$
	α	8.644940e-03	1.833503e-03	6.226313e-03	1.603395e-03	5.500000e-03	1.044832e-03
0.9	$E_{\alpha,\beta}$	7.089718e-01	5.828936e-01	6.887321e-01	5.850828e-01	6.930077e-01	5.312609e-01
	α	8.486190e-03	1.403682e-03	6.328550e-03	1.590334e-03	5.500000e-03	1.035226e-03
0.8	$E_{\alpha,\beta}$	6.870727e-01	5.232145e-01	6.772946e-01	5.556581e-01	6.695459e-01	4.935462e-01
	α	8.359327e-03	1.373818e-03	6.176976e-03	1.162224e-03	5.500000e-03	1.021165e-03
0.7	$E_{\alpha,\beta}$	6.646536e-01	4.834902e-01	6.472383e-01	4.841653e-01	6.467766e-01	4.473462e-01
	α	8.091878e-03	1.339788e-03	6.176017e-03	1.127667e-03	5.500000e-03	1.014329e-03
0.6	$E_{\alpha,\beta}$	6.360044e-01	4.353438e-01	6.333816e-01	4.379676e-01	6.163935e-01	4.041838e-01
	α	7.584882e-03	1.311958e-03	5.500000e-03	1.097198e-03	5.500000e-03	1.005180e-03
0.5	$E_{\alpha,\beta}$	5.988444e-01	3.889747e-01	5.792910e-01	3.853556e-01	5.856003e-01	3.567849e-01
	α	6.955816e-03	1.678801e-03	5.500000e-03	1.471606e-03	5.500000e-03	1.400771e-03
0.4	$E_{\alpha,\beta}$	5.567809e-01	3.701234e-01	5.687672e-01	3.787374e-01	5.465494e-01	3.529270e-01
	α	5.500000e-03	1.649122e-03	5.500000e-03	1.450032e-03	4.930524e-03	1.377632e-03
0.3	$E_{\alpha,\beta}$	5.315133e-01	3.184600e-01	5.386211e-01	3.219907e-01	4.785560e-01	2.990455e-01
	α	5.017099e-03	1.640921e-03	5.089394e-03	1.438126e-03	4.125522e-03	1.361925e-03
0.2	$E_{\alpha,\beta}$	5.548802e-01	2.729102e-01	5.166957e-01	2.729213e-01	4.360582e-01	2.523985e-01
	α	1.717764e-03	1.241515e-03	1.712944e-03	1.442362e-03	1.512160e-03	1.349326e-03
0.1	$E_{\alpha,\beta}$	1.919622e+00	2.883435e-01	1.310507e+00	2.644205e-01	1.287809e+00	2.335337e-01

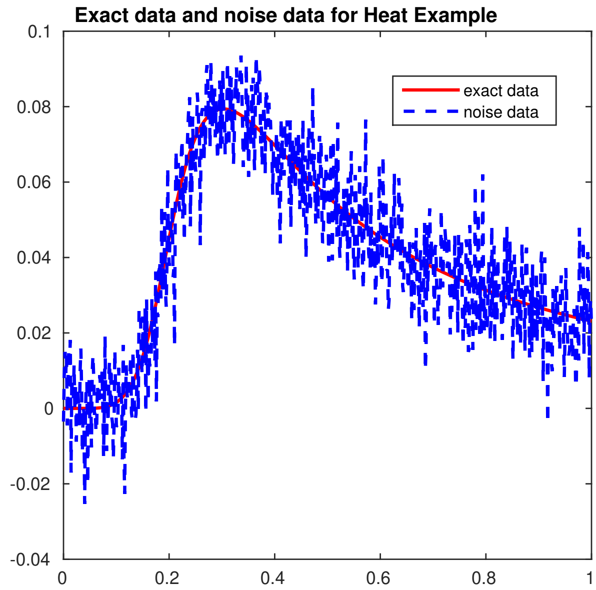

Figure 2.9: Exact data and noise data for $\delta = 0.01$ and $n = 700$ for the Heat example.

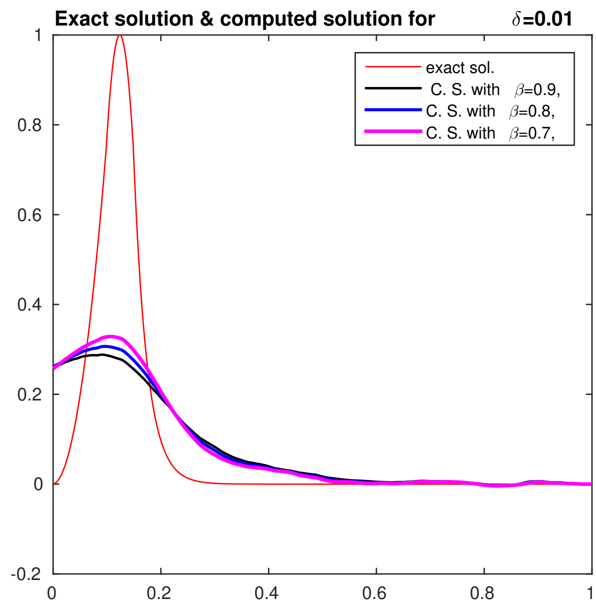

Figure 2.10: Solutions with $\delta = 0.01$ and $n = 700$ for the Heat example.

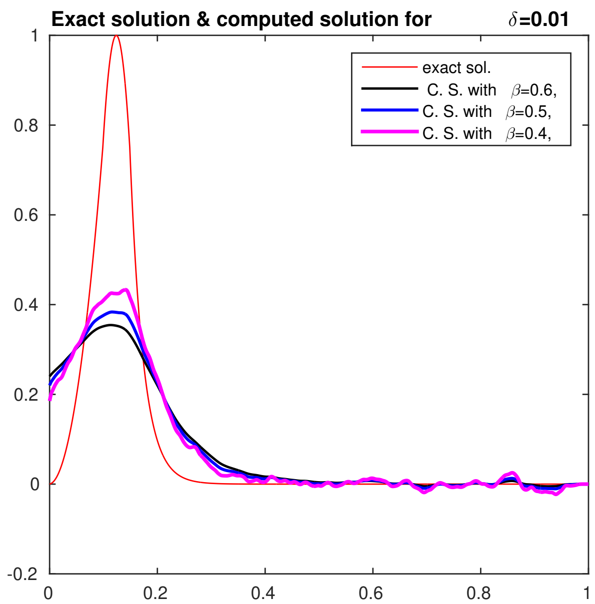

Figure 2.11: Solutions with $\delta = 0.01$ *and* $n = 700$ *for the Heat example.*

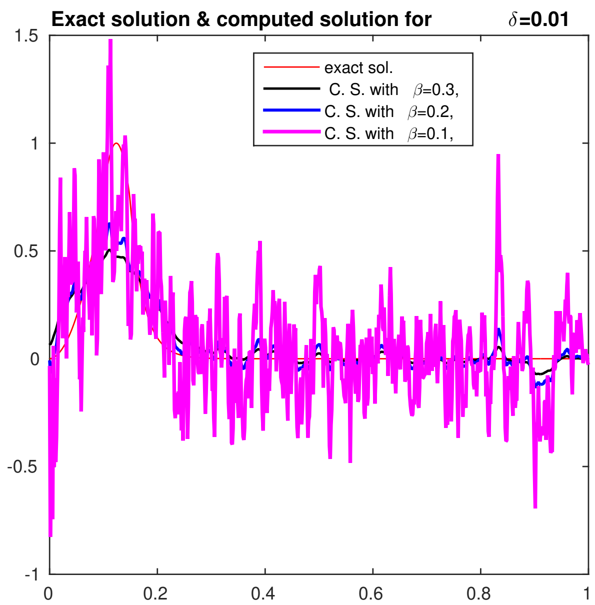

Figure 2.12: Solutions with $\delta = 0.01$ *and* $n = 700$ *for the Heat example*

EXAMPLE 2.6.3. *(Baart (1982)) Let*

$$[Tx](s) := \int_0^\pi k(s,t)x(t)dt = y(s), \;\; 0 \le s \le \frac{\pi}{2} \tag{2.6.4}$$

where $k(s,t) = \exp(s * \cos(t)), y(s) = 2 * \sinh(s)/s$. *The solution* $\hat{x}$ *is given by* $\hat{x}(t) = \sin(t)$. *Relative error and* α *values for different values of* β *and* δ *are given in Table 2.3. Exact data and noise data are given in Fig:2.13 and Fig:2.14 - 2.16 for various values of* β.

Table 2.3: Relative errors for Baart example.

β		$n = 300$		$n = 500$		$n = 700$	
		$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$
	α	1.253837e-03	9.327035e-04	1.196487e-03	8.642081e-04	1.167442e-03	8.235553e-04
0.9	$E_{\alpha,\beta}$	2.563487e-01	2.412385e-01	2.139743e-01	2.310873e-01	2.432770e-01	2.395962e-01
	α	1.321123e-03	9.431709e-04	1.266124e-03	8.742179e-04	1.239594e-03	8.332588e-04
0.8	$E_{\alpha,\beta}$	2.500173e-01	2.274841e-01	1.964376e-01	2.165434e-01	2.310267e-01	2.232504e-01
	α	1.397719e-03	9.535444e-04	1.346093e-03	8.844179e-04	1.323374e-03	8.432902e-04
0.7	$E_{\alpha,\beta}$	2.507527e-01	2.161988e-01	1.797863e-01	2.045156e-01	2.227577e-01	2.080711e-01
	α	1.081184e-03	9.635923e-04	1.437265e-03	8.945877e-04	1.420265e-03	8.534285e-04
0.6	$E_{\alpha,\beta}$	2.727533e-01	2.076083e-01	1.630856e-01	1.948424e-01	2.197371e-01	1.931986e-01
	α	1.172276e-03	9.733860e-04	1.134290e-03	9.047290e-04	1.128236e-03	8.636210e-04
0.5	$E_{\alpha,\beta}$	3.121324e-01	2.020755e-01	1.296917e-01	1.870055e-01	2.259017e-01	1.769675e-01
	α	1.283139e-03	9.834866e-04	1.245362e-03	9.152257e-04	1.251916e-03	8.741161e-04
0.4	$E_{\alpha,\beta}$	3.742698e-01	2.005388e-01	1.101827e-01	1.800166e-01	2.473384e-01	1.565571e-01
	α	1.423333e-03	9.951484e-04	1.380010e-03	9.270251e-04	1.408987e-03	8.856188e-04
0.3	$E_{\alpha,\beta}$	4.664453e-01	2.055357e-01	1.127394e-01	1.720981e-01	2.819807e-01	1.274094e-01
	α	1.603647e-03	1.010405e-03	1.546143e-03	9.416682e-04	1.618173e-03	8.993169e-04
0.2	$E_{\alpha,\beta}$	6.074621e-01	2.242173e-01	1.299478e-01	1.602976e-01	3.420304e-01	1.474094e-01
	α	1.800377e-03	1.031690e-03	1.715031e-03	9.605143e-04	1.837841e-03	9.161871e-04
0.1	$E_{\alpha,\beta}$	1.170143e+00	3.088854e-01	1.043151e+00	2.066513e-01	1.037584e+00	1.988448e-01

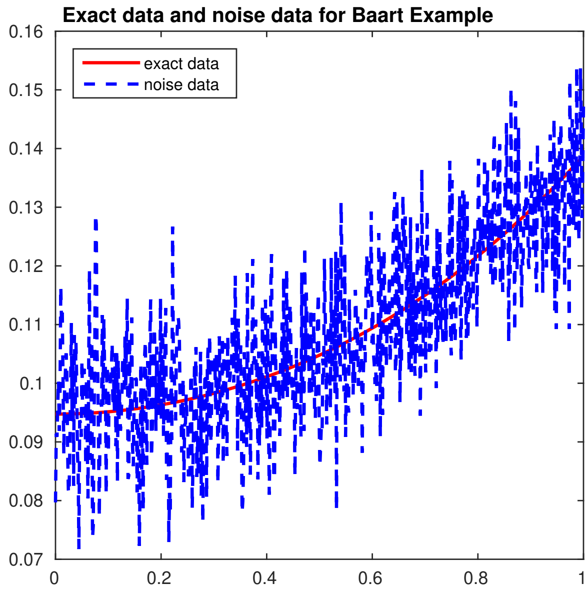

Figure 2.13: Exact data and noise data for $\delta = 0.01$ *and* $n = 700$ *for the Baart example.*

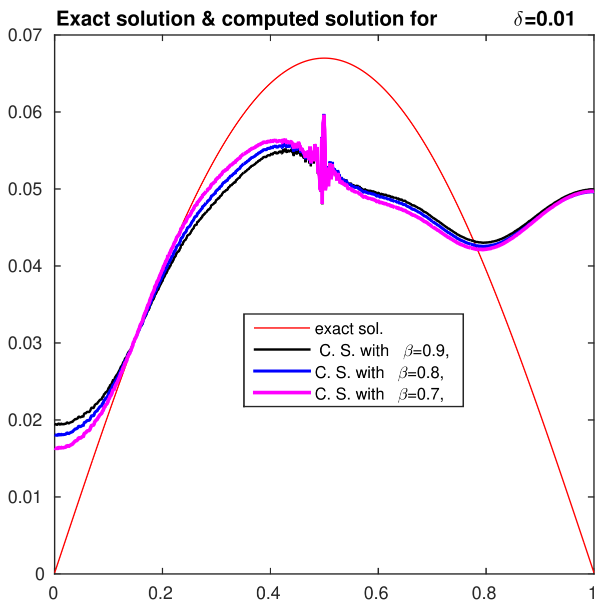

Figure 2.14: Solutions with $\delta = 0.01$ *and* $n = 700$ *for the Baart example.*

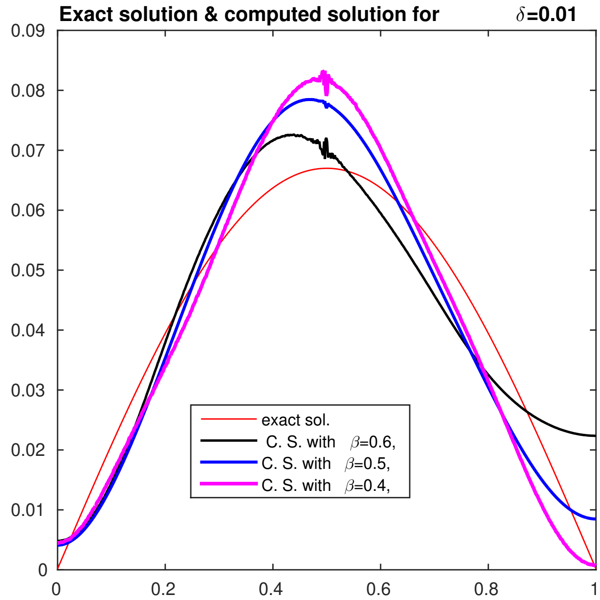

Figure 2.15: Solutions with $\delta = 0.01$ *and* $n = 700$ *for the Baart example.*

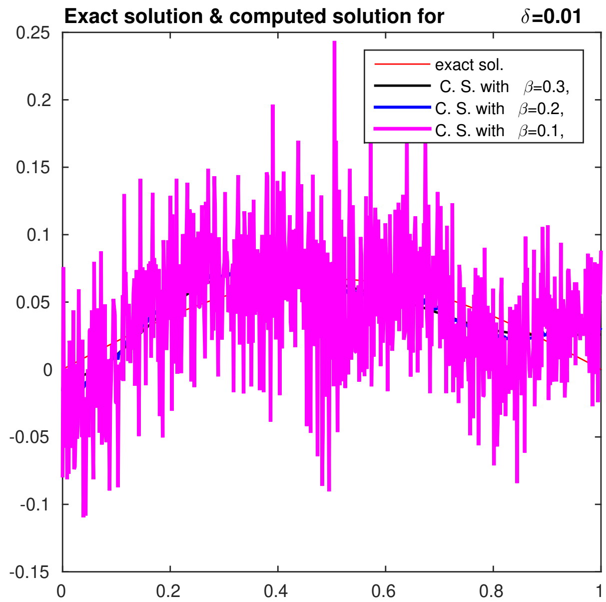

Figure 2.16: Solutions with $\delta = 0.01$ *and* $n = 700$ *for the Baart example*

In the next example, we compute the parameter α using the discrepancy principle (2.5.5) and the adaptive method considered in Section 2.5.1.

EXAMPLE 2.6.4. *(Shaw (1972)) Let*

$$[Tx](s) := \int_{-\pi}^{\pi} k(s,t)x(t)dt = y(s), \ -\pi \leq s \leq \pi, \tag{2.6.5}$$

where $k(s,t) = (cos(s)+cos(t))^2(\frac{sin(u)}{u})^2$, $u = \pi(sin(s)+sin(t))$. *We take* $y = T\hat{x}$, *where* $\hat{x}$ *is given by* $\hat{x}(t) = a_1 exp(-c_1(t-t_1)^2) + a_2 exp(-c_2(t-t_2)^2)$. *We have taken* $a_1 = 2 = a_2$, $t_1 = 0.8$, $t_2 = 0.5$, $c_1 = 6$ *and* $c_2 = 2$ *in our computation. Relative error and* α *values chosen according to (2.5.5) and the adaptive method for different values of* β *and* δ *are given in Table 2.4 and Table 2.5 respectively. Exact data and noise data are given in Fig:2.17 and Fig: 2.18 - 2.20 for various values of* β *when* α *is chosen according to (2.5.5). For the case when* α *is chosen according to the adaptive parameter choice strategy exact data and noise data are given in Fig:2.21 and Fig: 2.22 - 2.26.*

Table 2.4: Relative errors for Shawn example with α *chosen by (2.5.5).*

β		$n = 300$		$n = 500$		$n = 700$	
		$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$	$\delta = 0.01$	$\delta = 0.001$
	α	1.006267e-03	9.708538e-04	9.221944e-04	8.942839e-04	8.722542e-04	8.483506e-04
0.9	$E_{\alpha,\beta}$	1.413859e-01	1.376549e-01	1.337910e-01	1.351783e-01	1.389400e-01	1.375055e-01
	α	1.020155e-03	9.793453e-04	9.333764e-04	9.013028e-04	8.820363e-04	8.544705e-04
0.8	$E_{\alpha,\beta}$	1.288415e-01	1.244267e-01	1.193689e-01	1.226492e-01	1.306375e-01	1.245173e-01
	α	1.037061e-03	9.898842e-04	9.471880e-04	9.101683e-04	8.939000e-04	8.623115e-04
0.7	$E_{\alpha,\beta}$	1.171825e-01	1.126929e-01	1.078020e-01	1.093487e-01	1.145260e-01	1.119480e-01
	α	1.056594e-03	1.001994e-03	9.634052e-04	9.206038e-04	9.084711e-04	8.716568e-04
0.6	$E_{\alpha,\beta}$	1.017549e-01	9.940003e-02	9.055903e-02	9.667828e-02	1.056650e-01	9.852877e-02
	α	1.078096e-03	1.014853e-03	9.815291e-04	9.319441e-04	9.242003e-04	8.819936e-04
0.5	$E_{\alpha,\beta}$	9.304808e-02	8.870781e-02	8.311389e-02	8.390637e-02	9.063262e-02	8.665986e-02
	α	1.099918e-03	1.027352e-03	1.000319e-03	9.433314e-04	9.416443e-04	8.925363e-04
0.4	$E_{\alpha,\beta}$	7.697616e-02	7.742740e-02	6.298939e-02	7.280360e-02	8.215500e-02	7.473445e-02
	α	1.120677e-03	1.038683e-03	1.019503e-03	9.539402e-04	9.583312e-04	9.026130e-04
0.3	$E_{\alpha,\beta}$	8.546921e-02	6.995460e-02	5.927276e-02	6.127571e-02	5.931022e-02	6.481339e-02
	α	1.141456e-03	1.048665e-03	1.038583e-03	9.635010e-04	9.753445e-04	9.117726e-04
0.2	$E_{\alpha,\beta}$	8.942887e-02	6.348320e-02	5.378102e-02	5.223437e-02	5.115249e-02	5.625024e-02
	α	1.162299e-03	1.058060e-03	1.058797e-03	9.723786e-04	9.922042e-04	9.202463e-04
0.1	$E_{\alpha,\beta}$	1.370792e-01	6.171721e-02	1.076590e-01	4.563679e-02	8.392974e-02	4.939491e-02

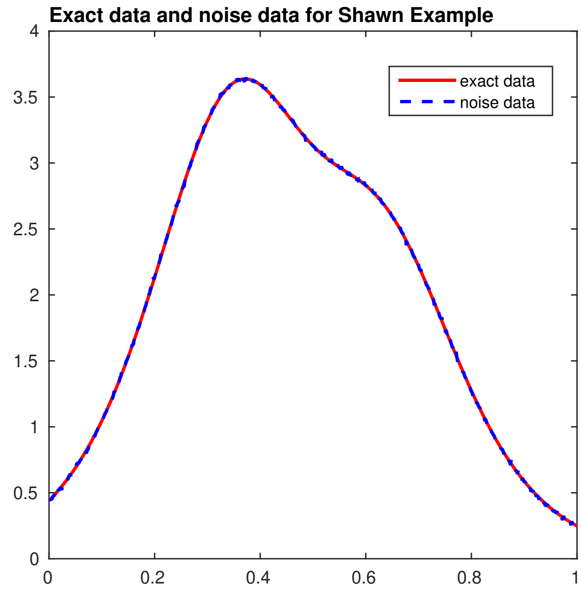

Figure 2.17: Exact data and noise data for $\delta = 0.01$, $n = 700$ and α chosen by (2.5.5) for the Shawn example.

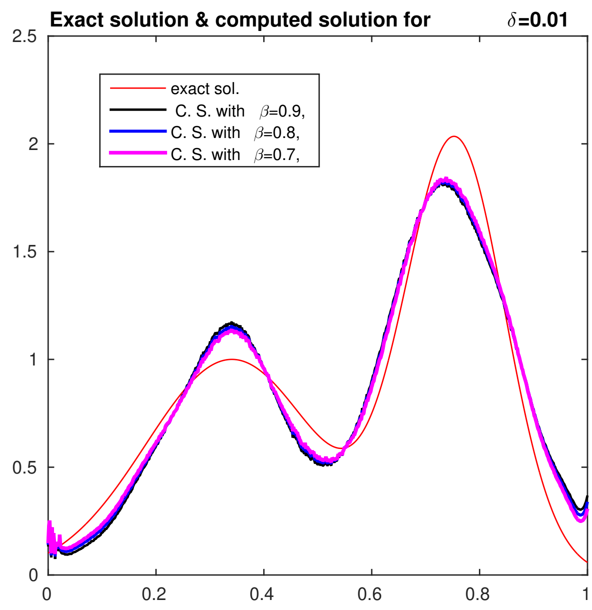

Figure 2.18: Solutions with $\delta = 0.01$, $n = 700$ and α chosen by (2.5.5) for the Shawn example.

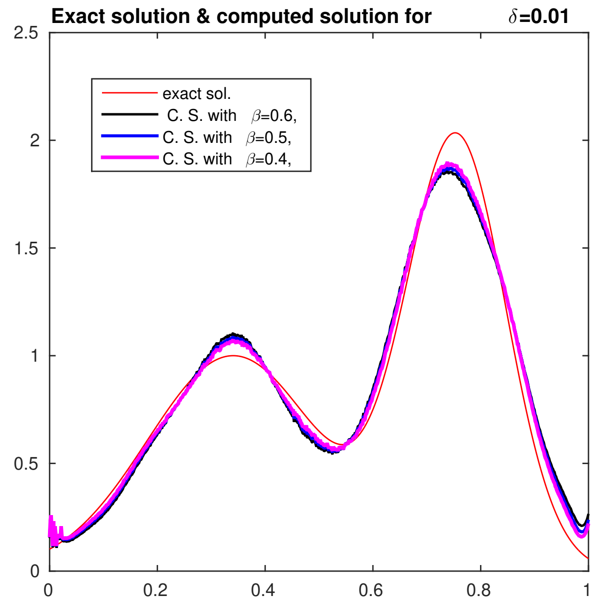

Figure 2.19: Solutions with $\delta = 0.01$, $n = 700$ and α chosen by (2.5.5) for the Shawn example.

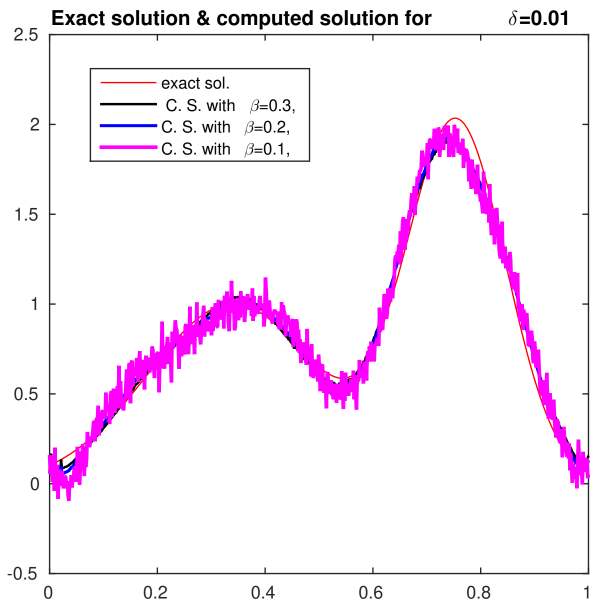

Figure 2.20: Solutions with $\delta = 0.01$, $n = 700$ and α chosen by (2.5.5) for the Shawn example

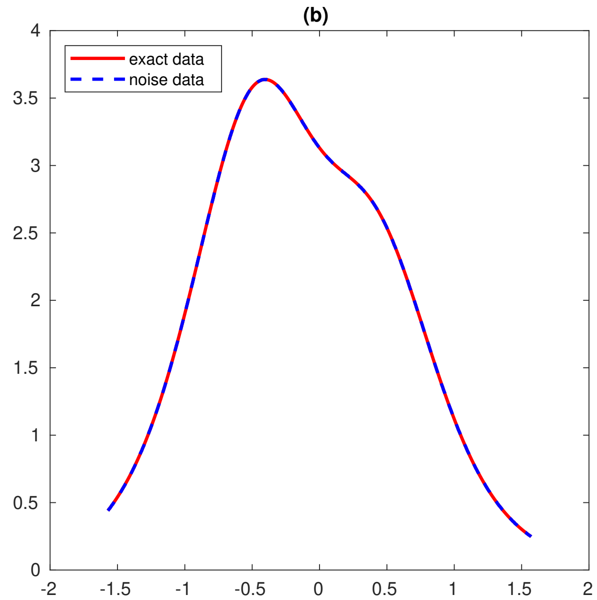

Figure 2.21: Exact data and noise data for $\delta = 0.01$ and $n = 700$ for the Shawn example using adaptive parameter choice strategy.

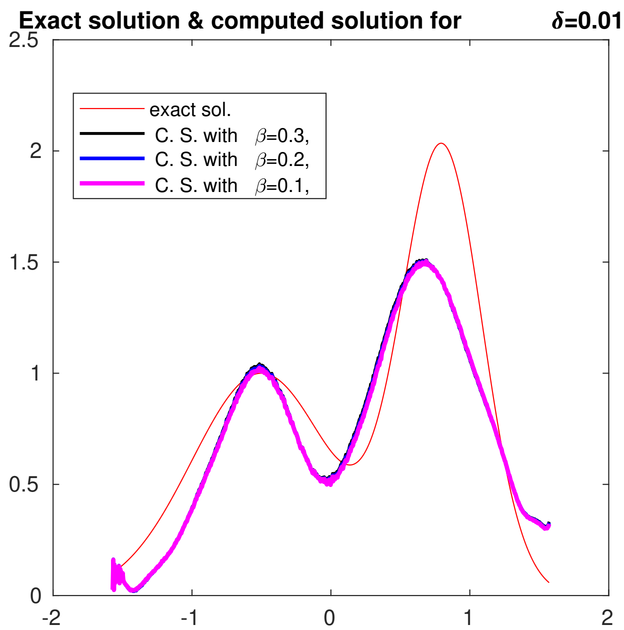

Figure 2.22: Solutions with $\delta = 0.01$ and $n = 700$ for the Shawn example using adaptive parameter choice strategy.

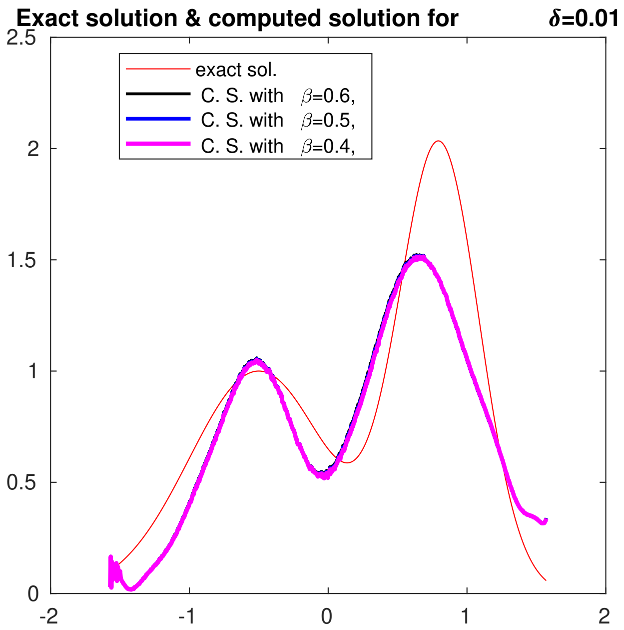

Figure 2.23: Solutions with $\delta = 0.01$ *and* $n = 700$ *for the Shawn example using adaptive parameter choice strategy.*

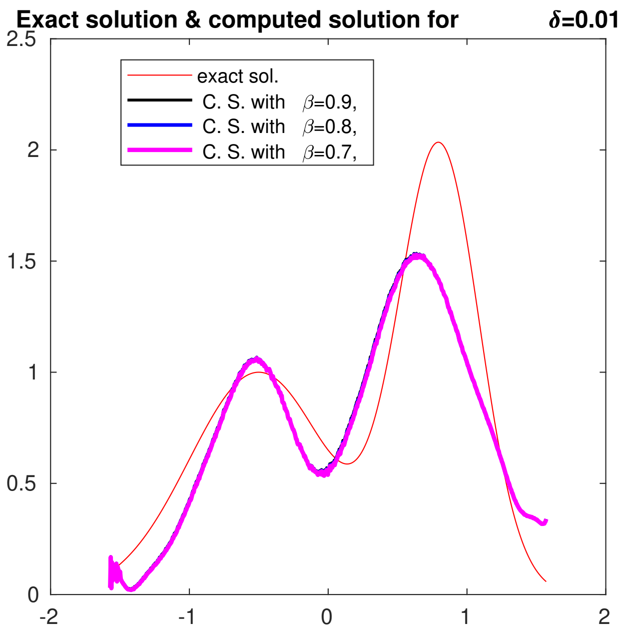

Figure 2.24: Solutions with $\delta = 0.01$ *and* $n = 700$ *for the Shawn example using adaptive parameter choice strategy.*

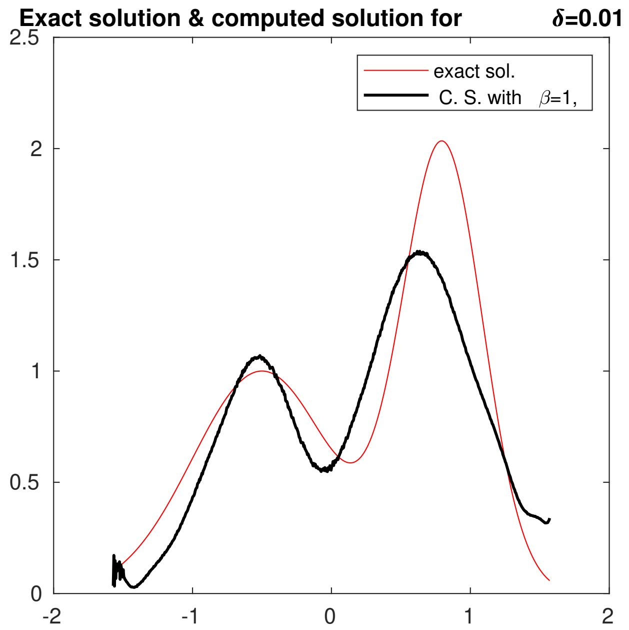

Figure 2.25: Solutions with $\delta = 0.01$ and $n = 700$ for the Shawn example using adaptive parameter choice strategy.

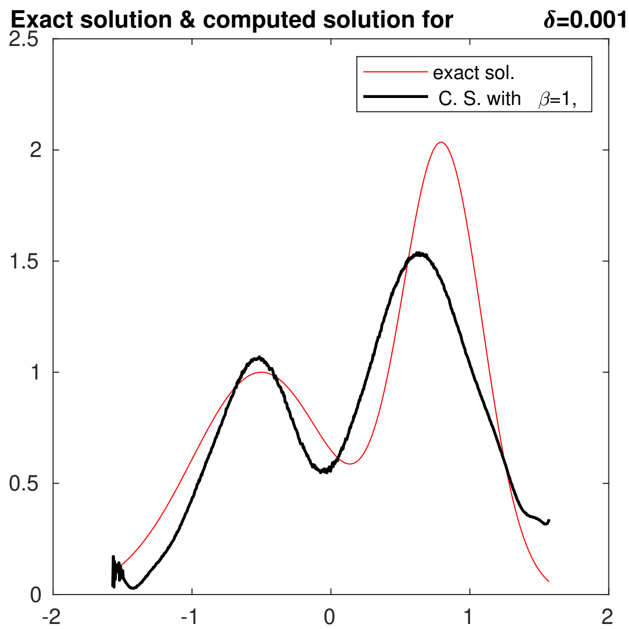

Figure 2.26: Solutions with $\delta = 0.01$ and $n = 700$ for the Shawn example using adaptive parameter choice strategy.

Table 2.5: Shawn example with adaptive parameter choice.

β		$n=300$		$n=500$		$n=700$	
		$\delta=0.01$	$\delta=0.001$	$\delta=0.01$	$\delta=0.001$	$\delta=0.01$	$\delta=0.001$
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
1	$E_{\alpha,\beta}$	2.510211e-01	2.513059e-01	2.565997e-01	2.563700e-01	2.620544e-01	2.620773e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.9	$E_{\alpha,\beta}$	2.504491e-01	2.501763e-01	2.553481e-01	2.553361e-01	2.613309e-01	2.611969e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.8	$E_{\alpha,\beta}$	2.493065e-01	2.490020e-01	2.543349e-01	2.543061e-01	2.604288e-01	2.602863e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.7	$E_{\alpha,\beta}$	2.480771e-01	2.477313e-01	2.532700e-01	2.532161e-01	2.594919e-01	2.593387e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.6	$E_{\alpha,\beta}$	2.461930e-01	2.463764e-01	2.518656e-01	2.520317e-01	2.582635e-01	2.583146e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.5	$E_{\alpha,\beta}$	2.446017e-01	2.447880e-01	2.505180e-01	2.507173e-01	2.571769e-01	2.572206e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.4	$E_{\alpha,\beta}$	2.427531e-01	2.429419e-01	2.489849e-01	2.492214e-01	2.559690e-01	2.559989e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.3	$E_{\alpha,\beta}$	2.393326e-01	2.406563e-01	2.473081e-01	2.475465e-01	2.548616e-01	2.547073e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.2	$E_{\alpha,\beta}$	2.362439e-01	2.379960e-01	2.451814e-01	2.454880e-01	2.532187e-01	2.530980e-01
	α	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02	7.200000e-02
0.1	$E_{\alpha,\beta}$	2.362439e-01	2.347846e-01	2.425772e-01	2.430249e-01	2.512441e-01	2.511911e-01

REMARK 2.6.5. *From Table 2.1- Table 2.4, one can observe that the relative error $E_{\alpha,\beta}$ decreases with β to a certain limit and increases thereafter. So one may fix a β, which yield minimum relative error $E_{\alpha,\beta}$. The same behavior (i.e., the computed solution get closer to the exact solution as β decreases upto certain limit and shoots out thereafter) can be observed in the Figures, Fig: 2.5–Fig: 2.20.*

In the next example, we compute the regularization parameter using adaptive method.

EXAMPLE 2.6.6. *(cf. Phillips (1962))*
Define the function

$$\phi(x) = \begin{cases} 1+\cos\left(\frac{x\pi}{3}\right) & |x|<3 \\ 0 & |x|\geq 3. \end{cases}$$

Consider the problem of solving integral equation

$$[Tx](s) := \int_{-6}^{6} k(s,t)\ x(t)dt = g(s), \quad -6\leq s\leq 6, \tag{2.6.6}$$

where $k(s,t) = \phi(s-t)$, $g(s) = (6-|s|)\left(1+\frac{1}{2}\cos\left(\frac{s\pi}{3}\right)\right)+\frac{9}{2\pi}\sin\left(\frac{|s|\pi}{3}\right)$ *and* $\tilde{h} = \frac{12}{n}$. *The solution of this problem* $\hat{x}(t)$ *is given by* $\hat{x}(t) = \phi(t)$. *We have introduced the random noise level* $\delta = 0.01$ *and* 0.001 *in the exact data. Relative errors and* α *values are showcased in Tables 2.6 obtained using adaptive method for different values of* β, n *and* δ. *Noise data and exact data are given in Fig:2.27 and Fig:2.28 - 2.30 display the computed solution and exact solution, respectively for various values of* β. *In Fig:2.31 and Fig:2.32, we have demonstrated the exact solution and computed solution for Standard Tikhonov Regularization Method(STRM).*

Table 2.6: RE for Phillips example with adaptive parameter choice.

β		$n=300$		$n=500$		$n=700$	
		$\delta=0.01$	$\delta=0.001$	$\delta=0.01$	$\delta=0.001$	$\delta=0.01$	$\delta=0.001$
	α	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02	1.791590e-01	7.200000e-02
1	$E_{\alpha,\beta}$	8.343761e-02	4.270015e-02	8.633825e-02	4.743883e-02	8.400357e-02	4.967453e-02
	α	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02
0.9	$E_{\alpha,\beta}$	8.728298e-02	4.327196e-02	9.441347e-02	4.820018e-02	9.966629e-02	5.039264e-02
	α	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02
0.8	$E_{\alpha,\beta}$	8.728298e-02	4.410263e-02	9.441347e-02	4.921456e-02	9.966629e-02	5.159880e-02
	α	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02
0.7	$E_{\alpha,\beta}$	8.728298e-02	4.523591e-02	9.441347e-02	5.056741e-02	9.966629e-02	5.318465e-02
	α	2.149908e-01	8.640000e-02	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02
0.6	$E_{\alpha,\beta}$	9.827280e-02	5.289967e-02	1.111988e-01	5.201912e-02	1.186109e-01	5.544085e-02
	α	2.149908e-01	8.640000e-02	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02
0.5	$E_{\alpha,\beta}$	9.827280e-02	5.289967e-02	1.111988e-01	5.431881e-02	1.186109e-01	5.805947e-02
	α	2.149908e-01	8.640000e-02	2.149908e-01	7.200000e-02	2.149908e-01	7.200000e-02
0.4	$E_{\alpha,\beta}$	9.827280e-02	5.289967e-02	1.111988e-01	5.725935e-02	1.186109e-01	6.137480e-02
	α	2.149908e-01	8.640000e-02	2.149908e-01	8.640000e-02	2.149908e-01	7.200000e-02
0.3	$E_{\alpha,\beta}$	1.314530e-01	6.299845e-02	1.424821e-01	7.048210e-02	1.510811e-01	6.550096e-02
	α	2.149908e-01	8.640000e-02	2.149908e-01	8.640000e-02	2.149908e-01	7.200000e-02
0.2	$E_{\alpha,\beta}$	1.314530e-01	6.299845e-02	1.424821e-01	7.048210e-02	1.510811e-01	6.550096e-02
	α	2.149908e-01	8.640000e-02	2.149908e-01	8.640000e-02	2.149908e-01	7.200000e-02
0.1	$E_{\alpha,\beta}$	1.314530e-01	6.299845e-02	1.424821e-01	7.048210e-02	1.510811e-01	6.550096e-02

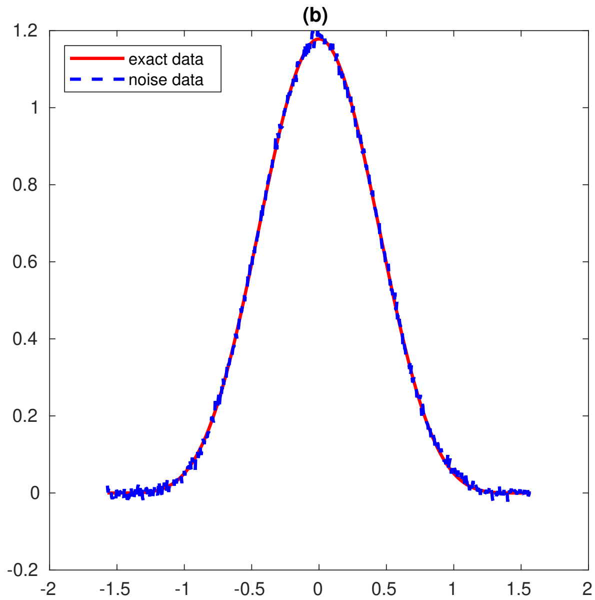

Figure 2.27: Exact data and noise data for $\delta = 0.01$ and $n = 700$ for the Phillips example with adaptive method.

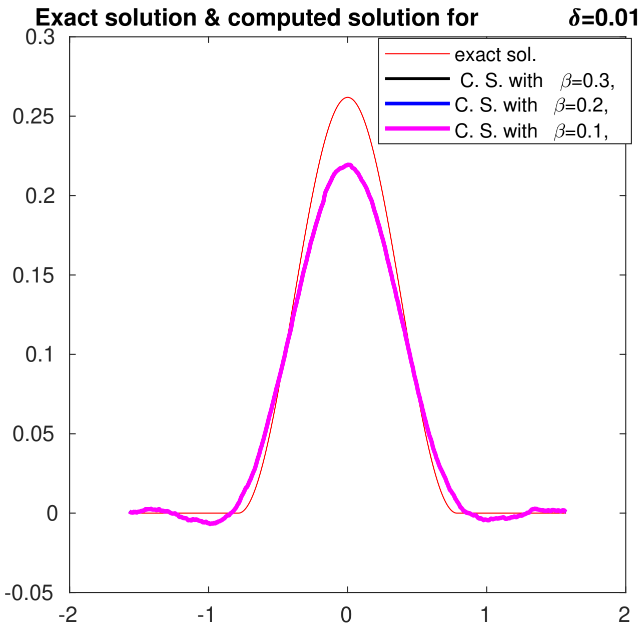

Figure 2.28: Solutions with $\delta = 0.01$ and $n = 700$ for the Phillips example using adaptive parameter choice strategy.

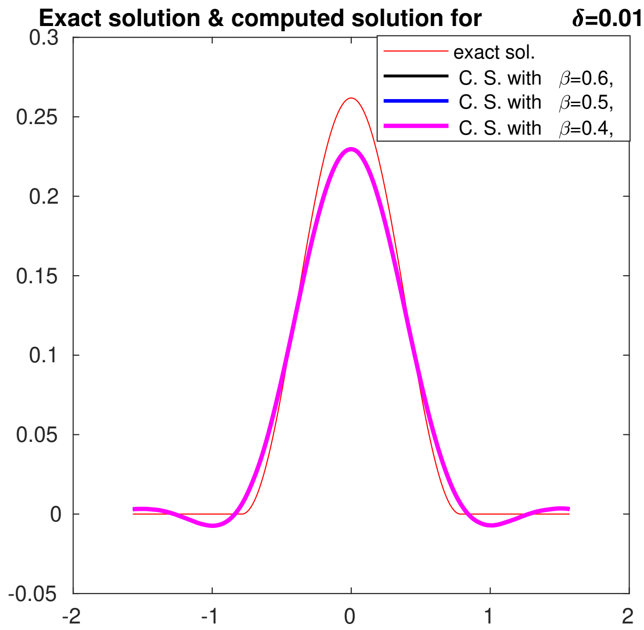

Figure 2.29: Solutions with $\delta = 0.01$ and $n = 700$ for the Phillips example using adaptive parameter choice strategy.

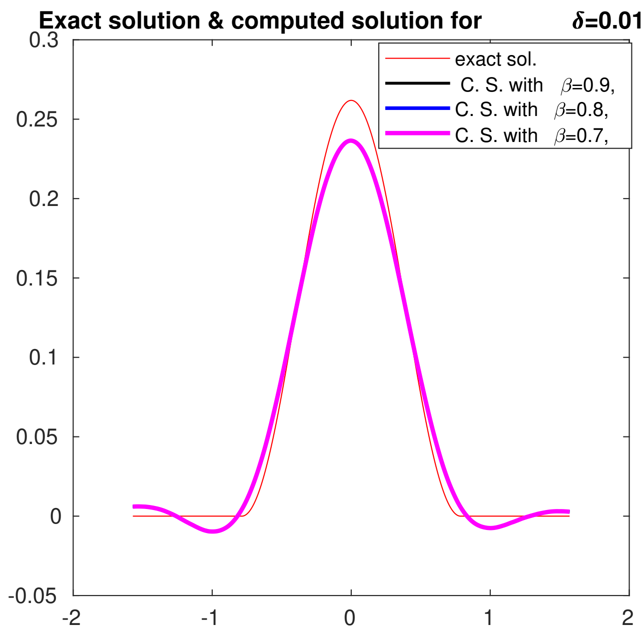

Figure 2.30: Solutions with $\delta = 0.01$ and $n = 700$ for the Phillips example using adaptive parameter choice strategy.

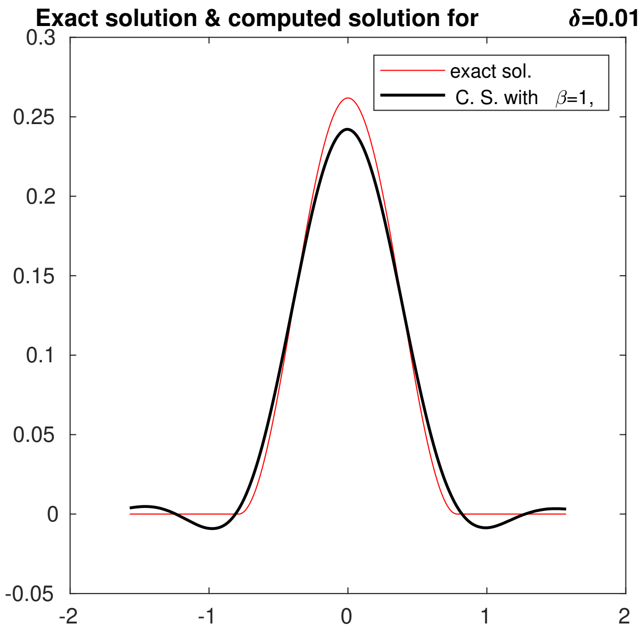

Figure 2.31: Solutions for standard Tikhonov regularization with $\delta = 0.01$ *and* $n = 700$ *for Phillips example using adaptive parameter choice strategy.*

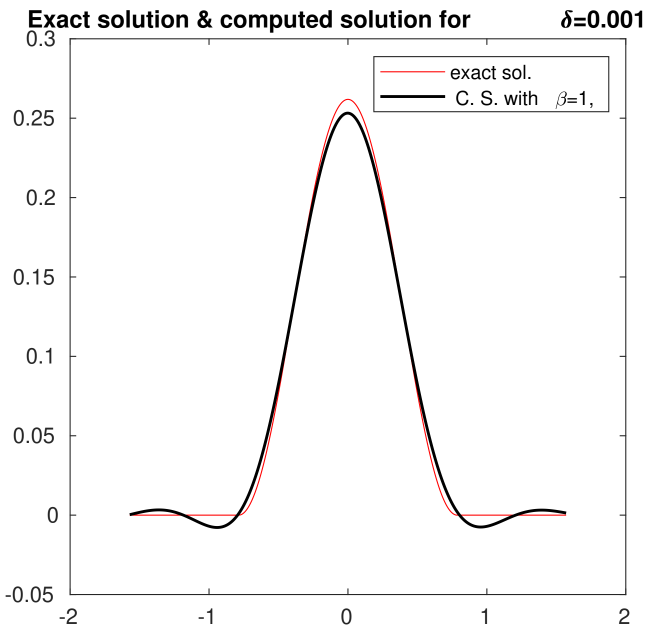

Figure 2.32: Solutions for standard Tikhonov regularization with $\delta = 0.001$ *and* $n = 700$ *for Phillips example using adaptive parameter choice strategy.*

Chapter 3

FINITE DIMENSIONAL REALIZATION OF FRACTIONAL TIKHONOV REGULARIZATION IN HILBERT SCALES

3.1 INTRODUCTION

In Chapter 2, we studied the convergence of $x_{\alpha,\beta}^{s,\delta}$ to $\hat{x}$. But performing the numerical computation of $x_{\alpha,\beta}^{s,\delta}$ in an infinite dimensional space is not easy. So one has to consider the finite dimensional realization of $x_{\alpha,\beta}^{s,\delta}$. The aim of this chapter is to study the finite dimensional realization of the method considered in Chapter 2.

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections of X onto $R(P_h)$, range of P_h. We impose the condition

$$\varepsilon_h = \|T(I - P_h)\|. \tag{3.1.1}$$

Assume $\lim_{h\to 0} \varepsilon_h = 0$. This is satisfied if $P_h \to I$ pointwise and T is a compact operator. Let $T_h = TP_h$ and let $h_0 > 0$ be such that

$$\varepsilon_h \leq \frac{b_1\|x\|_{-a}}{2\|x\|} \ \ \forall x \neq 0, h \leq h_0. \tag{3.1.2}$$

LEMMA 3.1.1. *Let $\bar{b}_1 = \frac{b_1}{2}$, $\bar{b}_2 = b_2 + \frac{b_1}{2}$ and $h \leq h_0$. Then*

$$\bar{b}_1\|x\|_{-a} \leq \|T_hx\| \leq \bar{b}_2\|x\|_{-a}. \tag{3.1.3}$$

Proof. Let (3.1.1) and (3.1.2) hold, then

$$\begin{aligned} \|T_h x\| &\leq \|Tx\| + \|T(P_h - I)x\| \\ &\leq b_2\|x\|_{-a} + \varepsilon_h\|x\| \\ &\leq \bar{b}_2\|x\|_{-a}. \end{aligned}$$

Similarly,

$$\begin{aligned} \|T_h x\| &\geq \|Tx\| - \|T(P_h - I)x\| \\ &\geq b_1\|x\|_{-a} - \varepsilon_h\|x\| \\ &\geq \bar{b}_1\|x\|_{-a}. \end{aligned}$$

□

Clearly $\bar{b}_1 \leq \bar{b}_2$. Let $\bar{f}(t) := \begin{cases} \bar{b}_1^t, & \text{if } 0 \leq t \\ \bar{b}_2^t, & \text{if } t < 0 \end{cases}$ and $\bar{g}(t) := \begin{cases} \bar{b}_2^t, & \text{if } 0 \leq t \\ \bar{b}_1^t, & \text{if } t < 0. \end{cases}$

PROPOSITION 3.1.2. *Suppose Lemma 3.1.1 holds and* $|\nu| \leq 1$, *then*

$$\bar{f}(\nu)\|x\|_{-\nu a} \leq \|(T_h^* T_h)^{\nu/2} x\| \leq \bar{g}(\nu)\|x\|_{-\nu a}, \; x \in D((T_h^* T_h)^{\nu/2}). \tag{3.1.4}$$

Proof. *The proof follows from the Proposition 2.2.1 by replacing* T *by* T_h..

PROPOSITION 3.1.3. *Let* T_h *be a bounded linear operator satisfying (3.1.3). Then, with the hypothesis of Proposition 3.1.2 and for* $0 \leq \beta \leq 1$, *the following hold:*

$$\bar{F}(\nu)\|x\|_{-\frac{\nu}{2}((1+\beta)a+2s)} \leq \|(L^{-s}(T_h^* T_h)^{\frac{1+\beta}{2}} L^{-s})^{\nu/2} x\| \leq \bar{G}(\nu)\|x\|_{-\frac{\nu}{2}((1+\beta)a+2s)},$$

$x \in D((L^{-s}(T_h^* T_h)^{\frac{1+\beta}{2}} L^{-s})^{\nu/2}), s > 0, |\nu| \leq 1,$ *where* $\bar{F}(t) := \begin{cases} \bar{b}_1^{(\frac{1+\beta}{2})t}, & \text{if } 0 \leq t \\ \bar{b}_2^{(\frac{1+\beta}{2})t}, & \text{if } t < 0 \end{cases}$
and
$\bar{G}(t) := \begin{cases} \bar{b}_2^{(\frac{1+\beta}{2})t}, & \text{if } 0 \leq t \\ \bar{b}_1^{(\frac{1+\beta}{2})t}, & \text{if } t < 0. \end{cases}$

Proof. Using Proposition 3.1.2, with $\nu = \frac{1+\beta}{2}$, we obtain

$$\bar{f}(\frac{1+\beta}{2})\|x\|_{-(\frac{1+\beta}{2})a} \leq \|(T_h^*T_h)^{\frac{1+\beta}{4}}x\| \leq \bar{g}(\frac{1+\beta}{2})\|x\|_{-(\frac{1+\beta}{2})a},\ x \in D((T_h^*T_h)^{\frac{1+\beta}{4}}).$$

Further, the proof follows by taking first, $x = L^{-s}x$ in the above equation and then applying Proposition 3.1.2 for the operator $(T_h^*T_h)^{\frac{1+\beta}{4}}L^{-s}$.

□

We consider the unique solution $x_{\alpha,\beta,h}^{s,\delta}$ of the equation

$$((T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s})x_{\alpha,\beta,h}^{s,\delta} = (T_h^*T_h)^{\frac{\beta}{2}}y^\delta, \tag{3.1.5}$$

as an approximation for $\hat{x}$.

Let

$$A_{s,\beta} := L^{-s}(T^*T)^{\frac{1+\beta}{2}}L^{-s},$$

and

$$A_{s,\beta,h} := L^{-s}(T_h^*T_h)^{\frac{1+\beta}{2}}L^{-s}.$$

Using the above notation, we define

$$x_{\alpha,\beta}^s = L^{-s}(A_{s,\beta} + \alpha I)^{-1}L^{-s}(T^*T)^{\frac{\beta}{2}}y, \tag{3.1.6}$$

$$x_{\alpha,\beta,h}^s = L^{-s}(A_{s,\beta,h} + \alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y \tag{3.1.7}$$

and

$$x_{\alpha,\beta,h}^{s,\delta} = L^{-s}(A_{s,\beta,h} + \alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta. \tag{3.1.8}$$

By spectral properties of the self-adjoint operator,
$A_{s,\beta},\ A_{s,\beta,h},\ s > 0,\ \beta \in [0, 1]$, we have

$$\|(A_{s,\beta} + \alpha I)^{-1}A_{s,\beta}^\mu\| \leq \alpha^{\mu-1}, \alpha > 0,\ 0 \leq \mu \leq 1. \tag{3.1.9}$$

$$\|(A_{s,\beta,h} + \alpha I)^{-1}A_{s,\beta,h}^\mu\| \leq \alpha^{\mu-1}, \alpha > 0,\ 0 \leq \mu \leq 1. \tag{3.1.10}$$

LEMMA 3.1.4. *Let $x^s_{\alpha,\beta,h}, x^{s,\delta}_{\alpha,\beta,h}$ be as in (3.1.7) and (3.1.8), respectively. If the assumptions in Proposition 3.1.3 holds, then for* $0 \leq \beta \leq 1$,

$$\|x^s_{\alpha,\beta,h} - x^{s,\delta}_{\alpha,\beta,h}\| \leq \varphi(s,a,\beta,h)\alpha^{\frac{-a}{(1+\beta)a+2s}}\delta,$$

where $\varphi(s,a,\beta,h) := \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{2s}{(1+\beta)a+2s}\right)\bar{f}(-\beta)} = \frac{(\bar{b}_1^{-\beta a-2s})^{\frac{1+\beta}{(1+\beta)a+2s}}}{\bar{b}_2^{-\beta}}$.

Proof. By (3.1.7) and (3.1.8), we have

$$\begin{aligned}
\|x^{s,\delta}_{\alpha,\beta,h} - x^s_{\alpha,\beta,h}\| &= \|L^{-s}(A_{s,\beta,h}+\alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\| \\
&= \|(A_{s,\beta,h}+\alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\|_{-s}.
\end{aligned}$$

Therefore, by using Proposition 3.1.3, with $\nu = \frac{2s}{(1+\beta)a+2s}$, $x = (A_{s,\beta,h}+\alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)$ and (3.1.10) we obtain in turn that

$$\begin{aligned}
&\|x^{s,\delta}_{\alpha,\beta,h} - x^s_{\alpha,\beta,h}\| \\
\leq\ & \frac{1}{\bar{F}\left(\frac{2s}{(1+\beta)a+2s}\right)}\|A_{s,\beta,h}^{\frac{s}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\| \\
=\ & \frac{1}{\bar{F}\left(\frac{2s}{(1+\beta)a+2s}\right)} \\
& \times\|A_{s,\beta,h}^{\frac{2s+\beta a}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-1}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\| \qquad (3.1.11) \\
\leq\ & \frac{1}{\bar{F}\left(\frac{2s}{(1+\beta)a+2s}\right)}\|A_{s,\beta,h}^{\frac{2s+\beta a}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-1}\|\|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\| \\
\leq\ & \frac{1}{\bar{F}\left(\frac{2s}{(1+\beta)a+2s}\right)}\alpha^{\frac{-a}{(1+\beta)a+2s}}\|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\|.
\end{aligned}$$

So the Lemma is proved, if we prove

$$\|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\| \leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-\beta)}\delta. \qquad (3.1.12)$$

But this can be seen by taking $\nu = \frac{-2(\beta a+s)}{(1+\beta)a+2s}$ and

$x = L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)$ in Proposition 3.1.3

$$\begin{aligned}
\|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\| &\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\|L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\|_{s+\beta a} \\
&= \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\|(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\|_{\beta a} \\
&\leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-\beta)}\|y^\delta - y\| \\
&\leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-\beta)}\delta.
\end{aligned}$$

The last but one step follows again from Proposition 3.1.2, by taking $\nu = -\beta$.

□

To obtain an estimate for $\|x_{\alpha,\beta,h}^s - x_{\alpha,\beta}^s\|$, we make use of the following formula;

$$\begin{aligned}
B^z(x) &= \frac{\sin \pi z}{\pi}\int_0^\infty t^z[(B+tI)^{-1}x - \frac{\theta(t)}{t}x + \cdots + (-1)^n\frac{\theta(t)}{t^n}B^{n-1}x]dt \\
&\quad + \frac{\sin \pi z}{\pi}[\frac{x}{z} - \frac{Bz}{z-1} + \cdots + (-1)^{n-1}\frac{B^{n-1}x}{z-n+1}], x \in X,
\end{aligned} \tag{3.1.13}$$

where

$$\theta(t) = \begin{cases} 0, & \text{if } 0 \leq t \leq 1 \\ 1, & \text{if } 1 < t \leq \infty \end{cases}$$

for any positive self-adjoint operator B and for any complex number z such that $0 < Rez < n$. Taking $z = \frac{1+\beta}{2}$, $0 \leq \beta < 1$, in (3.1.13) one can see that

$$B^{\frac{1+\beta}{2}}x = \frac{\sin \pi(\frac{1+\beta}{2})}{\pi}\left[\frac{2x}{1+\beta} + \int_0^\infty \lambda^{\frac{1+\beta}{2}}(B+\lambda I)^{-1}x d\lambda - \int_1^\infty \frac{x}{\lambda^{1-(\frac{1+\beta}{2})}}d\lambda\right]. \tag{3.1.14}$$

Using (3.1.14), for $z \in X$ we have

$$\begin{aligned}
[(T_h^*T_h)^{\frac{1+\beta}{2}} - (T^*T)^{\frac{1+\beta}{2}}]z &= \frac{\sin \pi(\frac{1+\beta}{2})}{\pi}\int_0^\infty \lambda^{\frac{1+\beta}{2}}((T_h^*T_h)+\lambda I)^{-1}[(T^*T) - (T_h^*T_h)] \\
&\quad ((T^*T)+\lambda I)^{-1}z d\lambda.
\end{aligned} \tag{3.1.15}$$

LEMMA 3.1.5. *Let $x^s_{\alpha,\beta}$ and $x^s_{\alpha,\beta,h}$ be as in (3.1.6) and (3.1.7), respectively and Assumption 2.3.2 holds. Further, let the assumptions in Proposition 2.2.1, Proposition 3.1.2 and Proposition 3.1.3 hold. Then for $0 \le t \le \frac{(1+\beta)}{2}a + 2s$*

$$\|x^s_{\alpha,\beta,h} - x^s_{\alpha,\beta}\| \le \varphi_1(s,a,\beta,h,t)\alpha^{\frac{-a}{(1+\beta)a+2s}}\varepsilon_h,$$

where
$$\varphi_1(s,a,\beta,h,t) = \frac{\sin\pi(\frac{1+\beta}{2})}{\pi b_2^{-\beta}} \frac{\bar{b}_1^{\frac{-(1+\beta)(2s+\beta a)}{(1+\beta)a+2s}}}{b_1^{\frac{(1+\beta)s}{(1+\beta)a+2s}}} \left(C_h b_1^{\frac{-t}{a}} \frac{2aE}{t} + b_2^{\frac{(1+\beta)s}{(1+\beta)a+2s}} 2\|T\|\|\hat{x}\| \right) + \frac{(\bar{b}_1)^{\frac{-(1+\beta)(\beta a+2s)}{(1+\beta)a+2s}}}{\bar{b}_2^{-\beta}}\|\hat{x}\|,$$
with $C_h = \begin{cases} b_2^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } t \le s \\ b_1^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } s < t. \end{cases}$

Proof. Note that

$$\begin{aligned} x^s_{\alpha,\beta} &= L^{-s}(A_{a,\beta} + \alpha I)^{-1}L^{-s}(T^*T)^{\frac{1+\beta}{2}}\hat{x} \\ x^s_{\alpha,\beta,h} &= ((T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T_h^*T_h)^{\frac{1+\beta}{2}}\hat{x} + ((T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1} \\ &\quad \times (T_h^*T_h)^{\frac{\beta}{2}}T[I - P_h]\hat{x} \end{aligned}$$

and hence

$$\begin{aligned} x^s_{\alpha,\beta,h} - x^s_{\alpha,\beta} &= [((T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T_h^*T_h)^{\frac{1+\beta}{2}} \\ &\quad -((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T^*T)^{\frac{1+\beta}{2}}]\hat{x} \\ &\quad +((T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T_h^*T_h)^{\frac{\beta}{2}}T[I - P_h]\hat{x}. \end{aligned} \tag{3.1.16}$$

So,

$$\|x^s_{\alpha,\beta,h} - x^s_{\alpha,\beta}\| \le \|K\| + \|((T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T_h^*T_h)^{\frac{\beta}{2}}T[I - P_h]\hat{x}\|, \tag{3.1.17}$$

where $K = [((T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T_h^*T_h)^{\frac{1+\beta}{2}} - ((T^*T)^{\frac{1+\beta}{2}} + \alpha L^{2s})^{-1}(T^*T)^{\frac{1+\beta}{2}}]\hat{x}$. We have,

$$\begin{aligned} &\|[(T_h^*T_h)^{\frac{1+\beta}{2}} + \alpha L^{2s}]^{-1}(T_h^*T_h)^{\frac{\beta}{2}}T[I - P_h]\hat{x}\| \\ = &\|[A_{s,\beta,h} + \alpha I]^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}T[I - P_h]\hat{x}\|_{-s} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\bar{F}(\frac{2s}{(1+\beta)a+2s})}\|A_{s,\beta,h}^{\frac{s}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-1}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}T[I-P_h]\hat{x}\| \\
&\leq \frac{1}{\bar{F}(\frac{2s}{(1+\beta)a+2s})}\|A_{s,\beta,h}^{\frac{2s+\beta a}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-1}A_{s,\beta,h}^{\frac{-(s+\beta a)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}T[I-P_h]\hat{x}\| \\
&\leq \frac{\bar{G}(\frac{-2(s+\beta a)}{(1+\beta)a+2s})}{\bar{F}(\frac{2s}{(1+\beta)a+2s})}\alpha^{\frac{-a}{(1+\beta)a+2s}}\|(T_h^*T_h)^{\frac{\beta}{2}}T[I-P_h]\hat{x}\|_{\beta a} \\
&\leq \frac{\bar{G}(\frac{-2(s+\beta a)}{(1+\beta)a+2s})}{\bar{F}(\frac{2s}{(1+\beta)a+2s})\bar{f}(-\beta)}\varepsilon_h\|\hat{x}\|\alpha^{\frac{-a}{(1+\beta)a+2s}}
\end{aligned}
\tag{3.1.18}
$$

and since

$$
\begin{aligned}
K &= L^{-s}[(A_{s,\beta,h}+\alpha I)^{-1}A_{s,\beta,h}-(A_{s,\beta}+\alpha I)^{-1}A_{s,\beta}]L^s\hat{x} \\
&= L^{-s}(A_{s,\beta,h}+\alpha I)^{-1}[A_{s,\beta,h}(A_{s,\beta}+\alpha I) \\
&\quad -(A_{s,\beta,h}+\alpha I)A_{s,\beta}](A_{s,\beta}+\alpha I)^{-1}L^s\hat{x} \\
&= L^{-s}(A_{s,\beta,h}+\alpha I)^{-1}\alpha[A_{s,\beta,h} \\
&\quad -A_{s,\beta}](A_{s,\beta}+\alpha I)^{-1}L^s\hat{x} \\
&= L^{-s}(A_{s,\beta,h}+\alpha I)^{-1}\alpha L^{-s}[(T_h^*T_h)^{\frac{1+\beta}{2}} \\
&\quad -(T^*T)^{\frac{1+\beta}{2}}]L^{-s}(A_{s,\beta}+\alpha I)^{-1}L^s\hat{x} \\
&= \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}L^{-s}(A_{s,\beta,h}+\alpha I)^{-1}\alpha L^{-s}\int_0^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1} \\
&\quad \times[T_h^*T_h-T^*T](T^*T+\lambda I)^{-1}L^{-s}(A_{a,\beta}+\alpha I)^{-1}L^s\hat{x}d\lambda.
\end{aligned}
$$

So, by (3.1.10), Proposition 3.1.2 and Proposition 3.1.3, we have

$$
\begin{aligned}
\|K\| &\leq \frac{1}{\bar{F}(\frac{2s}{(1+\beta)a+2s})}\frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\|\alpha(A_{s,\beta,h}+\alpha I)^{-1}A_{s,\beta,h}^{\frac{2s+\beta a}{(1+\beta)a+2s}} \\
&\quad \times A_{s,\beta,h}^{\frac{-s-\beta a}{(1+\beta)a+2s}}L^{-s}\int_0^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1} \\
&\quad [(T_h^*T_h)-(T^*T)]((T^*T)+\lambda I)^{-1}L^{-s}(A_{a,\beta}+\alpha I)^{-1}L^s\hat{x}d\lambda\|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \frac{\bar{G}(\frac{-2s}{(1+\beta)a+2s})}{\bar{F}(\frac{2s}{(1+\beta)a+2s})} \frac{\sin\pi(\frac{1+\beta}{2})}{\pi} \|\alpha(A_{s,\beta,h}+\alpha I)^{-1} A_{s,\beta,h}^{\frac{2s+\beta a}{(1+\beta)a+2s}}\| \\
& \times \| \int_0^\infty \lambda^{\frac{1+\beta}{2}} (T_h^* T_h + \lambda I)^{-1} \\
& [(T_h^* T_h) - (T^* T)]((T^*T)+\lambda I)^{-1} L^{-s}(A_{a,\beta}+\alpha I)^{-1} L^s \hat{x} d\lambda\|_{\beta a} \\
\leq\ & \frac{\bar{G}(\frac{-2s}{(1+\beta)a+2s})}{\bar{F}(\frac{2s}{(1+\beta)a+2s}) f(-\beta)} \frac{\sin\pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{2s+\beta a}{(1+\beta)a+2s}} \\
& \times \int_0^\infty \lambda^{\frac{1+\beta}{2}} \|(T_h^* T_h)^{-\frac{\beta}{2}} (T_h^* T_h + \lambda I)^{-1} [T_h^* T_h - T^* T](T^* T + \lambda I)^{-1} Z d\lambda\|
\end{aligned} \tag{3.1.19}
$$

where $Z = L^{-s}(A_{a,\beta}+\alpha I)^{-1} L^s \hat{x}$. Note that

$$
\begin{aligned}
& \int_0^\infty \lambda^{\frac{1+\beta}{2}} \|(T_h^* T_h)^{-\frac{\beta}{2}} (T_h^* T_h + \lambda I)^{-1} [T_h^* T_h - T^* T](T^* T + \lambda I)^{-1} Z d\lambda\| \\
=\ & \int_0^\infty \lambda^{\frac{1+\beta}{2}} \|(T_h^* T_h)^{-\frac{\beta}{2}} (T_h^* T_h + \lambda I)^{-1} [(I-P_h)T^*T + P_h T^* T(I-P_h)](T^*T+\lambda I)^{-1} Z\| d\lambda \\
\leq\ & \int_0^1 \lambda^{\frac{1+\beta}{2}} \|(T_h^* T_h)^{-\frac{\beta}{2}} (T_h^* T_h + \lambda I)^{-1} T_h^*\| \|T(I-P_h)\| \|(T^*T+\lambda I)^{-1} Z\| d\lambda \\
& + \int_1^\infty \lambda^{\frac{1+\beta}{2}} \|(T_h^* T_h)^{-\frac{\beta}{2}} (T_h^* T_h + \lambda I)^{-1}\| \|T_h^*\| \|T(I-P_h)\| \|(T^*T+\lambda I)^{-1} Z\| d\lambda \\
\leq\ & \int_0^1 \lambda^{\frac{1+\beta}{2}} \|(T_h^* T_h)^{-\frac{\beta}{2}} (T_h^* T_h + \lambda I)^{-1} T_h^*\| \|T(I-P_h)\| \\
& \|(T^*T+\lambda I)^{-1} (T^*T)^{\frac{t}{2a}} (T^*T)^{\frac{-t}{2a}} Z\| d\lambda \\
& + \int_1^\infty \lambda^{\frac{1+\beta}{2}} \|(T_h^* T_h)^{-\frac{\beta}{2}} (T_h^* T_h + \lambda I)^{-1}\| \|T_h^*\| \|T(I-P_h)\| \|(T^*T+\lambda I)^{-1} Z\| d\lambda \\
\leq\ & \int_0^1 \lambda^{\frac{t}{2a}-1} \varepsilon_h \|(T^*T)^{\frac{-t}{2a}} Z\| d\lambda + \int_1^\infty \frac{\|T\|\varepsilon_h}{\lambda^{\frac{3}{2}}} \|Z\| d\lambda \\
\leq\ & \frac{2a}{t} \varepsilon_h \|(T^*T)^{\frac{-t}{2a}} Z\| + 2\|T\|\varepsilon_h \|Z\|,
\end{aligned} \tag{3.1.20}
$$

where, we used $\|T_h^*\| \leq \|T\|$ and $(T_h^* T_h + \lambda I)^{-1}(I - P_h) = 0$.

Further, observe that

$$
\begin{aligned}
\|Z\| &= \|L^{-s}(A_{a,\beta}+\alpha I)^{-1}L^s\hat{x}\| \\
&\leq \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)}\|A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-1}L^s\hat{x}\| \\
&\leq \frac{G\left(\frac{2s}{(1+\beta)a+2s}\right)}{\alpha F\left(\frac{2s}{(1+\beta)a+2s}\right)}\|\hat{x}\|
\end{aligned}
$$

and

$$
\begin{aligned}
\|(T^*T)^{\frac{-t}{2a}}Z\| &\leq g(\frac{-t}{a})\|Z\|_t \\
&= g(\frac{-t}{a})\|L^{t-s}(A_{a,\beta}+\alpha I)^{-1}L^s\hat{x}\| \\
&\leq \frac{g(\frac{-t}{a})}{F\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}\|(A_{s,\beta}+\alpha I)^{-1}\|\|A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}}L^s\hat{x}\| \\
&\leq \frac{g(\frac{-t}{a})G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}\alpha^{-1}\|\hat{x}\|_t \\
&\leq \frac{g(\frac{-t}{a})G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{2s}{(1+\beta)a+2s}\right)}\alpha^{-1}E. \qquad (3.1.21)
\end{aligned}
$$

Therefore, by (3.1.19), (3.1.20) and (3.1.21), we have

$$
\begin{aligned}
\|K\| \leq\; & \frac{\bar{G}(\frac{-2s}{(1+\beta)a+2s})\sin\pi(\frac{1+\beta}{2})}{\bar{F}(\frac{2s}{(1+\beta)a+2s})f(-\beta)F\left(\frac{2s}{(1+\beta)a+2s}\right)\pi}\times\left(\frac{2a}{t}g(\frac{-t}{a})G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)E\right. \\
&+2\|T\|G\left(\frac{2s}{(1+\beta)a+2s}\right)\|\hat{x}\|\epsilon_h\alpha^{\frac{2s+\beta a}{(1+\beta)a+2s}-1}.
\end{aligned}
$$

(3.1.23)

Now the result follows from the fact that $\alpha^{\frac{2s+\beta a}{(1+\beta)a+2s}-1}=\alpha^{\frac{-a}{(1+\beta)a+2s}}$.

□

Combining the Lemma 3.1.4, Lemma 3.1.5 and Lemma 2.3.3 we have the following Theorem.

THEOREM 3.1.6. *Let $x^s_{\alpha,\beta}$, $x^s_{\alpha,\beta,h}$ and $x^{s,\delta}_{\alpha,\beta,h}$ be as in (3.1.6), (3.1.7) and (3.1.8), respectively, Assumption 2.3.2 holds. Further, let Lemma 3.1.4, Lemma 3.1.5 and Lemma 2.3.3 hold. Then*

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\| \leq 2\varphi_2(s,a,\beta,h)\alpha^{\frac{-a}{(1+\beta)a+2s}}(\delta + \varepsilon_h) + \psi_1(s,a,\beta,t)\alpha^{\frac{t}{(1+\beta)a+2s}}, \quad (3.1.24)$$

where $\varphi_2(s,a,\beta,h) = \max\{\varphi(s,a,\beta,h), \varphi_1(s,a,\beta,h)\}$. In particular, if $\alpha := \alpha(s,a,\beta,t) = c_0(\delta+\varepsilon_h)^{\frac{(1+\beta)a+2s}{t+a}}$ for some $c_0 > 0$, then

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\| \leq \eta(s,a,\beta,t)(\delta+\varepsilon_h)^{\frac{t}{t+a}},$$

where $\eta(s,a,\beta,t) = \max\{\varphi(s,a,\beta)c_0^{\frac{-a}{(1+\beta)a+2s}}, \psi_1(s,a,\beta,t)c_0^{\frac{t}{(1+\beta)a+2s}}\}$.

□

3.2 PARAMETER CHOICE STRATEGY

In Chapter 2, we considered the following parameter choice rule, i.e. choose α satisfying

$$\|\alpha^2(A_{s,\beta} + \alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s} = c\delta \quad (3.2.25)$$

where $c > 0$ is a constant for FTRM in Hilbert scales. In this section, we study the finite dimensional version of the parameter choice strategy (3.2.25). Let

$$\phi(\alpha, y^\delta, h) = \|\alpha^2(A_{s,\beta,h} + \alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s}. \quad (3.2.26)$$

The proof of the following theorem is analogous to the proof of the theorem in Chapter 2, but for the sake of completion we give the proof as well.

THEOREM 3.2.1. *For each non-zero y^δ, the function $\alpha \longrightarrow \phi(\alpha, y^\delta, h)$ for $\alpha > 0$, as defined in (3.2.26), is continuous and increasing. In addition*

$$\lim_{\alpha\to 0}\phi(\alpha, y^\delta, h) = 0, \lim_{\alpha\to\infty}\phi(\alpha, y^\delta, h) = \|(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a}. \quad (3.2.27)$$

Proof. Let $\{E_\lambda : 0 \le \lambda \le \|A_{s,\beta,h}\|\}$ be the spectral family of $A_{s,\beta,h}$. Then

$$\phi(\alpha, y^\delta, h)^2 = \int_0^{\|A_{s,\beta,h}\|} \left(\frac{\alpha}{\lambda+\alpha}\right)^4 d\langle E_\lambda L^{-s}(T_h^*T_h)^{\beta/2}y^\delta, L^{-s}(T_h^*T_h)^{\beta/2}y^\delta\rangle_{\beta a+s}. \tag{3.2.28}$$

Now since $\alpha \longrightarrow \left(\frac{\alpha}{\lambda+\alpha}\right)^4$ for $\lambda > 0$ is strictly increasing, $\lim_{\alpha\to 0}\left(\frac{\alpha}{\lambda+\alpha}\right)^4 = 0$ and $\lim_{\alpha\to\infty}\left(\frac{\alpha}{\lambda+\alpha}\right)^4 = 1$, by Dominated Convergence Theorem, we have

$$\lim_{\alpha\to 0}\phi(\alpha, y^\delta, h) = 0, \lim_{\alpha\to\infty}\phi(\alpha, y^\delta, h) = \|(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a}. \tag{3.2.29}$$

□

THEOREM 3.2.2. *Suppose (2.1.2) holds and*

$$\|(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a} \ge c\delta + d\epsilon_h > 0 \tag{3.2.30}$$

for some $c > 0$ and $d > 0$. Then there exists a unique $\alpha = \alpha(\delta, h)$ satisfying

$$\phi(\alpha, y^\delta, h) = c\delta + d\epsilon_h. \tag{3.2.31}$$

Proof. Follows from Intermediate Value Theorem and Theorem 3.2.2. □

REMARK 3.2.3. *Note that, by (2.1.2) and Proposition 3.1.2, we have*

$$\|y^\delta\| = \|(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\| \le \bar{g}(-\beta)\|(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a}. \tag{3.2.32}$$

Now, since $\|y^\delta\| \ge \|y\| - \delta$, if

$$\delta \le \frac{\|y\|}{\bar{g}(-\beta)(c + d\epsilon_h/\delta) + 1}, \tag{3.2.33}$$

then equation (3.2.30) is satisfied.

For convenience we use the following notations

$$
\begin{aligned}
c_1 &= \frac{\bar{b}_1^{\frac{-(a+s)(1+\beta)}{(1+\beta)a+2s}} b_1^{\frac{-t}{a}} 2a}{\bar{b}_2^{-1}((\beta-1)a+t)} \|\hat{x}\|_t \begin{cases} (\frac{b_2}{b_1})^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } t \le s \\ (\frac{b_1}{b_2})^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } s < t \end{cases} \\
c_2 &= \frac{\bar{b}_1^{\frac{-(1+\beta)(a+s)}{(1+\beta)a+2s}} b_2^{\frac{(1+\beta)s}{(1+\beta)a+2s}} 2}{\bar{b}_2^{-1} b_1^{\frac{(1+\beta)s}{(1+\beta)a+2s}} (1-\beta)} \|\hat{x}\| \\
c_3 &= \frac{b_1^{\frac{-t}{a}} 2a}{((\beta-1)a+t)} \|\hat{x}\|_t \begin{cases} (\frac{1}{b_1})^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } t \le s \\ (\frac{1}{b_2})^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } s < t \end{cases} \\
c_4 &= \frac{2}{b_1^{\frac{(1+\beta)s}{(1+\beta)a+2s}} (1-\beta)} \|\hat{x}\| \\
\tilde{C}_1 &= \frac{\sin \pi (\frac{\beta}{2})}{\pi \bar{b}_2^{-\beta}} \left(b_1^{\frac{-t}{a}} \frac{2a}{t} E + 2\|y\| \right) \\
Ç_1 &= (\frac{\bar{b}_1}{\bar{b}_2})^{\frac{-(1+\beta)(\beta a+s)}{(1+\beta)a+2s}} \tilde{C}_1 \\
Ç_2 &= \frac{\sin(\frac{1+\beta}{2})}{\pi} (c_1 + c_2) c_{(\beta-1)a,0}
\end{aligned}
$$

$$
Ç_3 = \frac{\bar{b}_1^{\frac{-(1+\beta)(a+s)}{(1+\beta)a+2s}} \sin \pi (\frac{1+\beta}{2}) b_2^{\frac{(1+\beta)(1-\beta)a}{(1+\beta)a+2s}}}{\bar{b}_2^{-1} \pi} c_{(\beta-1)a,0} (c_3 + c_4) \begin{cases} (b_2)^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } t \le s \\ (b_1)^{\frac{(1+\beta)(s-t)}{(1+\beta)a+2s}}, & \text{if } s < t. \end{cases}
$$

and

$$
Ç_4 = \bar{b}_2^{\frac{(1+\beta)(\beta a+s)}{(1+\beta)a+2s}} (Ç_2 + Ç_3). \tag{3.2.34}
$$

LEMMA 3.2.4. *Suppose Assumption 2.3.2 holds and $\alpha := \alpha(\delta) > 0$ is the unique solution of (3.2.31) with $c \ge c_0$ and $d \ge d_0$ where $c_0 = \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}$, $d_0 = Ç_1 + Ç_4$. Then,*

$$
\alpha \ge c_{\beta,a,s,h} (\delta + \epsilon_h)^{\frac{(1+\beta)a+2s}{a+t}} \tag{3.2.35}
$$

where $c_{\beta,a,s,h} = \frac{\bar{G}\left(\frac{-(\beta a+s)}{(1+\beta)a+2s}\right) G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-(\beta a+s)}{(1+\beta)a+2s}\right) \bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)} E \min\{c - c_0, d - d_0\}.$

Proof. Note that, from Proposition 3.1.2, Proposition 3.1.3, (3.1.10) and

(3.2.26) we have

$$\begin{aligned}
c\delta + d\epsilon_h &= \phi(\alpha, y^\delta, h) \\
&\leq \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\| \\
&\leq \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y^\delta - y)\| \\
&+ \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\| \\
&\leq \frac{\bar{G}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\frac{\|y^\delta - y\|}{\bar{f}(-\beta)} \\
&\quad + \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\| \\
&\leq \frac{\bar{G}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})\bar{f}(-\beta)}\delta + G, \qquad (3.2.36)
\end{aligned}$$

where $G = \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|$. Note that

$$\begin{aligned}
G &\leq \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}[(T_h^*T_h)^{\frac{\beta}{2}} - (T^*T)^{\frac{\beta}{2}}]y\| \\
&\quad + \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}\alpha^2\|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y\| \\
&\leq \frac{\bar{G}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}G_1 + \frac{1}{\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}G_2, \qquad (3.2.37)
\end{aligned}$$

where $G_1 = \|[(T_h^*T_h)^{\frac{\beta}{2}} - (T^*T)^{\frac{\beta}{2}}]y\|_{\beta a}$ and
$G_2 = \alpha^2\|(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|$.

Using the fomula (3.1.13) and Proposition 3.1.2, we have

$$G_1 \leq \frac{1}{\bar{f}(-\beta)}\|(T_h^*T_h)^{\frac{-\beta}{2}}[(T_h^*T_h)^{\frac{\beta}{2}} - (T^*T)^{\frac{\beta}{2}}]y\|$$

$$= \frac{\sin\pi(\frac{\beta}{2})}{\bar{f}(-\beta)\pi}$$
$$\|\int_0^\infty \lambda^{\frac{\beta}{2}}(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}[(T^*T)-(T_h^*T_h)](T^*T+\lambda I)^{-1}yd\lambda\|$$
$$\leq \frac{\sin\pi(\frac{\beta}{2})}{\bar{f}(-\beta)\pi}$$
$$\|\int_0^\infty \lambda^{\frac{\beta}{2}}(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}[T_h^*(T-T_h)+(T^*-T_h^*)T](T^*T+\lambda I)^{-1}T\hat{x}d\lambda\|$$
$$\leq \frac{\sin\pi(\frac{\beta}{2})}{\bar{f}(-\beta)\pi}\|\int_0^\infty \lambda^{\frac{\beta}{2}}(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}[T_h^*(T-T_h)](T^*T+\lambda I)^{-1}T\hat{x}d\lambda\|,$$

where (here and below) we used $(T_h^*T_h+\lambda I)^{-1}\equiv(T_h^*T_h+\lambda P_h)^{-1}\equiv(T_h^*T_h+\lambda)^{-1}P_h$ and $(T_h^*T_h+\lambda)^{-1}P_h[T^*-T_h^*]=(T_h^*T_h+\lambda I)^{-1}P_h[I-P_h]T^*=0$. Observe that

$$\begin{aligned}
&\|\int_0^\infty \lambda^{\frac{\beta}{2}}(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}T\hat{x}d\lambda\|\\
&\leq \|\int_0^1 \lambda^{\frac{\beta}{2}}(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}T\hat{x}d\lambda\|\\
&+\|\int_1^\infty \lambda^{\frac{\beta}{2}}(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}T\hat{x}d\lambda\|\\
&=\Gamma_1+\Gamma_2
\end{aligned} \tag{3.2.38}$$

where $\Gamma_1 = \|\int_0^1 \lambda^{\frac{\beta}{2}}(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}T\hat{x}d\lambda\|$ and $\Gamma_2 = \|\int_1^\infty \lambda^{\frac{\beta}{2}}(T_h^*T_h)^{\frac{-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}T\hat{x}d\lambda\|$. Further, note that

$$\begin{aligned}
\Gamma_1 &\leq \int_0^1 \lambda^{\frac{\beta}{2}}\|(T_h^*T_h)^{\frac{1-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}\|\|[T-T_h]\|\\
&\quad \|(T^*T+\lambda I)^{-1}(T^*T)^{(\frac{t}{2a}+\frac{1}{2})}\|\|(T^*T)^{\frac{-t}{2a}}\hat{x}\|d\lambda\\
&\leq g(\frac{-t}{a})\epsilon_h\int_0^1 \lambda^{\frac{t}{2a}-1}\|\hat{x}\|_t d\lambda\\
&= g(\frac{-t}{a})\frac{2a}{t}E\epsilon_h
\end{aligned} \tag{3.2.39}$$

and

$$\begin{aligned}
\Gamma_2 &\leq \int_1^\infty \lambda^{\frac{\beta}{2}}\|(T_h^*T_h)^{\frac{1-\beta}{2}}(T_h^*T_h+\lambda I)^{-1}\|\\
&\quad \|[T-T_h]\|\|(T^*T+\lambda I)^{-1}\|\|(T^*T)^{\frac{1}{2}}\hat{x}\|d\lambda\\
&\leq \int_1^\infty \frac{\lambda^{\frac{\beta}{2}}}{\lambda^{2-\frac{1}{2}+\frac{\beta}{2}}}\epsilon_h\|y\|d\lambda\\
&\leq 2\|y\|\epsilon_h.
\end{aligned} \tag{3.2.40}$$

Therefore, by (3.2.38), (3.2.39) and (3.2.40), we have

$$G_1 \leq \tilde{C}_1 \epsilon_h. \tag{3.2.41}$$

Again, we have

$$\begin{aligned} G_2 &\leq \alpha^2 \| A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} [(A_{s,\beta,h}+\alpha I)^{-2} - (A_{s,\beta}+\alpha I)^{-2}] L^{-s} (T^*T)^{\frac{\beta}{2}} y \| \\ &\quad + \alpha^2 \| A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta}+\alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y \|. \\ &=: G_{21} + G_{22}, \end{aligned} \tag{3.2.42}$$

where $G_{21} = \alpha^2 \| A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} [(A_{s,\beta,h}+\alpha I)^{-2} - (A_{s,\beta}+\alpha I)^{-2}] L^{-s} (T^*T)^{\frac{\beta}{2}} y \|$ and $G_{22} = \alpha^2 \| A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta}+\alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y \|$. Note that

$$\begin{aligned} G_{21} &= \| \alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta,h}+\alpha I)^{-2} [A_{s,\beta}^2 - A_{s,\beta,h}^2 + 2\alpha (A_{s,\beta} - A_{s,\beta,h})] \\ &\quad \times (A_{s,\beta}+\alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y \| \\ &\leq \| \alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta,h}+\alpha I)^{-1} (A_{s,\beta} - A_{s,\beta,h}) (A_{s,\beta}+\alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y \| \\ &\quad + \| \alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta,h}+\alpha I)^{-2} (A_{s,\beta} - A_{s,\beta,h}) (A_{s,\beta}+\alpha I)^{-1} L^{-s} (T^*T)^{\frac{\beta}{2}} y \| \\ &=: \Gamma_3 + \Gamma_4 \end{aligned} \tag{3.2.43}$$

where $\Gamma_3 = \| \alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta,h}+\alpha I)^{-1} (A_{s,\beta} - A_{s,\beta,h}) (A_{s,\beta}+\alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y \|$ and $\Gamma_4 = \| \alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta,h}+\alpha I)^{-2} (A_{s,\beta} - A_{s,\beta,h}) (A_{s,\beta}+\alpha I)^{-1} L^{-s} (T^*T)^{\frac{\beta}{2}} y \|$. Let $Z = (A_{s,\beta}+\alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y$. Then

$$\begin{aligned} \Gamma_3 &= \alpha^2 \| (A_{s,\beta,h}+\alpha I)^{-1} A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta} - A_{s,\beta,h}) Z \| \\ &= \alpha^2 \| (A_{s,\beta,h}+\alpha I)^{-1} A_{s,\beta,h}^{\frac{(1-\beta)a}{(1+\beta)a+2s}} A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}} L^{-s} [(T^*T)^{\frac{1+\beta}{2}} - (T_h^*T_h)^{\frac{1+\beta}{2}}] L^{-s} Z \| \\ &\leq \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \frac{\sin \pi (\frac{1+\beta}{2})}{\pi} \\ &\quad \times \| A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}} L^{-s} \int_0^\infty \lambda^{\frac{1+\beta}{2}} (T_h^*T_h + \lambda I)^{-1} [T^*T - T_h^*T_h] (T^*T + \lambda I)^{-1} L^{-s} Z \| \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \\
&\quad \|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}}L^{-s}\int_0^1 \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Zd\lambda\| \\
&+ \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1}\|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}}L^{-s} \\
&\quad \int_1^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Zd\lambda\| \\
&= \frac{\sin\pi(\frac{1+\beta}{2})}{\pi}(\Gamma_5+\Gamma_6) \qquad (3.2.44)
\end{aligned}
$$

where $\Gamma_5 = \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1}\|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}}L^{-s}\int_0^1 \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Zd\lambda\|$
and
$\Gamma_6 = \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1}\|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}}L^{-s}\int_1^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Zd\lambda\|$.
Using Proposition 3.1.2 and Proposition 3.1.3, we have

$$
\begin{aligned}
\Gamma_5 &\leq \bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \\
&\quad \|\int_0^1 \lambda^{\frac{1+\beta}{2}}(T_h^*T_h)^{\frac{1}{2}}(T_h^*T_h+\lambda I)^{-1}[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Zd\lambda\|_a \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \\
&\quad \times\int_0^1 \frac{\lambda^{\frac{1+\beta}{2}}}{\lambda^{2-\frac{t}{2a}}}\epsilon_h\|(T^*T)^{\frac{-t}{2a}}L^{-s}(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}T\hat{x}\|d\lambda \;\; \text{similar to (3.1.20)} \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)g(\frac{-t}{a})}{\bar{f}(-1)F\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}\frac{2a}{(\beta-1)a+t}\epsilon_h\|A_{s,\beta}^{\frac{(1-\beta)a}{(1+\beta)a+2s}}A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}}L^s\hat{x}\| \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)g(\frac{-t}{a})G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)F\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}\frac{2a}{(\beta-1)a+t}c_{(\beta-1)a,0}\epsilon_h\|\hat{x}\|_t \qquad (3.2.45)
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_6 &\leq \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1}\|A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}}L^{-s} \\
&\quad \int_1^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}T_h^*[T-T_h](T^*T+\lambda I)^{-1}L^{-s}Zd\lambda\| \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \\
&\quad \int_1^\infty \lambda^{\frac{1+\beta}{2}}\|(T_h^*T_h+\lambda I)^{-1}\|\|[T-T_h]\|\|(T^*T+\lambda I)^{-1}\|\|L^{-s}Z\|d\lambda. \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)F\left(\frac{2s}{(1+\beta)a+2s}\right)}\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}+1} \\
&\quad \int_1^\infty \lambda^{\frac{-3}{2}+\frac{\beta}{2}}\epsilon_h\|A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}}(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta+1}{2}}L^{-s}L^s\hat{x}\|d\lambda \\
&\leq \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)G\left(\frac{2s}{(1+\beta)a+2s}\right)}{\bar{f}(-1)F\left(\frac{2s}{(1+\beta)a+2s}\right)}\left(\frac{2}{1-\beta}\right)c_{(\beta-1)a,0}\|\hat{x}\|\epsilon_h. \qquad (3.2.46)
\end{aligned}
$$

Therefore, by (3.2.44),(3.2.45) and (3.2.46) we have

$$
\Gamma_3 \leq Ç_2\epsilon_h. \qquad (3.2.47)
$$

Similarly, we have for $Z_1 = (A_{s,\beta}+\alpha I)^{-1}L^{-s}(T^*T)^{\frac{\beta}{2}}y$,

$$
\begin{aligned}
\Gamma_4 &= \|\alpha^2 A_{s,\beta,h}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}}(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}[(T^*T)^{(\frac{1+\beta}{2})}-(T_h^*T_h)^{(\frac{1+\beta}{2})}]L^{-s}Z_1\| \\
&\leq \|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{(1-\beta)a}{(1+\beta)a+2s}}A_{s,\beta,h}^{\frac{-(a+s)}{(1+\beta)a+2s}}L^{-s}[(T_h^*T_h)^{(\frac{1+\beta}{2})}-(T^*T)^{(\frac{1+\beta}{2})}]L^{-s}Z_1\| \\
&\leq \bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)\alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times\|\frac{\sin\pi(\frac{1+\beta}{2})}{\pi}\int_0^\infty \lambda^{\frac{1+\beta}{2}}(T_h^*T_h+\lambda I)^{-1}[T^*T-T_h^*T_h](T^*T+\lambda I)^{-1}L^{-s}Z_1d\lambda\|_a
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)} \frac{\sin\pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \| \int_0^1 \lambda^{\frac{1+\beta}{2}} (T_h^* T_h + \lambda I)^{-1} [T - T_h] (T^*T + \lambda I)^{-1} L^{-s} Z_1 d\lambda \| \\
&\quad + \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)} \frac{\sin\pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \| \int_1^\infty \lambda^{\frac{1+\beta}{2}} (T_h^* T_h + \lambda I)^{-1} [T - T_h] (T^*T + \lambda I)^{-1} L^{-s} Z d\lambda \| \\
&\le \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)} \frac{\sin\pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \int_0^1 \frac{\lambda^{\frac{1+\beta}{2}}}{\lambda^{2-\frac{t}{2a}}} \epsilon_h \| (T^*T)^{\frac{-t}{2a}} L^{-s} (A_{s,\beta} + \alpha I)^{-1} L^{-s} (T^*T)^{\frac{\beta+1}{2}} L^{-s} L^s \hat{x} \| d\lambda \\
&\quad + \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)} \frac{\sin\pi(\frac{1+\beta}{2})}{\pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \int_1^\infty \frac{\lambda^{\frac{1+\beta}{2}}}{\lambda^2} \epsilon_h \| L^{-s} (A_{s,\beta} + \alpha I)^{-1} L^{-s} (T^*T)^{\frac{\beta}{2}} (T^*T)^{\frac{1}{2}} L^{-s} L^s \hat{x} \| d\lambda \\
&\le \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) \sin\pi(\frac{1+\beta}{2})}{\bar{f}(-1) F\left(\frac{2(s-t)}{(1+\beta)a+2s}\right) \pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \int_0^1 \lambda^{\frac{-3+\beta}{2} + \frac{t}{2a}} \epsilon_h \| (A_{s,\beta} + \alpha I)^{-1} A_{s,\beta}^{\frac{2\beta a + 2s}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{(1-\beta)a}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} L^s \hat{x} \| d\lambda \\
&\quad + \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) \sin\pi(\frac{1+\beta}{2})}{\bar{f}(-1) F\left(\frac{2s}{(1+\beta)a+2s}\right) \pi} \alpha^{\frac{(1-\beta)a}{(1+\beta)a+2s}} \\
&\quad \times \int_1^\infty \lambda^{\frac{-3+\beta}{2}} \epsilon_h \| (A_{s,\beta} + \alpha I)^{-1} A_{s,\beta}^{\frac{2\beta a + 2s}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{(1-\beta)a}{(1+\beta)a+2s}} A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} L^s \hat{x} \| d\lambda \\
&\le \frac{\bar{G}\left(\frac{-2(a+s)}{(1+\beta)a+2s}\right) \sin\pi(\frac{1+\beta}{2}) G\left(\frac{2(1-\beta)a}{(1+\beta)a+2s}\right) G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{\bar{f}(-1)\pi} c_{(\beta-1)a,0} \\
&\quad \times (c_3 + c_4)\epsilon_h \\
&= Ç_3 \epsilon_h.
\end{aligned}
\tag{3.2.48}
$$

Therefore, by (3.2.43), (3.2.47) and (3.2.48), we have

$$
G_{21} \le (Ç_2 + Ç_3)\epsilon_h. \tag{3.2.49}
$$

Also, we have

$$
\begin{aligned}
G_{22} &\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\alpha^2\|(A_{s,\beta}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}T\hat{x}\|_{\beta a+s} \\
&= \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}\alpha^2\|(A_{s,\beta}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{a+s}{(1+\beta)a+2s}}L^s\hat{x}\| \\
&= \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}\alpha^2\|(A_{s,\beta}+\alpha I)^{-2}A_{s,\beta,h}^{\frac{a+t}{(1+\beta)a+2s}}A_{s,\beta,h}^{\frac{s-t}{(1+\beta)a+2s}}L^s\hat{x}\| \\
&\leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}\alpha^{\frac{a+t}{(1+\beta)a+2s}}\|\hat{x}\|_t. \qquad (3.2.50)
\end{aligned}
$$

Thus, we have

$$
G_2 \leq (Ç_1+Ç_4)\epsilon_h + \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}E\alpha^{\frac{a+t}{(1+\beta)a+2s}}. \qquad (3.2.51)
$$

Hence from (3.2.37), (3.2.41) and (3.2.51), we have

$$
(c-c_0)\delta+(d-d_0)\epsilon_h \leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right)}{F\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{F}(\frac{-2(\beta a+s)}{(1+\beta)a+2s})}E\alpha^{\frac{a+t}{(1+\beta)a+2s}}, \qquad (3.2.52)
$$

which implies that

$$
\alpha \geq c_{\beta,a,s,h}(\delta+\epsilon_h)^{\frac{(1+\beta)a+2s}{a+t}}. \qquad (3.2.53)
$$

□

THEOREM 3.2.5. *Suppose conditions in Lemma 3.1.5 hold with $c \geq c_0$ and $d \geq d_0$ and $\alpha := \alpha(\delta) > 0$ is the unique solution of (3.2.31). Then*

$$
\|\hat{x}-x_{\alpha,\beta,h}^{s,\delta}\| = O((\delta+\epsilon_h)^{\frac{t}{t+a}}). \qquad (3.2.54)
$$

Proof. As in Lemma 2.3.3 , we have

$$
\begin{aligned}
\hat{x}-x_{\alpha,\beta}^s &= \hat{x}-((T^*T)^{\frac{1+\beta}{2}}+\alpha L^{2s})^{-1}(T^*T)^{\frac{\beta}{2}}y \\
&= \alpha((T^*T)^{\frac{1+\beta}{2}}+\alpha L^{2s})^{-1}L^{2s}\hat{x} \\
&= \alpha L^{-s}(A_{s,\beta}+\alpha I)^{-1}L^s\hat{x}. \qquad (3.2.55)
\end{aligned}
$$

By Proposition 2.2.2 we have

$$\|\hat{x} - x^s_{\alpha,\beta}\| \leq \frac{1}{F\left(\frac{2s}{(1+\beta)a+2s}\right)} \|\alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}\|. \tag{3.2.56}$$

In order to proceed with obtaining an error estimate for $\|\alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}\|$ we make use of the following moment inequality

$$\|B^u z\| \leq \|B^v z\|^{u/v} \|z\|^{1-u/v} \quad ,0 \leq u \leq v \tag{3.2.57}$$

where B is a positive self-adjoint operator.
Let $u = \frac{t}{p}$,$v = \frac{t+a}{p}$,$B = \alpha A_{s,\beta}^{\frac{p}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1}$ and $z = \alpha^{1-\frac{t}{p}} (A_{s,\beta} + \alpha I)^{-1+\frac{t}{p}} A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}} L^s \hat{x}$ where $p = (\frac{1+\beta}{2})a + 2s$.
Then $B^u z = \alpha A_{s,\beta}^{\frac{s}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-1} L^s \hat{x}$. Also, from (3.2.58) we have

$$\begin{aligned} \|B^u z\| &\leq \|B^v z\|^{\frac{t}{t+a}} \|z\|^{\frac{a}{t+a}} \\ &= \|\alpha^{1+\frac{a}{p}} (A_{s,\beta} + \alpha I)^{-(1+\frac{a}{p})} A_{s,\beta}^{\frac{s+a}{(1+\beta)a+2s}} L^s \hat{x}\|^{\frac{t}{t+a}} \\ &\quad \times \|\alpha^{1-\frac{t}{p}} (A_{s,\beta} + \alpha I)^{-1+\frac{t}{p}} A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}} L^s \hat{x}\|^{\frac{t}{t+a}}. \end{aligned}$$

We also have

$$\begin{aligned} \|B^v z\| &= \|\alpha^{\frac{a}{p}-1} (A_{s,\beta} + \alpha I)^{-(\frac{a}{p}-1)} \alpha^2 (A_{s,\beta} + \alpha I)^{-2} A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} A_{s,\beta} L^s \hat{x}\| \\ &\leq \|\alpha^2 (A_{s,\beta} + \alpha I)^{-2} A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} A_{s,\beta} L^s \hat{x}\| \\ &\leq \|\alpha^2 A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta} + \alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y\| \\ &\leq \|\alpha^2 A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} [(A_{s,\beta} + \alpha I)^{-2} - (A_{s,\beta,h} + \alpha I)^{-2}] L^{-s} (T^*T)^{\frac{\beta}{2}} y\| \\ &\quad + \|\alpha^2 A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta,h} + \alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y\| \\ &= G_{21} + Ģ \leq (Ç_2 + Ç_3)\epsilon_h + Ģ \quad \text{by (3.2.49)} \end{aligned} \tag{3.2.58}$$

where $Ģ = \|\alpha^2 A_{s,\beta}^{\frac{-(\beta a+s)}{(1+\beta)a+2s}} (A_{s,\beta,h} + \alpha I)^{-2} L^{-s} (T^*T)^{\frac{\beta}{2}} y\|$. Here we make use of

$\|\alpha^{\frac{a}{p}-1}(A_{s,\beta}+\alpha I)^{-(\frac{a}{p}-1)}\| \leq 1$ ($\frac{a}{p} \geq 1$ by the condition $a \geq \frac{4s}{1-\beta}$). Further,

$$\begin{aligned}
Ç &\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right) \|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T^*T)^{\frac{\beta}{2}}y\|_{\beta a+s} \\
&\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right) [\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}[(T^*T)^{\frac{\beta}{2}} - (T_h^*T_h)^{\frac{\beta}{2}}]y\|_{\beta a+s} \\
&\quad + \|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|_{\beta a+s}] \\
&\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right) (G_1 + \|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|_{\beta a+s}) \\
&\leq \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right) \\
&\quad (\tilde{C}_1\epsilon_h + \|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|_{\beta a+s}). \qquad (3.2.59)
\end{aligned}$$

Note that

$$\begin{aligned}
&\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y\|_{\beta a+s} \\
\leq\ &\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}(y-y^\delta)\|_{\beta a+s} \\
&+\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s} \\
\leq\ &\frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}\delta + \phi(\alpha, y^\delta, h) \text{ (see (3.2.36)).} \qquad (3.2.60)
\end{aligned}$$

Therefore from (3.2.58), (3.2.59) and (3.2.60) we have

$$\begin{aligned}
\|B^v z\| &\leq (Ç_2+Ç_3)\epsilon_h + \bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\tilde{C}_1\epsilon_h + \\
&\quad \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}\delta + \phi(\alpha, y^\delta, h) \\
&\leq \frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)}\delta + c\delta + \left(Ç_2+Ç_3+\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\tilde{C}_1 + d\right)\epsilon_h \\
&\leq Ļ(\delta+\epsilon_h) \qquad (3.2.61)
\end{aligned}$$

where

$$Ļ = \max\left\{\frac{\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)}{\bar{F}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\bar{f}(-\beta)} + c,\ Ç_2+Ç_3+\bar{G}\left(\frac{-2(\beta a+s)}{(1+\beta)a+2s}\right)\tilde{C}_1 + d\right\}.$$

We further have

$$\begin{aligned}
\|z\| &\leq \|A_{s,\beta}^{\frac{s-t}{(1+\beta)a+2s}} L^s \hat{x}\| \\
&\leq G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right) \|L^s \hat{x}\|_{t-s} \\
&\leq G\left(\frac{2(s-t)}{(1+\beta)a+2s}\right) \|\hat{x}\|_t. \qquad (3.2.62)
\end{aligned}$$

Hence by (3.2.56)-(3.2.62), we see that

$$\|\hat{x} - x_{\alpha,\beta}^s\| = O((\delta + \epsilon_h)^{\frac{t}{t+a}}). \qquad (3.2.63)$$

In Lemma 3.2.4 we have $\alpha \geq c_{\beta,a,s,h}(\delta + \epsilon_h)^{\frac{(1+\beta)a+2s}{a+t}}$ which further implies that

$$\begin{aligned}
\frac{1}{\alpha} &\leq \frac{1}{c_{\beta,a,s,h}(\delta + \epsilon_h)^{\frac{(1+\beta)a+2s}{a+t}}} \\
\frac{\delta + \epsilon_h}{\alpha^{\frac{a}{(1+\beta)a+2s}}} &\leq \frac{\delta + \epsilon_h}{c_{\beta,a,s,h}(\delta + \epsilon_h)^{\frac{a}{a+t}}} \\
&= \frac{1}{c_{\beta,a,s,h}}(\delta + \epsilon_h)^{\frac{t}{a+t}}. \qquad (3.2.64)
\end{aligned}$$

Therefore by Lemma 3.1.4, Lemma 3.1.5 and equations (3.2.63) and (3.2.64), we have

$$\|\hat{x} - x_{\alpha,\beta,h}^{s,\delta}\| = O((\delta + \epsilon_h)^{\frac{t}{t+a}}). \qquad (3.2.65)$$

□

3.3 NUMERICAL EXAMPLES

In this Section, we consider an academic example for the numerical discussion to validate our theoretical results.

Example 3.1 Let

$$[Tx](s) := \int_0^1 k(s,t)x(t)dt = y(s),\ 0 \leq s \leq 1, \tag{3.3.1}$$

where $k(s,t) = (cos(s)+cos(t))^2(\frac{sin(u)}{u})^2$, $u = \pi(sin(s)+sin(t))$. We take $y = T\hat{x}$, where $\hat{x}$ is given by $\hat{x}(t) = 2exp(-6(t-0.8)^2) + exp(-2(t+0.5)^2)$. We have introduced the random noise level $\delta = 0.01$ and 0.001 in the exact data.

We consider the Hilbert scales generated by the linear operator L defined by

$$Lx := \sum_{j=1}^{\infty} j^2 \langle x, u_j \rangle u_j,$$

where $u_j(t) = \sqrt{2}\sin(j\pi t)$, $j \in \mathbb{N}$, with domain of L as

$$D(L) := \left\{ x \in L^2[0,1] : \sum_{j=1}^{\infty} j^4 |\langle x, u_j \rangle|^2 < \infty \right\}.$$

In this case the Hilbert scale $\{\mathcal{X}\}_s$ generated by L is given by

$$\begin{aligned} \mathcal{X}_s &= \{x \in L^2[0,1] : \sum_{j=1}^{\infty} j^{4s} |\langle x, u_j \rangle|^2 < \infty\} \\ &= \{x \in H^{2s}(0,1) : x^{(2l)}(0) = x^{(2l)}(1) = 0, l = 0, 1, \ldots \lceil \frac{s}{2} - \frac{1}{4} \rceil\}, \end{aligned} \tag{3.3.2}$$

where $\lceil p \rceil$ denote the greatest integer less than or equal to p, $s \in \mathbb{R}$, and H^s is the usual Sobolev space. Also, one can see that $H^0 = L^2[0,1]$, and for $s \in \mathbb{N}, H_s \subset H^s$. We have taken $s = 1/2$ in our computation. The constants a, b_1 and b_2 are given by $a = 1, b_1 = b_2 = \frac{1}{\pi}$.

We use a sequence of finite dimensional subspaces of (V_n) of $\mathcal{X}$ and P_h $(h = \frac{1}{n})$ denote the orthogonal projection on $\mathcal{X}$ with $R(P_h) = V_n$. We choose V_n as the linear span of $\{v_1, v_2, \ldots, v_n\}$ with $v_i, i = 1, 2, \ldots, n-1$ are linear splines (Schroter and Tautenhahn (1994)) defined by

$$v_i(t) = \begin{cases} nt + 1 - i & (i-1)h \le t \le ih \\ -nt + 1 + i & ih \le t \le (i+1)h \\ 0 & otherwise. \end{cases}$$

In this case the matrix corresponding to $T_h^* T_h$ is given by

$$M_h := (\langle Tv_i, Tv_j \rangle)_{i,j} = (\int_0^1 [Tv_i(s)][Tv_j(s)]ds)_{i,j}, \; i,j = 1,2,\ldots,n-1,$$

where

$$Tv_i(s) = \int_{(i-1)h}^{ih} k(s,t)(\frac{t}{h} + 1 - i)dt + \int_{ih}^{(i+1)h} k(s,t)(\frac{-t}{h} + 1 + i)dt.$$

The matrix corresponding to L^{2s} for $s = \frac{1}{2}$ is given by

$$B_h = (\langle Lv_i, v_j \rangle)_{i,j} \tag{3.3.3}$$

$$= (\sum_{m=1}^{\infty} m^2 \langle v_i, u_m \rangle \langle v_j, u_m \rangle)_{i,j} \;, i,j = 1,2,\ldots,n-1. \tag{3.3.4}$$

One can see (Schroter and Tautenhahn (1994)[Page 165])

$$B_h := \frac{1}{h} \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{bmatrix}.$$

We take the singular value decomposition (SVD) of M_h as

$$M_h = U\Sigma V^T,$$

where $U = [u_1, u_2, \ldots, u_n] \in \mathbb{R}^{n\times n}$ and $V = [v_1, v_2, \ldots, v_n] \in \mathbb{R}^{n\times n}$ are orthogonal matrices, and

$$\Sigma = diag[\lambda_1, \lambda_2, \ldots \lambda_n] \in \mathbb{R}^{n\times n},$$

are the singular values of M_h ordered as

$$\lambda_1 \geq \lambda_2 \geq \ldots \lambda_r > \lambda_{r+1} = \ldots \lambda_n = 0.$$

Substituting the SVD in (3.1.8) yields

$$x^{s,\delta}_{\alpha,\beta,h} = V(\Sigma^{1+\beta} + \alpha V^T B_h V)^{-1}\Sigma^\beta V^T y^\delta. \tag{3.3.5}$$

Therefore the error between the actual solution and regularised solution is given by

$$\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\| = \|\hat{x} - V(\Sigma^{1+\beta} + \alpha V^T B_h V)^{-1}\Sigma^\beta V^T y^\delta\|. \tag{3.3.6}$$

For a particular β we wish to find the value of α using the parameter choice strategy in (4.3.2) i.e. we want an α satisfying $\|\alpha^2(A_{s,\beta,h}+\alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s} = c\delta + d\epsilon_h$. We do this using Newton's method on the function $f(\alpha) = \|\alpha^2(A_{s,\beta,h} + \alpha I)^{-2}L^{-s}(T_h^*T_h)^{\frac{\beta}{2}}y^\delta\|_{\beta a+s} - (c\delta + d\epsilon_h)$ and find the value of α when $f(\alpha) = 0$. Substituting the SVD we have

$$f(\alpha) = \alpha^2\|B_h^{\frac{3s+\beta a}{2s}} V^2(\Sigma^{1+\beta} + \alpha V^T B_h V)^{-2} V^T B_h^{\frac{1}{2}}\Sigma^\beta V^T y^\delta\| - (c\delta + d\epsilon_h). \tag{3.3.7}$$

We observe the values of α and the error $\|\hat{x} - x^{s,\delta}_{\alpha,\beta,h}\|$ for different values of $\beta(0.1, 0.2....0.9, 1)$ with random noise levels $\delta = 0.01$ and $\delta = 0.001$.

Relative errors and α values are showcased in Table 3.1 for different values of β, n and δ. In Fig: 3.2 - Fig: 3.4 and Fig: 3.6 - Fig: 3.8, contains the computed solution (C.S) and exact solution for different values of β and in Fig: 3.1 and Fig: 3.5, the exact data and noise data are plotted.

REMARK 3.3.1. *From Table 3.1 , one can observe that the relative error $E_{\alpha,\beta}$ obtained using the finite dimensional realization of FTRM in Hilbert scales for fractional values of β is smaller than that when $\beta = 1$. However, one can observe that the relative error $E_{\alpha,\beta}$ decreases with β to a certain limit and then increases thereafter. This can also be observed from the Figures (Fig:3.1-Fig:3.8).*

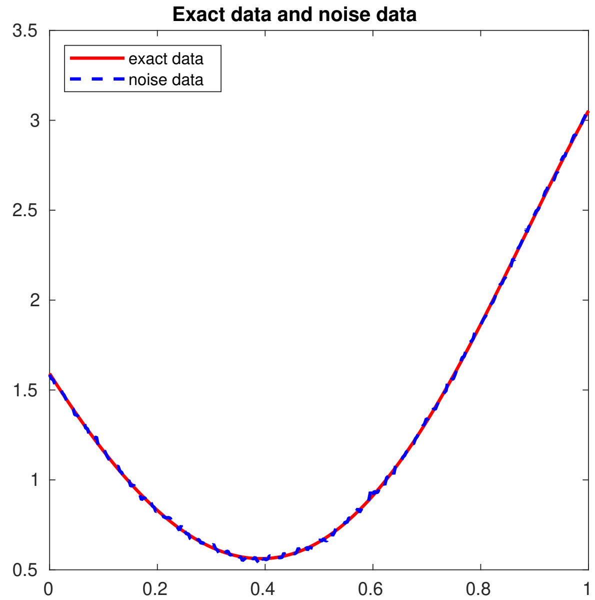

Figure 3.1: Exact data and noise data for $\delta = 0.01$ and $n = 300$

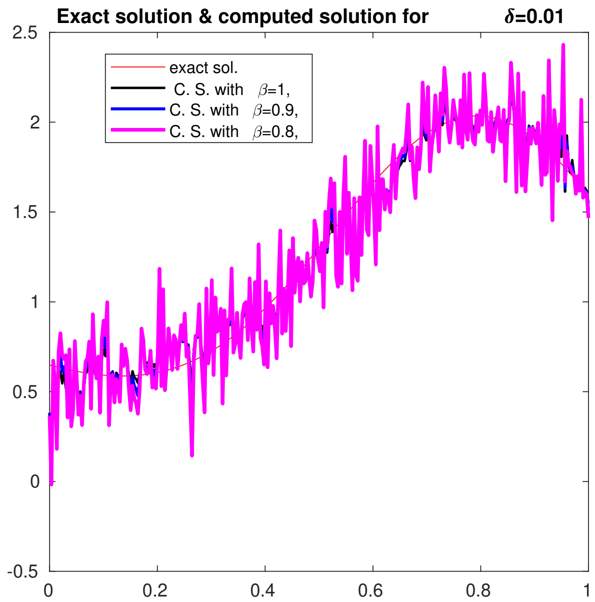

Figure 3.2: Solutions with $\delta = 0.01$ and $n = 300$

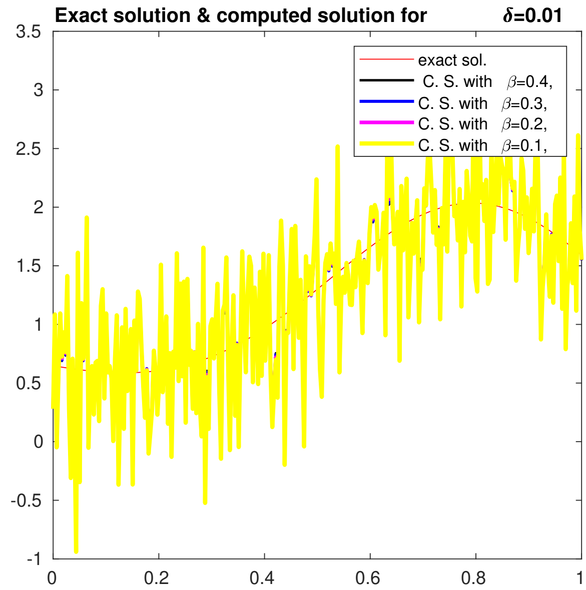

Figure 3.3: Solutions with $\delta = 0.01$ and $n = 300$

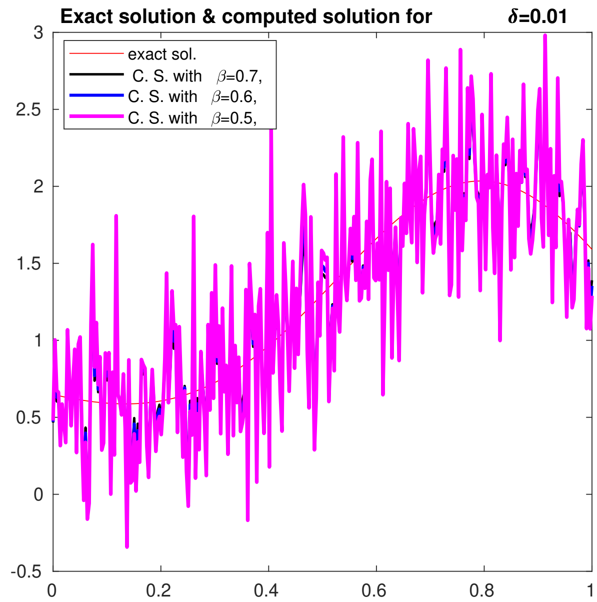

Figure 3.4: Solutions with $\delta = 0.01$ and $n = 300$

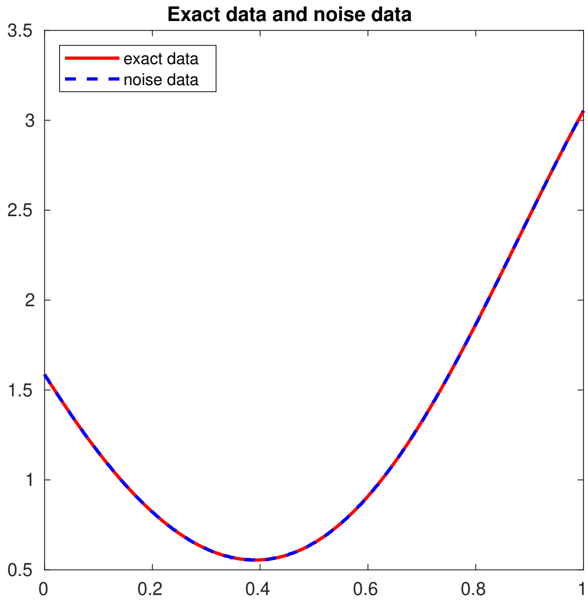

Figure 3.5: Exact data and noise data for $\delta = 0.001$ and $n = 500$

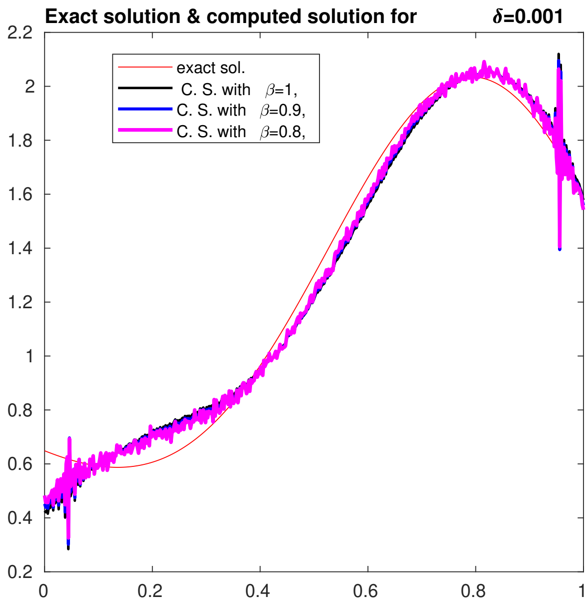

Figure 3.6: Solutions with $\delta = 0.001$ and $n = 500$

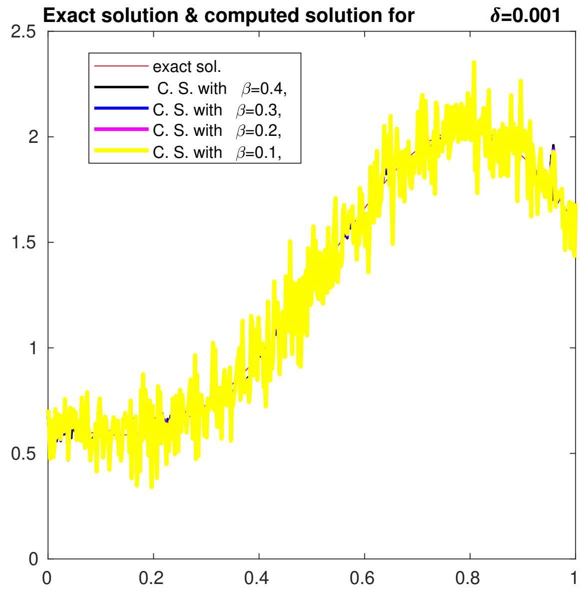

Figure 3.7: Solutions with $\delta = 0.001$ and $n = 500$

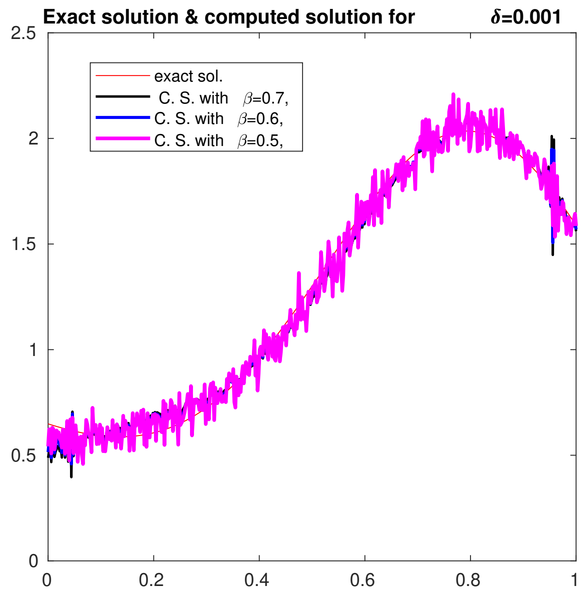

Figure 3.8: Solutions with $\delta = 0.001$ and $n = 500$

Table 3.1: Relative errors for the example.

β		$n=300$		$n=500$	
		$\delta=0.01$	$\delta=0.001$	$\delta=0.01$	$\delta=0.001$
1	α	2.157497e-03	2.111102e-03	2.359467e-03	2.045106e-03
	$E_{\alpha,\beta}$	8.086544e-02	5.580062e-02	6.979050e-02	5.984465e-02
	E_{ratio}	1.966310e+01	4.290699e+01	2.190834e+01	5.940711e+01
0.9	α	2.094812e-03	2.001696e-03	2.335022e-03	1.999903e-03
	$E_{\alpha,\beta}$	1.028586e-01	4.954536e-02	7.844515e-02	5.437206e-02
	E_{ratio}	2.501092e+01	3.809712e+01	2.462517e+01	5.397453e+01
0.8	α	2.002888e-03	1.838909e-03	2.289867e-03	1.925116e-03
	$E_{\alpha,\beta}$	1.423106e-01	4.333680e-02	1.017988e-01	4.881881e-02
	E_{ratio}	3.460401e+01	3.332314e+01	3.195624e+01	4.846188e+01
0.7	α	1.879297e-03	1.606383e-03	2.204036e-03	1.802906e-03
	$E_{\alpha,\beta}$	1.952871e-01	3.860271e-02	1.453034e-01	4.467454e-02
	E_{ratio}	4.748569e+01	2.968294e+01	4.561301e+01	4.434792e+01
0.6	α	2.149673e-03	1.296593e-03	2.073304e-03	1.610127e-03
	$E_{\alpha,\beta}$	2.373810e-01	3.835648e-02	2.223598e-01	4.313786e-02
	E_{ratio}	5.772115e+01	2.949360e+01	6.980223e+01	4.282247e+01
0.5	α	2.062660e-03	1.329601e-03	1.890583e-03	1.327042e-03
	$E_{\alpha,\beta}$	3.099169e-01	4.098500e-02	3.347035e-01	4.923642e-02
	E_{ratio}	7.535886e+01	3.151476e+01	1.050687e+02	4.887644e+01
0.4	α	2.526780e-03	1.333518e-03	2.112389e-03	1.327837e-03
	$E_{\alpha,\beta}$	3.677623e-01	4.264501e-02	4.479871e-01	5.680138e-02
	E_{ratio}	8.942446e+01	3.279120e+01	1.406302e+02	5.638609e+01
0.3	α	3.600039e-03	1.351594e-03	2.365584e-03	1.262176e-03
	$E_{\alpha,\beta}$	3.818477e-01	4.650115e-02	5.419249e-01	6.773099e-02
	E_{ratio}	9.284943e+01	3.575632e+01	1.701188e+02	6.723579e+01
0.2	α	4.540330e-03	1.472437e-03	2.741712e-03	1.168581e-03
	$E_{\alpha,\beta}$	4.142634e-01	4.931446e-02	6.257286e-01	7.788526e-02
	E_{ratio}	1.007316e+02	3.791956e+01	1.964261e+02	7.731582e+01
0.1	α	5.973373e-03	2.219342e-03	3.314624e-03	1.173023e-03
	$E_{\alpha,\beta}$	3.525842e-01	4.961016e-02	6.943178e-01	8.458691e-02
	E_{ratio}	8.573378e+01	3.814694e+01	2.179573e+02	8.396847e+01

Chapter 4

FRACTIONAL LAVRENTIEV REGULARIZATION METHOD IN HILBERT SCALES

4.1 INTRODUCTION

Let $A : \mathcal{X} \longrightarrow \mathcal{X}$ be injective, positive self-adjoint operator defined on a real Hilbert space $\mathcal{X}$. We are concerned with the problem of approximating a solution $\hat{x}$ (assumed to exist) of the ill-posed equation

$$Ax = y. \tag{4.1.1}$$

Our aim is to study finite dimensional realization of the fractional Lavrentiev regularization method for approximately solving (4.1.1) in the setting of the Hilbert scales $\{\mathcal{X}_s\}_{s\in\mathbb{R}}$ defined in the introduction. Note that the Hilbert scale generated by L connects $\mathcal{X}$ with $\mathcal{X}_s$ through the relation $\|x\|_s = \|x\|_{\mathcal{X}_s} = \|L^s x\|$ (Egger and Hofmann (2018), see also Kreĭn and Petunin (1966)[Page 145]). We assume throughout the study that, the operator A satisfies:

$$d_1\|x\|_{-a} \leq \|Ax\| \leq d_2\|x\|_{-a},\ x \in \mathcal{X} \tag{4.1.2}$$

for some $a > 0, d_1 > 0$ and $d_2 > 0$.

Let $f(t) := \min\{d_1^t, d_2^t\}$, $g(t) := \max\{d_1^t, d_2^t\}$, $t \in \mathbb{R}$ and $|t| \leq 1$.

We shall make use of the following proposition (George and Nair (1997)) in our analysis.

PROPOSITION 4.1.1. *(cf. Natterer (1984)[Proposition 1]) Let $A : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear operator that satisfies (4.1.2). Then, for $|\nu| \leq 1$,*

$$f(\nu)\|x\|_{-\nu a} \leq \|(A^*A)^{\nu/2}x\| \leq g(\nu)\|x\|_{-\nu a},\ x \in D((A^*A)^{\nu/2}).$$

For $|\tau| \leq 2$, let $F(t) := \min\{f(\frac{\tau}{2})^t, g(\frac{\tau}{2})^t\}$, $G(t) := \max\{f(\frac{\tau}{2})^t, g(\frac{\tau}{2})^t\}$. Using the above Proposition 4.1.1, and notation, we prove the following proposition, which will be used extensively in our analysis.

PROPOSITION 4.1.2. *Let $A : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear self-adjoint operator satisfying (4.1.2). Then, the following hold:*

(i)

$$f(\nu)\|x\|_{-\nu a} \leq \|A^\nu x\| \leq g(\nu)\|x\|_{-\nu a},\ x \in D(A^\nu),\ |\nu| \leq 1,$$

(ii)

$$f(\frac{\tau}{2})\|x\|_{-\frac{\tau a+s}{2}} \leq \|A^{\frac{\tau}{2}}L^{-s/2}x\| \leq g(\frac{\tau}{2})\|x\|_{-\frac{\tau a+s}{2}},\ x \in D(A^{\tau/2}L^{-s/2}),$$

$s > 0, |\tau| \leq 2,$

(iii)

$$F(\nu)\|x\|_{-\nu(\frac{\tau a+s}{2})} \leq \|(L^{-s/2}A^\tau L^{-s/2})^{\nu/2}x\| \leq G(\nu)\|x\|_{-\nu(\frac{\tau a+s}{2})},$$

$x \in D((L^{-s/2}A^\tau L^{-s/2})^{\nu/2}), s > 0, |\tau| \leq 2, |\nu| \leq 1.$

Proof. Proof of (i) follows from Proposition 4.1.1, since $A^*A = A^2$. Note that, if we take $\nu = \tau/2$ in (i), we obtain

$$f(\frac{\tau}{2})\|x\|_{-\frac{\tau a}{2}} \leq \|A^{\frac{\tau}{2}}x\| \leq g(\frac{\tau}{2})\|x\|_{-\frac{\tau a}{2}},\ x \in D(A^{\frac{\tau}{2}}).$$

Now, (ii) follows by taking $x = L^{-s/2}x$ in the above equation. The proof of (iii) follows by taking $A = A^{\tau/2}L^{-s/2}$ in Proposition 4.1.1.

□

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections of X onto $R(P_h)$, the range of P_h as we have seen in Chapter 3. For the results that follow, we impose the following conditions. Let

$$\epsilon_h := \|A(I - P_h)\|.$$

We assume that $\lim_{h \longrightarrow 0} \epsilon_h = 0$. The above assumption is satisfied if $P_h \to I$ point-wise and if A is a compact operator. Let $A_h := P_h A P_h$. Then

$$\|A_h - A\| \leq \|P_h A(P_h - I)\| + \|(P_h - I)A\| \leq 2\epsilon_h. \tag{4.1.3}$$

Let $h_0 > 0$ be such that

$$\epsilon_h \leq \frac{d_1 \|x\|_{-a}}{4\|x\|}, \text{ for all } x \neq 0 \text{ and } h \leq h_0. \tag{4.1.4}$$

Hereafter, we assume that $h \leq h_0$. Let $\bar{d}_1 = \frac{d_1}{2}$ and $\bar{d}_2 = d_2 + \frac{d_1}{2}$. Using, the above notation we have the following lemma;

LEMMA 4.1.3. *Let $\bar{d}_1$ and $\bar{d}_2$ be as above. Then,*

$$\bar{d}_1 \|x\|_{-a} \leq \|A_h x\| \leq \bar{d}_2 \|x||_{-a}. \tag{4.1.5}$$

Proof. Using (4.1.3) and (4.1.4), we have

$$\begin{aligned} \|A_h x\| &\leq \|Ax\| + \|(A_h - A)x\| \\ &\leq d_2 \|x\|_{-a} + 2\epsilon_h \|x\| \\ &\leq \bar{d}_2 \|x\|_{-a} \end{aligned}$$

and

$$\begin{aligned} \|A_h x\| &\geq \|Ax\| - \|(A_h - A)x\| \\ &\geq d_1 \|x\|_{-a} - 2\epsilon_h \|x\| \\ &\geq \bar{d}_1 \|x\|_{-a}. \end{aligned}$$

□

Let $\bar{f}(t) := \min\{\bar{d}_1^t, \bar{d}_2^t\}$, $\bar{g}(t) := \max\{\bar{d}_1^t, \bar{d}_2^t\}$, $t \in \mathbb{R}$ and $|t| \leq 1$. Analogously to the proof of (Natterer (1984)[Proposition 1]), one can prove the following proposition.

PROPOSITION 4.1.4. *Let $A_h : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear operator that satisfies (4.1.5). Then, for $|\nu| \leq 1$,*

$$\bar{f}(\nu) \|x\|_{-\nu a} \leq \|(A_h^* A_h)^{\nu/2} x\| \leq \bar{g}(\nu) \|x\|_{-\nu a}, \; x \in D((A_h^* A_h)^{\nu/2}).$$

For $|\tau| \leq 2$, let $\bar{F}(t) := \min\{\bar{f}(\frac{\tau}{2})^t, \bar{g}(\frac{\tau}{2})^t\}$, $\bar{G}(t) := \max\{\bar{f}(\frac{\tau}{2})^t, \bar{g}(\frac{\tau}{2})^t\}$. Using the above Proposition 4.1.4, and above notation, analogous to the proof of Proposition 4.1.2, one can prove the following proposition.

PROPOSITION 4.1.5. *Let $A_h : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear self-adjoint operator satisfying (4.1.5). Then, the following hold:*

(i)

$$\bar{f}(\nu)\|x\|_{-\nu a} \leq \|A_h^\nu x\| \leq \bar{g}(\nu)\|x\|_{-\nu a},\ x \in D(A_h^\nu),\ |\nu| \leq 1,$$

(ii)

$$\bar{f}(\frac{\tau}{2})\|x\|_{-\frac{\tau a+s}{2}} \leq \|A_h^{\frac{\tau}{2}} L^{-s/2}x\| \leq \bar{g}(\frac{\tau}{2})\|x\|_{-\frac{\tau a+s}{2}},\ x \in D(A_h^{\tau/2}L^{-s/2}),$$

$s > 0, |\tau| \leq 2,$

(iii)

$$\bar{F}(\nu)\|x\|_{-\nu(\frac{\tau a+s}{2})} \leq \|(L^{-s/2}A_h^\tau L^{-s/2})^{\nu/2}x\| \leq \bar{G}(\nu)\|x\|_{-\nu(\frac{\tau a+s}{2})},$$

$x \in D((L^{-s/2}A_h^\tau L^{-s/2})^{\nu/2}), s > 0, |\tau| \leq 2, |\nu| \leq 1.$

□

4.2 FRACTIONAL LAVRENTIEV REGULARIZATION IN HILBERT SCALES: FINITE DIMENSIONAL REALIZATION

In this Section, we introduce the fractional Lavrentiev regularization method for approximately solving the ill-posed operator equation (4.1.1). We consider the minimizer $w^s_{\alpha,\beta}$ of the functional

$$J^s_{\alpha,\beta}(x) = \langle Ax, x\rangle - 2\langle y, x\rangle + \alpha\langle A^\beta x, x\rangle_{\frac{s}{2}}, \quad \alpha > 0, \tag{4.2.1}$$

where $0 \leq \beta \leq 1$ (to be precised later), as an approximation for $\hat{x}$. Note that the minimizer $w^s_{\alpha,\beta}$ satisfies the equation

$$(A + \alpha A^\beta L^s)w^s_{\alpha,\beta} = y. \tag{4.2.2}$$

Note that, for $\beta = s = 0$, (4.2.2) is Lavrentiev regularization of (4.1.1). Throughout this study we assume that the available data y^δ satisfies (2.1.2). In this case instead of (4.2.2), we consider $w^{s,\delta}_{\alpha,\beta,h}$ satisfying the equation

$$(A_h + \alpha A_h^\beta L^s) w^{s,\delta}_{\alpha,\beta,h} = P_h y^\delta, \tag{4.2.3}$$

as an approximation for $\hat{x}$.

Let

$$A_{s,\beta} := L^{-s/2} A^{1-\beta} L^{-s/2},$$

and

$$A_{s,\beta,h} := L^{-s/2} A_h^{1-\beta} L^{-s/2}.$$

Then

$$w^s_{\alpha,\beta} = L^{-s/2}(A_{s,\beta} + \alpha I)^{-1} L^{-s/2} A^{-\beta} y, \tag{4.2.4}$$

$$w^s_{\alpha,\beta,h} := L^{-s/2}(A_{s,\beta,h} + \alpha I)^{-1} L^{-s/2} A_h^{-\beta} P_h y \tag{4.2.5}$$

and

$$w^{s,\delta}_{\alpha,\beta,h} = L^{-s/2}(A_{s,\beta,h} + \alpha I)^{-1} L^{-s/2} A_h^{-\beta} P_h y^\delta. \tag{4.2.6}$$

Furthermore, by spectral properties of the self-adjoint operators $A_{s,\beta}$, $A_{s,\beta,h}$, $s > 0$, we have

$$\|(A_{s,\beta} + \alpha I)^{-1} A^\mu_{s,\beta}\| \leq \alpha^{\mu-1}, \alpha > 0, \ 0 \leq \mu \leq 1 \tag{4.2.7}$$

and

$$\|(A_{s,\beta,h} + \alpha I)^{-1} A^\mu_{s,\beta,h}\| \leq \alpha^{\mu-1}, \alpha > 0, \ 0 \leq \mu \leq 1. \tag{4.2.8}$$

Next, we present the error estimate for $\|w^s_{\alpha,\beta,h} - w^{s,\delta}_{\alpha,\beta,h}\|$ using the above notation and propositions.

LEMMA 4.2.1. *Let $w^s_{\alpha,\beta,h}$ and $w^{s,\delta}_{\alpha,\beta,h}$ be as in (4.2.5) and (4.2.6), respectively. Let A satisfy (4.1.2) and (4.1.4) hold. Then, for $0 \leq \beta \leq \frac{2s+a}{3a}$, $s \leq a$ we have*

$$\|w^s_{\alpha,\beta,h} - w^{s,\delta}_{\alpha,\beta,h}\| \leq \varphi_3(s,a,\beta,h)\alpha^{\frac{-a}{(1-\beta)a+s}}\delta,$$

where $\varphi_3(s,a,\beta,h) := \frac{\bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)\bar{f}(\beta)}$.

Proof.

By (4.2.5) and (4.2.6), we have

$$\begin{aligned}\|w^{s,\delta}_{\alpha,\beta,h} - w^{s}_{\alpha,\beta,h}\| &= \|L^{-s/2}(A_{s,\beta,h}+\alpha I)^{-1}L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\| \\ &= \|(A_{s,\beta,h}+\alpha I)^{-1}L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\|_{-s/2}.\end{aligned}$$

Therefore, by Proposition 4.1.5, (iii); with $\nu = \frac{s}{(1-\beta)a+s}, \tau = 1-\beta$ and $x = (A_{s,\beta,h}+\alpha I)^{-1}L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)$ and (4.2.8) we obtain in turn that

$$\begin{aligned}&\|w^{s,\delta}_{\alpha,\beta,h} - w^{s}_{\alpha,\beta,h}\| \\ \leq\ & \frac{1}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)}\|A_{s,\beta,h}^{\frac{s}{2[(1-\beta)a+s]}}(A_{s,\beta,h}+\alpha I)^{-1}L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\| \\ =\ & \frac{1}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)} \\ & \times\|A_{s,\beta,h}^{\frac{s-\beta a}{(1-\beta)a+s}}(A_{s,\beta,h}+\alpha I)^{-1}A_{s,\beta,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\| \qquad (4.2.9) \\ \leq\ & \frac{1}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)}\|A_{s,\beta,h}^{\frac{s-\beta a}{(1-\beta)a+s}}(A_{s,\beta,h}+\alpha I)^{-1}\| \\ & \times\|A_{s,\beta,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\| \\ \leq\ & \frac{1}{\bar{F}\left(\frac{s}{(1-\beta)a+s}\right)}\alpha^{\frac{-a}{(1-\beta)a+s}}\|A_{s,\beta,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\|.\end{aligned}$$

So the lemma is proved, if we prove

$$\|A_{s,\beta,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\| \leq \frac{\bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{\bar{f}(\beta)}\delta. \qquad (4.2.10)$$

But this can be seen as follows; by taking $\nu = \frac{-(s-2\beta a)}{(1-\beta)a+s}, \tau = 1-\beta$ and $x =$

$L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)$ in Proposition 4.1.5; (iii), we obtain

$$
\begin{aligned}
& \|A_{s,\beta,h}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}} L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\| \\
\leq\ & \bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)\|L^{-s/2}A_h^{-\beta}P_h(y^\delta - y)\|_{s/2-\beta a} \\
=\ & \bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)\|A_h^{-\beta}P_h(y^\delta - y)\|_{-\beta a} \\
\leq\ & \frac{\bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{\bar{f}(\beta)}\|y^\delta - y\| \\
\leq\ & \frac{\bar{G}\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{\bar{f}(\beta)}\delta.
\end{aligned}
$$

The last but one step follows again from Proposition 4.1.5 (i), by taking $\nu = \beta$.

□

In order to obtain an estimate for $\|w^s_{\alpha,\beta,h} - w^s_{\alpha,\beta}\|$, we shall make use of the formula in (3.1.13). Taking $z = 1-\beta$ for $0 \leq \beta \leq 1$ and $B = A_h$ and $B = A$ in (3.1.13) and subtracting the two, for any $Z \in \mathcal{X}$ we get

$$[(A_h^2)^{\frac{1-\beta}{2}} - (A^2)^{\frac{1-\beta}{2}}]Z = \frac{\sin\pi(1-\beta)}{\pi}\int_0^\infty \lambda^{\frac{1-\beta}{2}}(A_h^2+\lambda I)^{-1}(A^2 - A_h^2)(A^2+\lambda I)^{-1}Z d\lambda \tag{4.2.11}$$

The following assumption is used to estimate $\|\hat{x} - w^s_{\alpha,\beta}\|$.

ASSUMPTION 4.2.2. *There exists some $E > 0$ and $0 < t \leq (1-\beta)\frac{a}{2} + s$ such that $\hat{x} \in M_{t,E} = \{x \in \mathcal{X} : \|x\|_t \leq E\}$.*

LEMMA 4.2.3. *Let $w^s_{\alpha,\beta,h}, w^s_{\alpha,\beta}$ be as in (4.2.5) and (4.2.4), respectively. Let A satisfy (4.1.2) and (4.1.4) hold. Then,*

$$\|w^s_{\alpha,\beta,h} - w^s_{\alpha,\beta}\| \leq \varphi_4(s,a,\beta,h)\alpha^{\frac{-a}{(1-\beta)a+s}}\varepsilon_h,$$

where $\varphi_4(s,a,\beta,h) := \frac{1}{\bar{F}(\frac{2s}{(1-\beta)a+s})}\bar{G}\left(\frac{2\beta a}{(1-\beta)a+s}\right)\frac{1}{\bar{f}(\beta)}\|\hat{x}\| + C_h$, with
$C_h = 8\frac{\bar{G}(\frac{-s+2\beta a}{(1-\beta)a+s})}{\bar{F}(\frac{s}{(1-\beta)a+s})\bar{f}(\beta)}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\left[\frac{a}{t}\frac{g(\frac{-t}{a})G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})}\|\hat{x}\|_t + 2\|A\|\frac{G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})}\|\hat{x}\|\right].$

Proof. Note that

$$\begin{aligned} w^s_{\alpha,\beta,h} &= (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h y \\ &= (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A\hat{x} \\ &= (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{1-\beta}\hat{x} + (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A(I-P_h)\hat{x}, \\ w^s_{\alpha,\beta} &= (A^{1-\beta} + \alpha L^s)^{-1} A^{-\beta} y \\ &= (A^{1-\beta} + \alpha L^s)^{-1} A^{1-\beta}\hat{x} \end{aligned}$$

and hence

$$\begin{aligned} w^s_{\alpha,\beta,h} - w^s_{\alpha,\beta} &= [(A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{1-\beta} - (A^{1-\beta} + \alpha L^s)^{-1} A^{1-\beta}]\hat{x} \\ &\quad + (A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A(I-P_h)\hat{x}. \end{aligned}$$

So

$$\|w^s_{\alpha,\beta,h} - w^s_{\alpha,\beta}\| \le \|Ł\| + \|(A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A(I-P_h)\hat{x}\|, \tag{4.2.12}$$

where $Ł = [(A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{1-\beta} - (A^{1-\beta} + \alpha L^s)^{-1} A^{1-\beta}]\hat{x}$.

Further, we have

$$\begin{aligned} & \|(A_h^{1-\beta} + \alpha L^s)^{-1} A_h^{-\beta} P_h A(I-P_h)\hat{x}\| \\ \le\ & \|L^{-\frac{s}{2}} (A_{s,\beta,h} + \alpha I)^{-1} L^{-\frac{s}{2}} A_h^{-\beta} P_h A(I-P_h)\hat{x}\| \\ \le\ & \frac{1}{\bar{F}(\frac{2s}{(1-\beta)a+s})} \|(A_{s,\beta,h} + \alpha I)^{-1} A_{s,\beta,h}^{\frac{s-\beta a}{(1-\beta)a+s}} A_{s,\beta,h}^{\frac{\beta a}{(1-\beta)a+s}} A_h^{-\beta} P_h A(I-P_h)\hat{x}\| \\ \le\ & \frac{1}{\bar{F}(\frac{2s}{(1-\beta)a+s})} \alpha^{\frac{-a}{(1-\beta)a+s}} \\ & \times \|A_{s,\beta,h}^{\frac{\beta a}{(1-\beta)a+s}} A_h^{-\beta} P_h A(I-P_h)\hat{x}\| \\ \le\ & \frac{1}{\bar{F}(\frac{2s}{(1-\beta)a+s})} \alpha^{\frac{-a}{(1-\beta)a+s}} \bar{G}\left(\frac{2\beta a}{(1-\beta)a+s}\right) \|A_h^{-\beta} P_h A(I-P_h)\hat{x}\|_{-\beta a} \\ \le\ & \frac{1}{\bar{F}(\frac{2s}{(1-\beta)a+s})} \alpha^{\frac{-a}{(1-\beta)a+s}} \bar{G}\left(\frac{2\beta a}{(1-\beta)a+s}\right) \frac{1}{\bar{f}(\beta)} \|P_h A(I-P_h)\hat{x}\| \\ \le\ & \frac{1}{\bar{F}(\frac{2s}{(1-\beta)a+s})} \bar{G}\left(\frac{2\beta a}{(1-\beta)a+s}\right) \frac{1}{\bar{f}(\beta)} \epsilon_h \|\hat{x}\| \alpha^{\frac{-a}{(1-\beta)a+s}}, \end{aligned} \tag{4.2.13}$$

and

$$
\begin{aligned}
Ł &= [L^{-s/2}(A_{s,\beta,h}+\alpha I)^{-1}A_{s,\beta,h} - L^{-s/2}(A_{s,\beta}+\alpha I)^{-1}A_{s,\beta}]L^{\frac{s}{2}}\hat{x} \\
&= L^{-s/2}(A_{s,\beta,h}+\alpha I)^{-1}[A_{s,\beta,h}(A_{s,\beta}+\alpha I) - (A_{s,\beta,h}+\alpha I)A_{s,\beta}](A_{s,\beta}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x} \\
&= L^{-s/2}(A_{s,\beta,h}+\alpha I)^{-1}\alpha[A_{s,\beta,h} - A_{s,\beta}](A_{s,\beta}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x} \\
&= L^{-s/2}(A_{s,\beta,h}+\alpha I)^{-1}\alpha[L^{-s/2}A_h^{1-\beta}L^{-s/2} - L^{-s/2}A^{1-\beta}L^{-s/2}](A_{s,\beta}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x} \\
&= L^{-s/2}(A_{s,\beta,h}+\alpha I)^{-1}L^{-s/2}(A_h^{1-\beta} - A^{1-\beta})\alpha L^{-\frac{s}{2}}(A_{s,\beta}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x} \\
&= L^{-s/2}(A_{s,\beta,h}+\alpha I)^{-1}L^{-s/2}((A_h^2)^{\frac{1-\beta}{2}} - (A^2)^{\frac{1-\beta}{2}})\alpha L^{-\frac{s}{2}}(A_{s,\beta}+\alpha I)^{-1}L^{\frac{s}{2}}\hat{x}
\end{aligned}
$$

so by (4.2.11), we have

$$
\begin{aligned}
Ł = &\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}L^{-s/2}(A_{s,\beta,h}+\alpha I)^{-1} \\
&\times \int_0^\infty \lambda^{\frac{1-\beta}{2}}L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}\alpha Z d\lambda.
\end{aligned}
$$

where $Z = L^{-s/2}(A_{s,\beta}+\alpha I)^{-1}L^{s/2}\hat{x}$. Therefore,

$$
\begin{aligned}
\|Ł\| &\le \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\|A_{s,\beta,h}^{\frac{s}{2[(1-\beta)a+s]}}(A_{s,\beta,h}+\alpha I)^{-1} \\
&\quad \times \int_0^\infty \lambda^{\frac{1-\beta}{2}}L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}\alpha Z\|d\lambda \\
&\le \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\alpha\|A_{s,\beta,h}^{\frac{2s-2\beta a}{2[(1-\beta)a+s]}}(A_{s,\beta,h}+\alpha I)^{-1} \\
&\quad \times \int_0^\infty \lambda^{\frac{1-\beta}{2}}A_{s,\beta,h}^{\frac{-s+2\beta a}{2[(1-\beta)a+s]}}L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}Z\|d\lambda \\
&\le \frac{1}{\bar{F}(\frac{s}{(1-\beta)a+s})}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\alpha^{\frac{2s-2\beta a}{2[(1-\beta)a+s]}} \\
&\quad \times \int_0^\infty \lambda^{\frac{1-\beta}{2}}\|A_{s,\beta,h}^{\frac{-s+2\beta a}{2[(1-\beta)a+s]}}L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}Z\|d\lambda \\
&\le \frac{\bar{G}(\frac{-s+2\beta a}{(1-\beta)a+s})}{\bar{F}(\frac{s}{(1-\beta)a+s})}\frac{\sin\pi(\frac{1-\beta}{2})}{\pi}\alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \\
&\quad \times \int_0^\infty \lambda^{\frac{1-\beta}{2}}\|L^{-s/2}(A_h^2+\lambda I)^{-1}(A^2-A_h^2)(A^2+\lambda I)^{-1}Z\|_{\frac{s}{2}-\beta a}\ d\lambda
\end{aligned}
$$

$$\leq \frac{\bar{G}(\frac{-s+2\beta a}{(1-\beta)a+s})}{\bar{F}(\frac{s}{(1-\beta)a+s})\bar{f}(\beta)} \frac{\sin\pi(\frac{1-\beta}{2})}{\pi} \alpha^{\frac{s-\beta a}{(1-\beta)a+s}}$$
$$\times \int_0^\infty \lambda^{\frac{1-\beta}{2}} \|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}[A_h(A-A_h)+(A-A_h)A](A^2+\lambda I)^{-1}Z\| d\lambda$$
$$\leq \frac{\bar{G}(\frac{-s+2\beta a}{(1-\beta)a+s})}{\bar{F}(\frac{s}{(1-\beta)a+s})\bar{f}(\beta)} \frac{\sin\pi(\frac{1-\beta}{2})}{\pi}[L_1+L_2] \tag{4.2.14}$$

where $L_1 = \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_0^\infty \lambda^{\frac{1-\beta}{2}} \|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}[A_h(A-A_h)](A^2+\lambda I)^{-1}Z\| d\lambda$ and $L_2 = \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_0^\infty \lambda^{\frac{1-\beta}{2}} \|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}[(A-A_h)A](A^2+\lambda I)^{-1}Z\| d\lambda$.

$$\begin{aligned}
L_1 &\leq \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_0^1 \lambda^{\frac{1-\beta}{2}} \|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}A_h\| \|(A-A_h)\| \|(A^2+\lambda I)^{-1}A^{\frac{t}{a}}\| \|A^{-\frac{t}{a}}Z\| d\lambda \\
&+ \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_1^\infty \lambda^{\frac{1-\beta}{2}} \|(A_h^2)^{\frac{\beta}{2}}(A_h^2+\lambda I)^{-1}\| \|A_h\| \|(A-A_h)\| \|(A^2+\lambda I)^{-1}Z\| d\lambda \\
&\leq \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_0^1 \lambda^{\frac{t}{2a}-1} 2\varepsilon_h \|A^{-\frac{t}{a}}Z\| d\lambda \\
&+ \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \int_1^\infty \frac{\|A_h\| 2\varepsilon_h \|Z\|}{\lambda^{\frac{3}{2}}} d\lambda \\
&\leq \alpha^{\frac{s-\beta a}{(1-\beta)a+s}} \left[\frac{2a}{t} 2\varepsilon_h \| \|A^{-\frac{t}{a}}Z\| + 4\|A_h\| 2\varepsilon_h \|Z\|\right].
\end{aligned} \tag{4.2.15}$$

Further, observe that

$$\begin{aligned}
\|A^{\frac{-t}{a}}Z\| &= \|A^{\frac{-t}{a}}L^{-s/2}(A_{s,\beta}+\alpha I)^{-1}L^{s/2}\hat{x}\| \\
&\leq g(\frac{-t}{a}) \|(A_{s,\beta}+\alpha I)^{-1}L^{s/2}\hat{x}\|_{t-\frac{s}{2}} \\
&\leq \frac{g(\frac{-t}{a})}{F(\frac{s-2t}{(1-\beta)a+s})} \|(A_{s,\beta}+\alpha I)^{-1}A_{s,\beta}^{\frac{s-2t}{2[(1-\beta)a+s]}} L^{s/2}\hat{x}\| \\
&\leq \frac{g(\frac{-t}{a})G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \alpha^{-1} \|\hat{x}\|_t
\end{aligned} \tag{4.2.16}$$

and

$$\begin{aligned}
\|Z\| &= \|L^{-s/2}(A_{s,\beta}+\alpha I)^{-1}L^{s/2}\hat{x}\| \\
&\leq \frac{G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \alpha^{-1} \|\hat{x}\|.
\end{aligned} \tag{4.2.17}$$

Therefore, from (4.2.15), (4.2.16) and (4.2.17) it follows that

$$L_1 \leq 4\left[\frac{a}{t} \frac{g(\frac{-t}{a})G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \|\hat{x}\|_t + 2\|A_h\| \frac{G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})} \|\hat{x}\|\right] \varepsilon_h \ \alpha^{\frac{-a}{(1-\beta)a+s}}. \tag{4.2.18}$$

Proceeding in a similar manner for L_2 we get

$$L_2 \leq 4\left[\frac{a}{t}\frac{g(\frac{-t}{a})G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})}\|\hat{x}\|_t + 2\|A\|\frac{G(\frac{s-2t}{(1-\beta)a+s})}{F(\frac{s-2t}{(1-\beta)a+s})}\|\hat{x}\|\right]\varepsilon_h \ \alpha^{\frac{-a}{(1-\beta)a+s}}. \quad (4.2.19)$$

The result, follows from (4.2.12), (4.2.13), (4.2.14) (4.2.18), (4.2.19) and the fact that $\|A_h\| \leq \|A\|$.

□

LEMMA 4.2.4. *Let $w^s_{\alpha,\beta}$ be as in (4.2.4), A satisfies (4.1.2) and suppose that Assumption 4.2.2 holds. Then for $0 < \beta \leq \frac{2s+a}{3a}$, $s \leq a$, we have*

$$\|\hat{x} - w^s_{\alpha,\beta}\| \leq \psi_2(s,a,\beta,t)\alpha^{\frac{t}{(1-\beta)a+s}},$$

where

$$\psi_2(s,a,\beta,t) := \frac{G\left(\frac{s-2t}{(1-\beta)a+s}\right)}{F\left(\frac{s}{(1-\beta)a+s}\right)}E.$$

Proof. By (4.2.4) and Assumption 4.2.2, we have in turn that

$$\begin{aligned}
\hat{x} - w^s_{\alpha,\beta} &= \hat{x} - (A^{1-\beta} + \alpha L^s)^{-1}A^{-\beta}y \\
&= \alpha(A^{1-\beta} + \alpha L^s)^{-1}L^s\hat{x} \\
&= \alpha L^{-s/2}(A_{s,\beta} + \alpha I)^{-1}L^{s/2}\hat{x},
\end{aligned}$$

that is

$$\|\hat{x} - w^s_{\alpha,\beta}\| = \alpha\|(A_{s,\beta} + \alpha I)^{-1}L^{s/2}\hat{x}\|_{-s/2}.$$

So, by Proposition 4.1.2, (iii) (by taking $\nu = \frac{s}{(1-\beta)a+s}$) and (4.2.7), we have

$$\begin{aligned}
\|\hat{x} - w^s_{\alpha,\beta}\| &\leq \frac{1}{F\left(\frac{s}{(1-\beta)a+s}\right)}\|\alpha A_{s,\beta}^{\frac{s}{2[(1-\beta)a+s]}}(A_{s,\beta} + \alpha I)^{-1}L^{s/2}\hat{x}\| \\
&\leq \frac{1}{F\left(\frac{s}{(1-\beta)a+s}\right)}\|\alpha A_{s,\beta}^{\frac{t}{(1-\beta)a+s}}(A_{s,\beta} + \alpha I)^{-1}\|\|A_{s,\beta}^{\frac{s-2t}{2[(1-\beta)a+s]}}L^{s/2}\hat{x}\| \\
&\leq \frac{G\left(\frac{s-2t}{(1-\beta)a+s}\right)}{F\left(\frac{s}{(1-\beta)a+s}\right)}\alpha^{\frac{t}{(1-\beta)a+s}}\|\hat{x}\|_t \\
&\leq \frac{G\left(\frac{s-2t}{(1-\beta)a+s}\right)}{F\left(\frac{s}{(1-\beta)a+s}\right)}\alpha^{\frac{t}{(1-\beta)a+s}}E.
\end{aligned}$$

□

Combining the Lemma 4.2.1, Lemma 4.2.3 and Lemma 4.2.4, we obtain the following theorem.

THEOREM 4.2.5. *Let $w^{s,\delta}_{\alpha,\beta,h}$ be as in (4.2.6), A satisfies (4.1.2) and suppose that Assumption 4.2.2 and (4.1.4) hold. Then for $0 \leq \beta \leq \frac{2s+a}{3a}$, $s \leq a$ we have*

$$\|\hat{x} - w^{s,\delta}_{\alpha,\beta,h}\| \leq \varphi_2(s,a,\beta,h)\alpha^{\frac{-a}{(1-\beta)a+s}}(\delta+\epsilon_h) + \psi_5(s,a,\beta,t)\alpha^{\frac{t}{(1-\beta)a+s}},$$

where $\varphi_5(s,a,\beta,h) = \max\{\varphi_4(s,a,\beta,h), \varphi_3(s,a,\beta,h)\}$. In particular, if $\alpha := \alpha(s,a,\beta,h,t) = c_0(\delta+\epsilon_h)^{\frac{(1-\beta)a+s}{t+a}}$ for some $c_0 > 0$, then

$$\|\hat{x} - w^{s,\delta}_{\alpha,\beta,h}\| \leq \eta_1(s,a,\beta,t)(\delta+\epsilon_h)^{\frac{t}{t+a}},$$

where $\eta_1(s,a,\beta,h,t) = \max\{\varphi_5(s,a,\beta,h)c_0^{\frac{-a}{(1-\beta)a+s}}, \psi_2(s,a,\beta,t)c_0^{\frac{t}{(1-\beta)a+s}}\}$.

□

4.2.1 ORDER OPTIMALITY

As in Micchelli and Rivlin (1977), we define the best possible worst error for identifying the solution $\hat{x}$ of (4.1.1) from $y^\delta \in \mathcal{X}$ satisfying (2.1.2) and $\hat{x}$ satisfying Assumption 4.2.2 as

$$\Theta(M_{t,E},\delta) = \inf_R \sup\{\|\hat{x} - Ry^\delta\| : \hat{x} \in M_{t,E}, y^\delta \in \mathcal{X}, \|A\hat{x} - y^\delta\| \leq \delta\}.$$

Here, the minimum is taken over all regularization methods $R : \mathcal{X} \longrightarrow \mathcal{X}$. Let

$$e(M_{t,E},\delta) := \sup\{\|x\| : x \in M_{t,E}, \|Ax\| \leq \delta\}.$$

Then, since $\mathcal{X}$ is Hilbert space and A is positive self-adjoint, we have (see Melkman and Micchelli (1979)) $e(M_{t,E},\delta) = \Theta(M_{t,E},\delta)$.

A regularization method R_α together with a parameter choice strategy $\alpha = \alpha(\delta)$ is said to be of optimal order if

$$\|R_\alpha y^\delta - \hat{x}\| = O(e(M_{t,E},\delta)).$$

Using the Interpolation Inequality (see Kreĭn and Petunin (1966))

$$\|x\|_s \leq \|x\|_r^{\theta}\|x\|_t^{1-\theta},\ x \in \mathcal{X}_t$$

where $r \leq s \leq t$ and $\theta = \frac{t-s}{t-r}$ with $r = -a$ and $s = 0$, we obtain

$$\begin{aligned}\|x\| &\leq \|x\|_{-a}^{\frac{t}{t+a}}\|x\|_t^{\frac{a}{t+a}} \\ &\leq \left(\frac{\|Ax\|}{d_1}\right)^{\frac{t}{t+a}}\|x\|_t^{\frac{a}{t+a}} \\ &\leq \left(\frac{\delta}{d_1}\right)^{\frac{t}{t+a}}\|x\|_t^{\frac{a}{t+a}},\ x \in M_{t,E},\end{aligned}$$

and the above estimate is sharp (c.f. Vainikko (1987)).

In view of the above observation, a regularization method is called optimal order yielding regularization method with respect to $M_{t,E}$ and (4.1.2), if it yields an approximation, say $R_\alpha y^\delta$ with $\|y - y^\delta\| \leq \delta$ and satisfies

$$\|R_\alpha y^\delta - \hat{x}\| = O(\delta^{\frac{t}{t+a}}).$$

Theorem 4.2.5, shows that we obtained the optimal order for the choice of $\alpha := \alpha(s, a, \beta, t) = c_0\delta^{\frac{(1-\beta)a+s}{t+a}}$ for some $c_0 > 0$.

4.3 STANDARD LAVRENTIEV METHOD VS FRACTIONAL LAVRENTIEV REGULARIZATION METHOD IN HILBERT SCALES

In this Section, we compare the filter factors (Hochstenbach et al. (2015)) of Lavrentiev regularization method and fractional Lavrentiev regularization method in the Hilbert scales. Recall (George and Nair (1997, 2003); George et al. (2013); Lu et al. (2010)), the Lavrentiev regularized solution for (4.1.1) in Hilbert scales is given by

$$w_\alpha^s = L^{-s/2}(L^{-s/2}AL^{-s/2} + \alpha I)^{-1}L^{-s/2}y. \tag{4.3.1}$$

So, using Proposition 4.1.2 (iii) with $\tau = 1$, we have

$$\begin{aligned}\|w_\alpha^s\| &\leq \frac{1}{F(\frac{s}{s+a})}\|(L^{-s/2}AL^{-s/2})^{\frac{s}{2(s+a)}}(L^{-s/2}AL^{-s/2}+\alpha I)^{-1}L^{-s/2}y\| \\ &\leq \frac{1}{F(\frac{s}{s+a})}\|(L^{-s/2}AL^{-s/2})^{\frac{s}{(s+a)}}(L^{-s/2}AL^{-s/2}+\alpha I)^{-1} \\ &\quad \times(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y\|\end{aligned}$$

and hence

$$\begin{aligned}\|w_\alpha^s\|^2 &\leq \frac{1}{F(\frac{s}{s+a})^2}\int_0^{\|L^{-s/2}AL^{-s/2}\|}\left(\frac{\lambda^{\frac{s}{(s+a)}}}{\lambda+\alpha}\right)^2 \qquad (4.3.2) \\ &\quad \times d\langle E_\lambda(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y,(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y\rangle,\end{aligned}$$

where $\{E_\lambda : 0 \leq \lambda \leq \|L^{-s/2}AL^{-s/2}\|\}$ is the spectral family of $L^{-s/2}AL^{-s/2}$. Similarly, we have

$$\begin{aligned}\|w_{\alpha,\beta}^s\| &\leq \frac{1}{F\left(\frac{s}{(1-\beta)a+s}\right)} \\ &\quad \times\|A_{s,\beta}^{\frac{2s-2\beta a}{2[(1-\beta)a+s]}}(A_{s,\beta}+\alpha I)^{-1}A_{s,\beta}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A^{-\beta}y\| \qquad (4.3.3)\end{aligned}$$

and hence

$$\begin{aligned}\|w_{\alpha,\beta}^s\|^2 &\leq \frac{1}{F\left(\frac{s}{(1-\beta)a+s}\right)^2}\int_0^{\|A_{s,\beta}\|}\left(\frac{\lambda^{\frac{s-\beta a}{(1-\beta)a+s}}}{\lambda+\alpha}\right)^2 \qquad (4.3.4) \\ &\quad \times d\langle F_\lambda A_{s,\beta}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A^{-\beta}y, A_{s,\beta}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A^{-\beta}y\rangle,\end{aligned}$$

where $\{F_\lambda : 0 \leq \lambda \leq \|A_{s,\beta}\|\}$ is the spectral family of $A_{s,\beta}$. Further, note that

$$\begin{aligned}&d\langle E_\lambda(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y,(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y\rangle \\ &= \|E_\lambda(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y\|^2 \\ &\leq \|(L^{-s/2}AL^{-s/2})^{\frac{-s}{2(s+a)}}L^{-s/2}y\|^2 \\ &\leq G^2(\frac{-s}{s+a})\|y\|^2.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}&d\langle F_\lambda A_{s,\beta}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A^{-\beta}y, A_{s,\beta}^{\frac{-(s-2\beta a)}{2[(1-\beta)a+s]}}L^{-s/2}A^{-\beta}y\rangle \\ &\leq \left(\frac{G\left(\frac{-(s-2\beta a)}{(1-\beta)a+s}\right)}{f(\beta)}\right)^2\|y\|^2.\end{aligned}$$

Therefore, the quality of the approximate solution w_α^s and $w_{\alpha,\beta}^s$ are depending on the integrands in (4.3.2) and (4.3.4), respectively. Let $\bar{\varphi}_1(t) := \frac{t^{\frac{s}{(s+a)}}}{t+\alpha}$ and $\bar{\varphi}_2(t) := \frac{t^{\frac{s-\beta a}{(1-\beta)a+s}}}{t+\alpha}$. We call the functions $\bar{\varphi}_1$ and $\bar{\varphi}_2$ the filter factors (Hochstenbach et al. (2015); Klann and Ramlau (2008)) associated with the standard Lavrentiev regularization method in Hilbert scales and fractional Lavrentiev regularization method in Hilbert scales, respectively. Fig. 4.1, displays the filter function $t \longrightarrow \bar{\varphi}_1(t)$ for standard Lavrentiev regularization method in Hilbert scales. The Fig. 4.2, displays the filter function $t \longrightarrow \bar{\varphi}_2(t)$ for fractional Lavrentiev regularization method in Hilbert scales for $\beta = 0.5, 0.35, 0.25, 0.2, 0.1$.

Note that, when the desired solution $\hat{x}$ is smooth, one would like the filter functions to satisfy

$$\lim_{t \longrightarrow 0} \bar{\varphi}_1(t) = 0 \text{ and } \lim_{t \longrightarrow 0} \bar{\varphi}_2(t) = 0.$$

We observed that (see Figs. 4.1 and 4.2) the filter function $\bar{\varphi}_2(t)$ is smoother than the filter function $\bar{\varphi}_1(t)$ near 0. So we expect the computed solution obtained by fractional Lavrentiev regularization method in Hilbert scales approximate the desired solution $\hat{x}$ better than the standard Lavrentiev regularization method in Hilbert scales.

REMARK 4.3.1. *(cf. Bianchi et al. (2015)[Proposition 10]) Note that,* $\frac{\delta+\varepsilon_h}{\alpha^{\frac{a}{(1-\beta)a+s}}}$ *is increasing for* $\beta \in [0, \frac{s}{a}]$*, whereas* $\alpha^{\frac{t}{(1-\beta)+s}}$ *(see Theorem 4.2.5) is decreasing for* $\beta \in [0, \frac{s}{a}]$*. Therefore, one has to choose* $\beta \in [0, \frac{s}{a}]$*, such that* $\frac{\delta+\varepsilon_h}{\alpha^{\frac{a}{(1-\beta)a+s}}} = \alpha^{\frac{t}{(1-\beta)+s}}$ *in order to obtain an optimal order error estimate for* $\|\hat{x} - w_{\alpha,\beta,h}^{s,\delta}\|$*. For a fixed,* $\delta > 0$*,* $t > 0, s > 0, a > 0$ *and* $\alpha \in [\delta^{\frac{s+a}{a+t}}, \delta^{\frac{a}{a+t}}]$*, the best possible choice for* β *is*

$$\beta = 1 + \frac{s}{a} - \left(\frac{a+t}{a}\right) \frac{\log \alpha}{\log(\delta + \varepsilon_h)}.$$

In this case $\beta \in [0, \frac{s}{a}]$ *and* $\alpha = (\delta + \varepsilon_h)^{\frac{(1-\beta)a+s}{a+t}}$*. But such a choice of* β *and* α *is not possible in practice, because t is unknown. Therefore, in Section 4.4 we study George and Nair type (see George and Nair (1993)) discrepancy principle for choosing* α*, for a fixed* β*.*

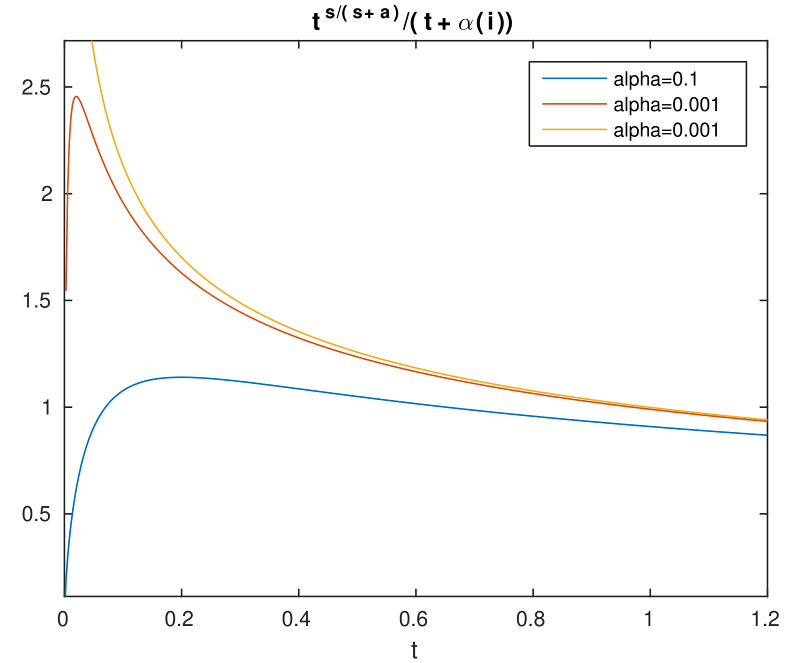

Figure 4.1: Filter function $\bar{\varphi}_1(t)$ as a function of t.

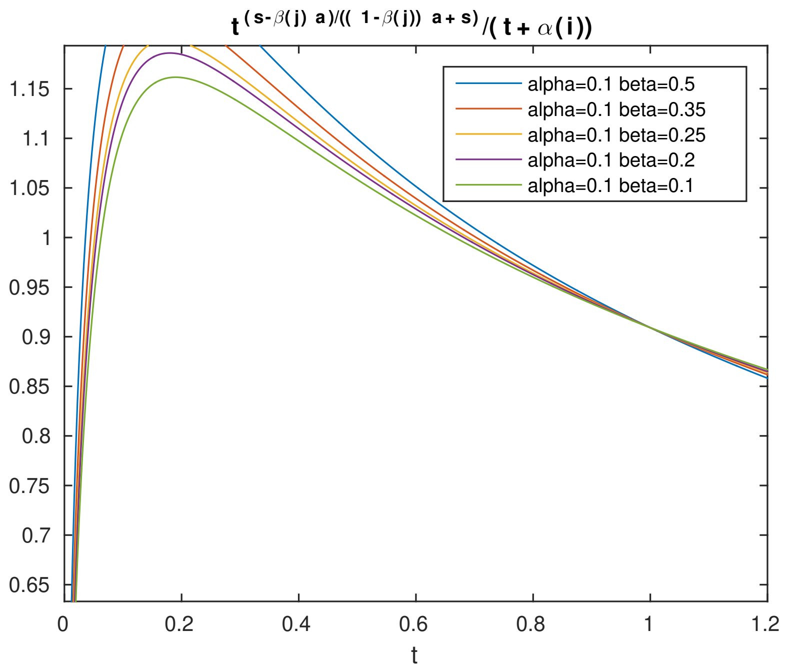

Figure 4.2: Filter function $\bar{\varphi}_2(t)$ for $\alpha = 0.1$ and $\beta = 0.5, 0.35, 0.25, 0.2, 0.1$.

4.4 ERROR BOUNDS UNDER DISCREPANCY PRINCIPLE

In this section we study the analogous of the discrepancy principle considered in George and Nair (1993). For a fixed $s \geq 0, \beta \geq 0$ and $\rho > -1$, let

$$F_{s,\beta,h}(\alpha, x) = \alpha^{\rho+1}\|(A_{s,\beta,h} + \alpha I)^{-(\rho+1)}L^{-s/2}x\|, \;\; x \in \mathcal{X}, \;\; \alpha > 0.$$

Let $c > c_{\frac{-s}{2},0}$ and $y^\delta \in \mathcal{X}$ be such that

$$0 < c(\delta + \epsilon_h) \leq \|L^{-s/2}y^\delta\|. \tag{4.4.1}$$

Next, we prove the existence of unique solution for $F_{s,\beta,h}(\alpha, y^\delta) = c(\delta + \epsilon_h)$.

PROPOSITION 4.4.1. *Let c be as in (4.4.1). Then, there exists a unique $\alpha := \alpha(\delta, \beta, h, s, y^\delta) > 0$ such that*

$$F_{s,\beta,h}(\alpha, y^\delta) = c(\delta + \epsilon_h). \tag{4.4.2}$$

Furthermore,

$$(c - c_{\frac{-s}{2},0})(\delta + \epsilon_h) \leq F_{s,\beta,h}(\alpha, y) \leq (c + c_{\frac{-s}{2},0})(\delta + \epsilon_h). \tag{4.4.3}$$

Proof. Note that

$$\begin{aligned} F^2_{s,\beta,h}(\alpha, y^\delta) &= \alpha^{2(\rho+1)}\langle (A_{s,\beta,h} + \alpha I)^{-(\rho+1)}L^{-s/2}y^\delta, \\ &\quad (A_{s,\beta,h} + \alpha I)^{-(\rho+1)}L^{-s/2}y^\delta\rangle \\ &= \int_0^{\|A_{s,\beta,h}\|}\left(\frac{\alpha}{\lambda + \alpha}\right)^{2(\rho+1)} d\langle F_\lambda L^{-s/2}y^\delta, L^{-s/2}y^\delta\rangle, \end{aligned} \tag{4.4.4}$$

where $\{F_\lambda : 0 \leq \lambda \leq \|A_{s,\beta,h}\|\}$ is the spectral family of $A_{s,\beta,h}$. Now since the map $\alpha \longrightarrow \varphi(\alpha, \lambda) := \left(\frac{\alpha}{\lambda+\alpha}\right)^{2(\rho+1)}$ is strictly increasing for $\lambda > 0$,

$$\varphi(\alpha, \lambda) \longrightarrow 0 \text{ as } \alpha \longrightarrow 0$$

and

$$\varphi(\alpha, \lambda) \longrightarrow 1 \text{ as } \alpha \longrightarrow \infty$$

by Dominated Convergence Theorem, there exists a unique $\alpha := \alpha(\delta, \beta, h, s, y^\delta) > 0$ satisfying (4.4.2).

The second part of the proposition follows by noting that

$$\begin{aligned} F_{s,\beta,h}(\alpha, y) &\leq F_{s,\beta,h}(\alpha, y - y^\delta) + F_{s,\beta,h}(\alpha, y^\delta) \\ &\leq \|L^{-s/2}(y - y^\delta)\| + F_{s,\beta,h}(\alpha, y^\delta) \\ &\leq (c_{\frac{-s}{2},0} + c)(\delta + \epsilon_h) \end{aligned}$$

and

$$\begin{aligned} F_{s,\beta,h}(\alpha, y) &\geq F_{s,\beta,h}(\alpha, y^\delta) - F_{s,\beta,h}(\alpha, y - y^\delta) \\ &\geq c(\delta + \epsilon_h) - \|L^{-s/2}(y - y^\delta)\| \\ &\geq (c - c_{\frac{-s}{2},0})(\delta + \epsilon_h) \end{aligned}$$

where we used the relation $\|L^{-s/2}x\| \leq c_{\frac{-s}{2},0}\|x\|$, $x \in \mathcal{X}$.

□

The proof of the following proposition is analogous to the proof of Proposition 3.5 in George and Nair (1993), so details are ignored.

PROPOSITION 4.4.2. *(c.f George and Nair (1993)[Proposition 3.5]) Let y^δ satisfy (4.4.1) and $0 \neq y \in \mathcal{X}$. Let $(\delta + \epsilon_h) > 0$ and $\alpha := \alpha(\delta, \beta, h, s, y^\delta) > 0$ be chosen according to (4.4.2). Then, there exists a, $\delta_0 + \epsilon_{h,0} > 0$ such that*

$$\begin{aligned} S: &= \{\alpha(\delta, \beta, h, s, y^\delta) : 0 < (\delta + \epsilon_h) \leq \delta_0 + \epsilon_{h,0} \text{ and} \\ &\quad 0 < c(\delta_0 + \epsilon_{h,0}) \leq \|L^{-s/2}y^\delta\|, \|y - y^\delta\| \leq \delta\} \end{aligned}$$

is a bounded set.

Next, we state and prove the main results of this section.

THEOREM 4.4.3. *Let A satisfy (4.1.2) and (4.1.4) hold. Suppose $\hat{x} \in \mathcal{X}$, $0 < t \leq \min\{s + \rho(s + (1-\beta)a), \frac{s-(1+\beta)a}{2}\}$, y^δ satisfies (2.1.2), (4.4.1); and $\alpha := \alpha(\delta, \beta, h, s, y^\delta)$ satisfies (4.4.2). Then for $0 < \beta \leq \frac{2s+a}{3a}$, $s \leq a$, we have,*

$$\|\hat{x} - w_{\alpha,\beta,h}^{s,\delta}\| \leq \Phi(s, a, \beta, h, t)(\delta + \epsilon_h)^{\frac{t}{(\rho+1)[(1-\beta)a+s]}},$$

where

$$\begin{aligned}\Phi(s,a,\beta,h,t) &= \max\{\varphi_5(s,a,\beta,h)C_{s,\beta,h,a,t}c_{s,\beta,\rho}^{\frac{s+2t}{2(\rho+1)[(1-\beta)a+s]}},\\ &\qquad \psi_2(s,a,\beta,t)Ec_{s,\beta,\rho}^{\frac{t}{2(\rho+1)(\rho+1)[(1-\beta)a+s]}}\},\end{aligned}$$

$c_{s,\beta,\rho} = \frac{\|(A_{s,\beta,h}+\alpha I)^{(\rho+1)}\|}{\|y\|_{-s/2}}(c+c_{\frac{-s}{2},0})$, *and* $C_{s,\beta,h,a,t} = \frac{\bar{G}(\frac{-2(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})}g(\frac{s-2t}{2a})c_{t-s/2,t}E$. *In particular, if* $\rho = \frac{t+\beta a-s}{(1-\beta)a+s}$, *then we have*

$$\|\hat{x}-w_{\alpha,\beta,h}^{s,\delta}\| \leq \Phi(s,a,\beta,h,t)(\delta+\epsilon_h)^{\frac{t}{t+a}}.$$

Proof. Note that from (4.4.3), we have

$$\frac{\alpha^{\rho+1}\|y\|_{-s/2}}{\|(A_{s,\beta,h}+\alpha I)^{(\rho+1)}\|} \leq F_{s,\beta,h}(\alpha,y) \leq (c+c_{\frac{-s}{2},0})(\delta+\epsilon_h)$$

so that

$$\alpha \leq c_{s,\beta,\rho}^{1/(\rho+1)}(\delta+\epsilon_h)^{1/(\rho+1)}. \tag{4.4.5}$$

Again by (4.4.3), and (4.2.8), we have

$$\begin{aligned}(c-c_{\frac{-s}{2},0})(\delta+\epsilon_h) &\leq \alpha^{\rho+1}\|(A_{s,\beta,h}+\alpha I)^{-(\rho+1)}L^{-s/2}y\|\\ &\leq \alpha^{\rho+1}\|(A_{s,\beta,h}+\alpha I)^{-(\rho+1)}A_{s,\beta,h}^{\frac{a+t}{(1-\beta)a+s}}\|\|A_{s,\beta,h}^{\frac{-(a+t)}{(1-\beta)a+s}}L^{-s/2}y\|\\ &\leq \alpha^{\frac{a+t}{(1-\beta)a+s}}\|A_{s,\beta,h}^{\frac{-(a+t)}{(1-\beta)a+s}}L^{-s/2}y\|. \end{aligned}\tag{4.4.6}$$

Now using Proposition 4.1.5, (iii) and Proposition 4.1.2 (i), we have

$$\begin{aligned}\|A_{s,\beta,h}^{\frac{-(a+t)}{(1-\beta)a+s}}L^{-s/2}y\| &= \|A_{s,\beta,h}^{\frac{-(a+t)}{(1-\beta)a+s}}L^{-s/2}A\hat{x}\|\\ &\leq \bar{G}\left(\frac{-2(a+t)}{(1-\beta)a+s}\right)\|L^{-s/2}A\hat{x}\|_{a+t}\\ &\leq \frac{\bar{G}(\frac{-2(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})}\|A^{\frac{s-2t}{2a}}\hat{x}\|\\ &\leq \frac{\bar{G}(\frac{-2(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})}g(\frac{s-2t}{2a})\|\hat{x}\|_{t-s/2}\\ &\leq \frac{\bar{G}(\frac{-2(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})}g(\frac{s-2t}{2a})c_{t-s/2,t}\|x\|_t \end{aligned}\tag{4.4.7}$$

here $c_{t-s/2,t}$ is the constant in the Definition 1.4.1. Combining (4.4.6) and (4.4.7), we obtain

$$(c - c_{\frac{-s}{2},0})\delta \leq \frac{\bar{G}(\frac{-(a+t)}{(1-\beta)a+s})}{f(\frac{s-2(a+t)}{2a})} g(\frac{s-2t}{2a}) c_{t-s/2,t} E \alpha^{\frac{a+t}{(1-\beta)a+s}},$$

so that

$$\delta \alpha^{\frac{-a}{(1-\beta)a+s}} \leq C_{s,\beta,h,a,t} \alpha^{\frac{t}{(1-\beta)a+s}}. \tag{4.4.8}$$

The result now follows from Theorem 4.2.5, (4.4.5) and (4.4.8).

4.5 NUMERICAL EXAMPLES

In this section we consider the Hilbert scales generated by the linear operator L defined by

$$Lx := \sum_{j=1}^{\infty} j^2 \langle x, u_j \rangle u_j,$$

where $u_j(t) = \sqrt{2}\sin(j\pi t)$, $j \in \mathbb{N}$, with domain of L as

$$D(L) := \left\{ x \in L^2[0,1] : \sum_{j=1}^{\infty} j^4 |\langle x, u_j \rangle|^2 < \infty \right\}.$$

In this case the Hilbert scale $\{\mathcal{X}\}_s$ generated by L is given by

$$\begin{aligned} \mathcal{X}_s &= \{x \in L^2[0,1] : \sum_{j=1}^{\infty} j^{4s} |\langle x, u_j \rangle|^2 < \infty\} \\ &= \{x \in H^{2s}(0,1) : x^{(2l)}(0) = x^{(2l)}(1) = 0, l = 0, 1, \ldots \lceil \frac{s}{2} - \frac{1}{4} \rceil\}, \end{aligned} \tag{4.5.9}$$

where $\lceil p \rceil$ denote the greatest integer less than or equal to p, $s \in \mathbb{R}$, and H^s is the usual Sobolev space. Also, one can see that $H^0 = L^2[0,1]$, and for $s \in \mathbb{N}, H_s \subset H^s$.

We consider four examples for the numerical discussion to validate our theoretical results. We use a sequence of finite dimensional subspaces of (V_n) of $\mathcal{X}$ and P_h ($h = \frac{1}{n}$) denote the ortogonal projection on $\mathcal{X}$ with $R(P_h) = V_n$. We choose V_n as the linear span of $\{v_1, v_2, \ldots, v_n\}$ with $v_i, i = 1, 2, \ldots, n$ as the L^2-orthogonalized characteristic functions of the interval $[\frac{i-1}{n}, \frac{i}{n}]$. Then since $w^{s,\delta}_{\alpha,\beta,h} \in V_n$, it is of the form $\sum_{i=1}^{n} \lambda_i v_i$ for some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$. It can be seen that $w^{s,\delta}_{\alpha,\beta,h} = \sum_{i=1}^{n} \lambda_i v_i$ is the solution of (4.2.3) if and only if $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n)^T$ is the unique solution of

$$(M_n + \alpha B_n)\bar{\lambda} = W_n,$$

where

$$M_n = \langle A_h v_i, v_j \rangle, i, j = 1, 2, \ldots, n,$$

$$B_n = \langle L^s v_i, A_h^{\beta} v_j \rangle, i, j = 1, 2, \ldots, n$$

and

$$W_n = (\langle y^{\delta}, v_1 \rangle, \ldots, \langle y^{\delta}, v_n \rangle)^T.$$

Here and below $(a_1, a_2, \ldots, a_n)^T$ denote the transpose of $(a_1, a_2, \ldots, a_n)$.

We use the Newton's method to solve the nonlinear equations (4.4.2) for α with different values β, δ and ρ. Relative error $E_{\alpha,\beta} := \left(\frac{\|\hat{x} - w^{s,\delta}_{\alpha,\beta}\|}{\|\hat{x}\|} \right)$, and α are presented in the tables for different values of β, $\rho = \frac{t+\beta a - s}{(1-\beta)a+s}$ with $t = -\beta a + s/3, n = 300$(size of the mesh) and noise level δ. We have introduced the random noise level $\delta = 0.1$, 0.01 and 0.001 and $\epsilon_h = \epsilon_{\frac{1}{n}} = \frac{1}{n^2}$ in the exact data.

EXAMPLE 4.5.1. *(Shaw (1972)) Let*

$$[Tx](s) := \int_{-\pi}^{\pi} k(s,t)x(t)dt = g(s), \; -\pi \le s \le \pi, \tag{4.5.10}$$

where $k(s,t) = (cos(s) + cos(t))^2 (\frac{sin(u)}{u})^2$, $u = \pi(sin(s) + sin(t))$. We take $A := T^*T|_{N(T^*T)^{\perp}}$ and $y = T^*g$ for our computation. The solution $\hat{x}$ is given by $\hat{x}(t) = a_1 exp(-c_1(t-t_1)^2) + a_2 exp(-c_2(t-t_2)^2)$. We have taken $s = a = \frac{1}{2}$, $d_1 = d_2 = \frac{1}{\pi^2}$ in our computation. The relative error and α values for different values of β and δ are given in Table 4.1 and Table 4.2. The figures for exact data and noise data for $\delta = 0.01$ is given in Fig. 4.3 , solutions with $\delta = 0.01$ and for $\beta = 0, 0.05, 0.1, 0.15$ and $\beta = 0.2$ are given in Figs. 4.4 - 4.8.

Table 4.1: Relative errors for fixed α.

β	0	0.05	0.1	0.15	0.2
$E_{\alpha,\beta}$	$6.801195e-01$	$6.766239e-01$	$6.726160e-01$	$6.679884e-01$	$6.626066e-01$

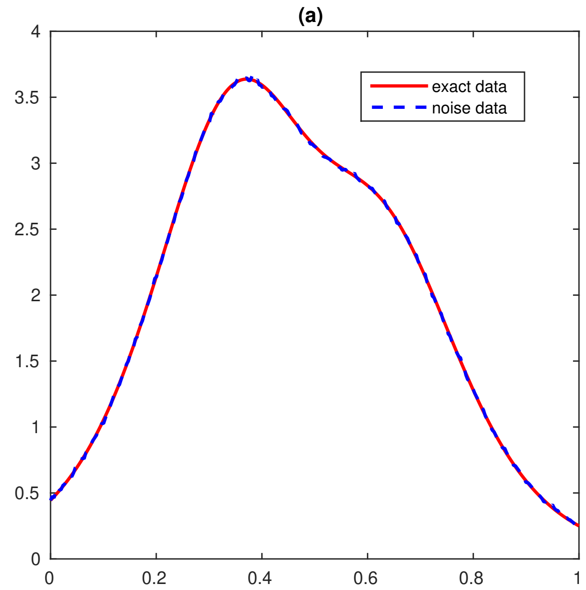

Figure 4.3: Exact data and noise data when $\delta = 0.01$ *for Shawn example*

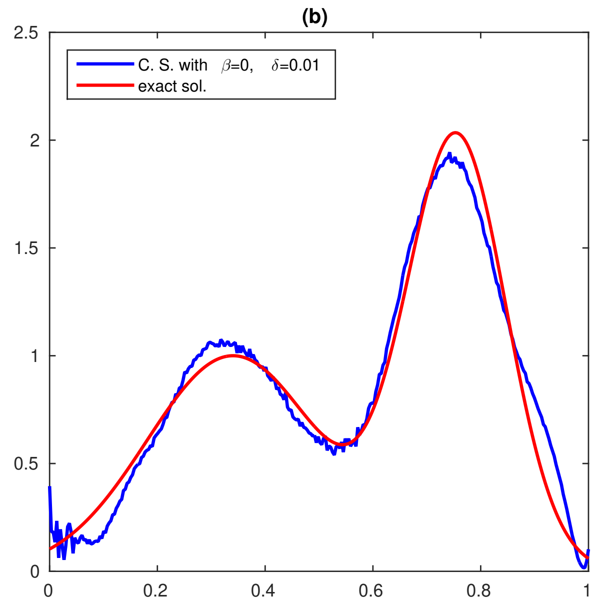

Figure 4.4: Solutions with $\delta = 0.01$ *and* $\beta = 0$ *for Shawn example.*

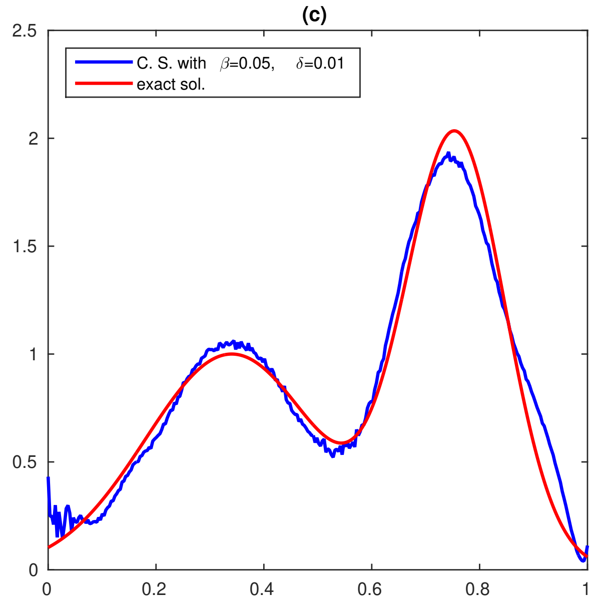

Figure 4.5: Solutions with $\delta = 0.01$ and $\beta = 0.05$ for Shawn example.

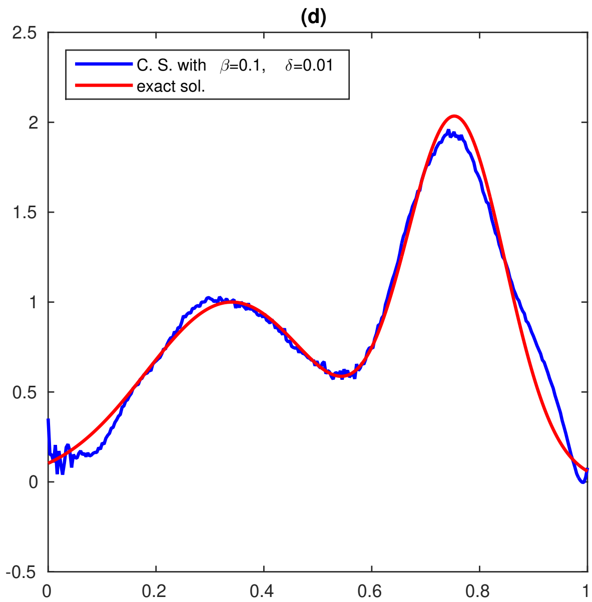

Figure 4.6: Solutions with $\delta = 0.01$ and $\beta = 0.1$ for Shawn example.

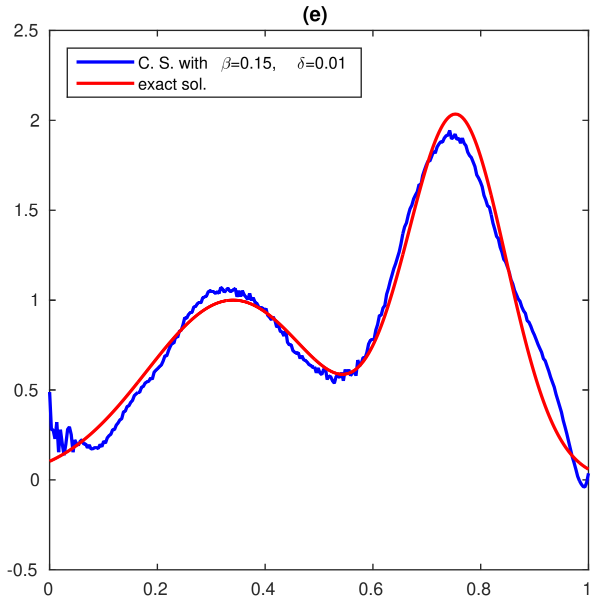

Figure 4.7: Solutions with $\delta = 0.01$ *and* $\beta = 0.15$ *for Shawn example.*

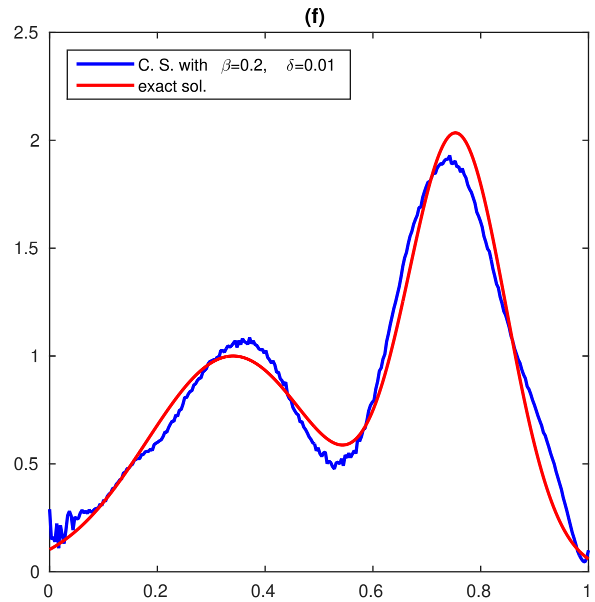

Figure 4.8: Solutions with $\delta = 0.01$ *and* $\beta = 0.2$ *for Shawn example.*

Table 4.2: *Relative errors under discrepancy principle for Shawn example.*

δ	β	0	0.05	0.1	0.15	0.2
	ρ	5.000000e-01	5.128205e-01	5.263158e-01	5.405405e-01	5.555556e-01
0.1	$\alpha(k)$	5.931937e-04	6.229828e-04	6.892837e-04	7.511112e-04	8.620988e-04
	$E_{\alpha,\beta}$	2.016406e-01	1.936678e-01	2.040882e-01	1.878488e-01	4.261247e-01
0.01	$\alpha(k)$	5.929687e-04	6.232274e-04	6.893498e-04	7.513056e-04	8.627321e-04
	$E_{\alpha,\beta}$	9.689599e-02	9.355126e-02	8.139111e-02	9.526484e-02	9.511034e-02
0.001	$\alpha(k)$	5.929932e-04	6.232488e-04	6.893604e-04	7.513294e-04	8.627484e-04
	$E_{\alpha,\beta}$	9.617521e-02	8.861288e-02	8.175922e-02	7.492941e-02	7.269160e-02

EXAMPLE 4.5.2. *(Phillips (1962)) Let*

$$\int_{-6}^{6} k(s,t)x(t)dt = g(s), \; -6 \le u \le 6, \tag{4.5.11}$$

where $k(s,t) = \phi(s-t)$, *with*

$$\phi(x) = \begin{cases} 1+\cos(x*\pi/3), & |x| < 3 \\ 0, & |x| \ge 3 \end{cases}.$$

We take $A := T^*T|_{N(T^*T)^\perp}$ *and* $y = T^*g$, *where* $g(s) = (6-|s|)*(1+.5*\cos(s*\pi/3)) + 9/(2*\pi)*\sin(|s|*\pi/3)$ *for our computation. The solution* $\hat{x}$ *is given by* $\hat{x}(t) = \phi(t)$. *We have taken* $s = a = \frac{1}{2}$, $d_1 = d_2 = \frac{1}{36}$ *in our computation. The relative error and* α *values for different values of* β *and* δ *are given in Table 4.3 and Table 4.4. The figures for exact data and noise data for* $\delta = 0.01$ *is given in Fig. 4.9, solutions with* $\delta = 0.01$ *and for* $\beta = 0, 0.05, 0.1, 0.15$ *and* $\beta = 0.2$ *are given in Figs. 4.9 - 4.14. The figure 4.9 contains the exact data and noise data and remaining figures contains the computed solution (C.S) and exact solution (exact sol.).*

Table 4.3: *Relative errors for fixed* α.

β	0	0.05	0.1	0.15	0.2
$E_{\alpha,\beta}$	$6.303972e-01$	$6.253094e-01$	$6.174618e-01$	$6.046162e-01$	$5.830876e-01$

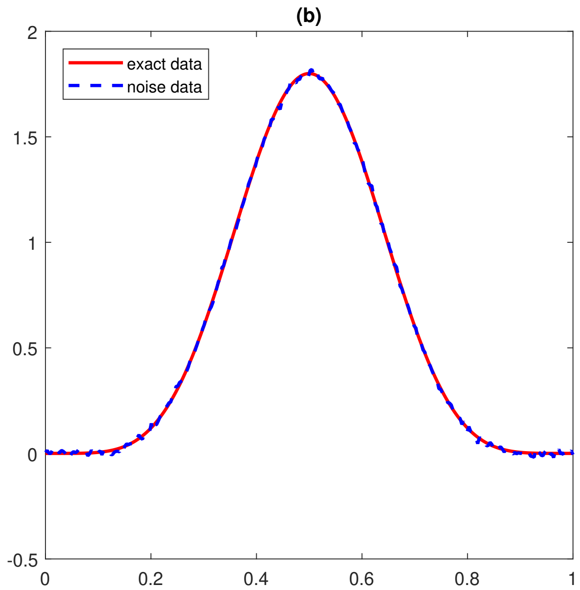

Figure 4.9: Exact data and noise data when $\delta = 0.01$ for Phillips example.

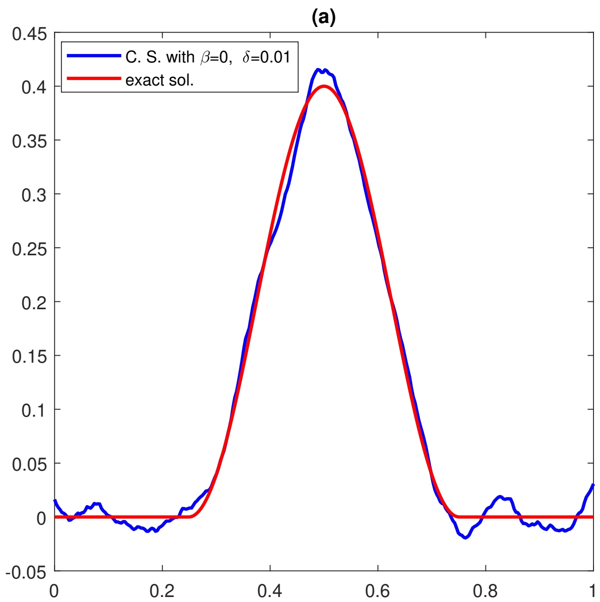

Figure 4.10: Solutions with $\delta = 0.01$ and $\beta = 0$ for Phillips example.

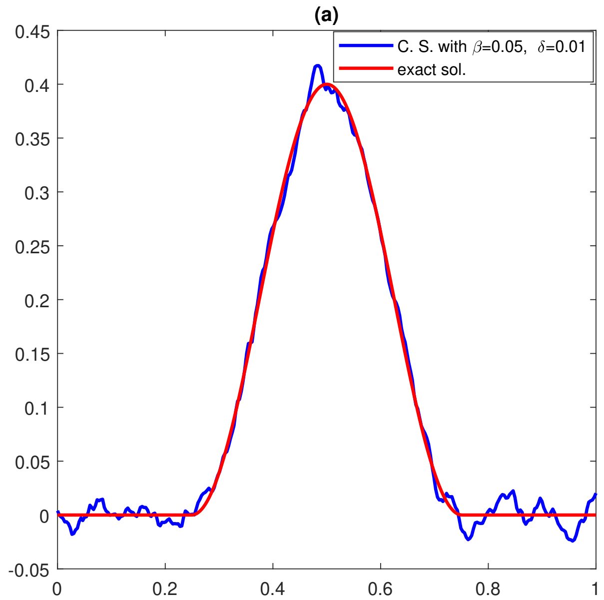

Figure 4.11: Solutions with $\delta = 0.01$ and $\beta = 0.05$ for Phillips example.

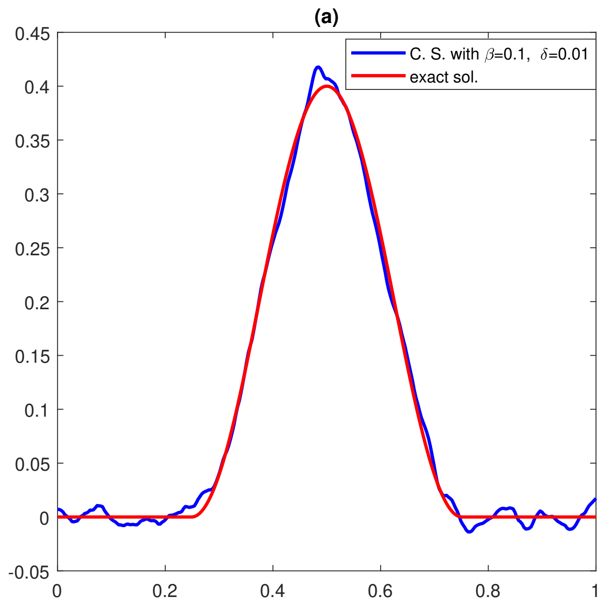

Figure 4.12: Solutions with $\delta = 0.01$ and $\beta = 0.1$ for Phillips example.

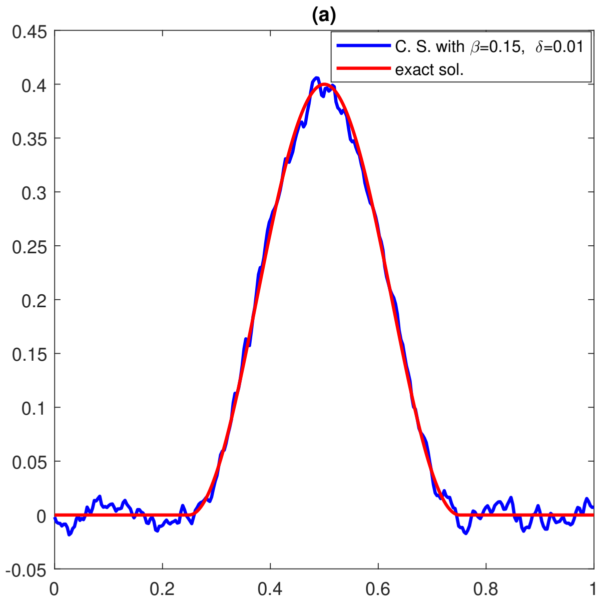

Figure 4.13: Solutions with $\delta = 0.01$ and $\beta = 0.15$ for Phillips example.

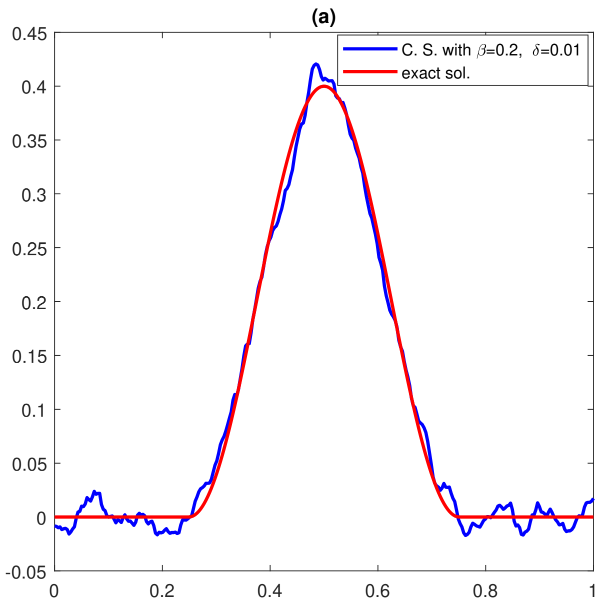

Figure 4.14: Solutions with $\delta = 0.01$ and $\beta = 0.2$ for Phillips example.

Table 4.4: Relative errors under discrepancy principle for Phillips example.

δ	β	*0*	*0.05*	*0.1*	*0.15*	*0.2*
	ρ	*5.000000e-01*	*5.263158e-01*	*5.405405e-01*	*5.555556e-01*	*5.555556e-01*
0.1	$\alpha(k)$	*1.805922e-02*	*2.575646e-02*	*3.576999e-02*	*4.785191e-02*	*6.326033e-02*
	$E_{\alpha,\beta}$	*3.515146e-01*	*2.315353e-01*	*3.511560e-01*	*3.068689e-01*	*2.925603e-01*
0.01	$\alpha(k)$	*1.807031e-02*	*2.576193e-02*	*4.825071e-02*	*4.823351e-02*	*6.338610e-02*
	$E_{\alpha,\beta}$	*6.661885e-02*	*5.778094e-02*	*2.703130e-02*	*4.506687e-02*	*3.913097e-02*
0.001	$\alpha(k)$	*1.806768e-02*	*2.578385e-02*	*3.576453e-02*	*4.824909e-02*	*6.336271e-02*
	$E_{\alpha,\beta}$	*2.007506e-02*	*1.797936e-02*	*2.382614e-02*	*2.033829e-02*	*2.208742e-02*

EXAMPLE 4.5.3. *(Non-smooth Signal:) In this example we generate a square wave with sharp edges to analyse the performance of the Lavrentiev and fractional Lavrentiev method. The Lavrentiev regularization results in smoothing of sharp discontinuities whereas the fractional Lavrentiev retains the sharpness in the signal thus reducing the over-smoothing effect. We have taken* $s = a = \frac{1}{2}$ *in our computation. The results are shown in Figures 4.16 - 4.18 . The results of the proposed fractional Lavrentiev regularization model are shown in Figures 4.17 - 4.18 for different* β *values.*

In the next example, we consider an image restoration problem.

EXAMPLE 4.5.4. *(Image Restoration Example) Here we show some examples to demonstrate the restoration ability of the method when applied to different images. IR Tool: a Matlab package for iterative inverse problems in Gazzola et al. (2019) and Algebraic IR Tools in Hansen and Jø rgensen (2018) are being used here for the numerical implementation of the model for 2D images (both gray-scale and color). Two test images (a satellite image and a synthetic image) given along with the IR/AIR tools (package) are tested and the results are demonstrated below. The test image is synthetically corrupted by Gaussian blur with standard deviation 2 and Gaussian white noise with zero mean and noise variance 0.05. The test results are shown for standard Lavrentiev regularization and the proposed model (fractional Lavrentiev model). The standard Lavrentiev model tends to perform denoising by penalizing the image details resulting in an over-smoothed data as observed from the results. Nevertheless, the proposed model restores the images without*

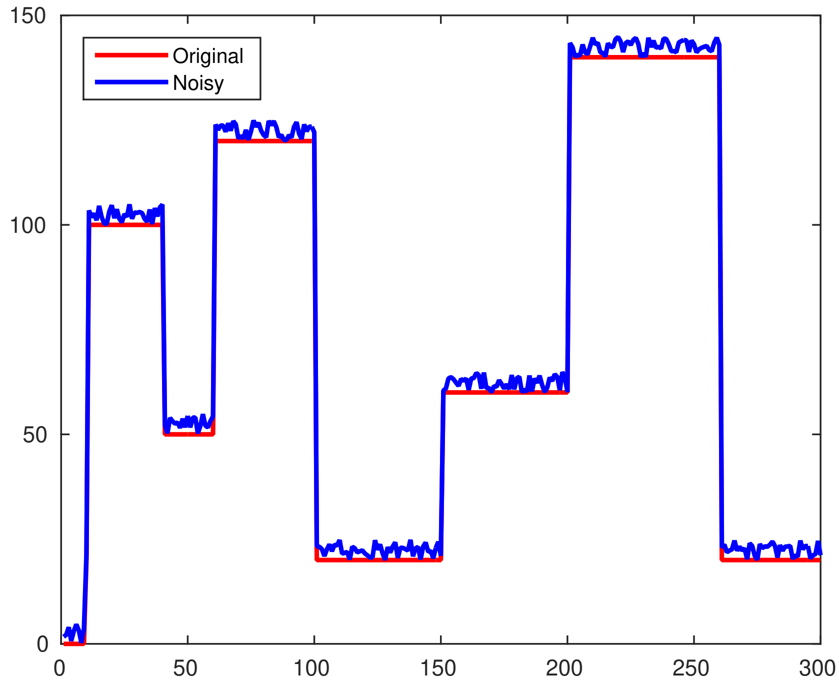

Figure 4.15: Original and noisy input signal: Contaminated with Gaussian noise with standard deviation $\sigma = 0.15$.

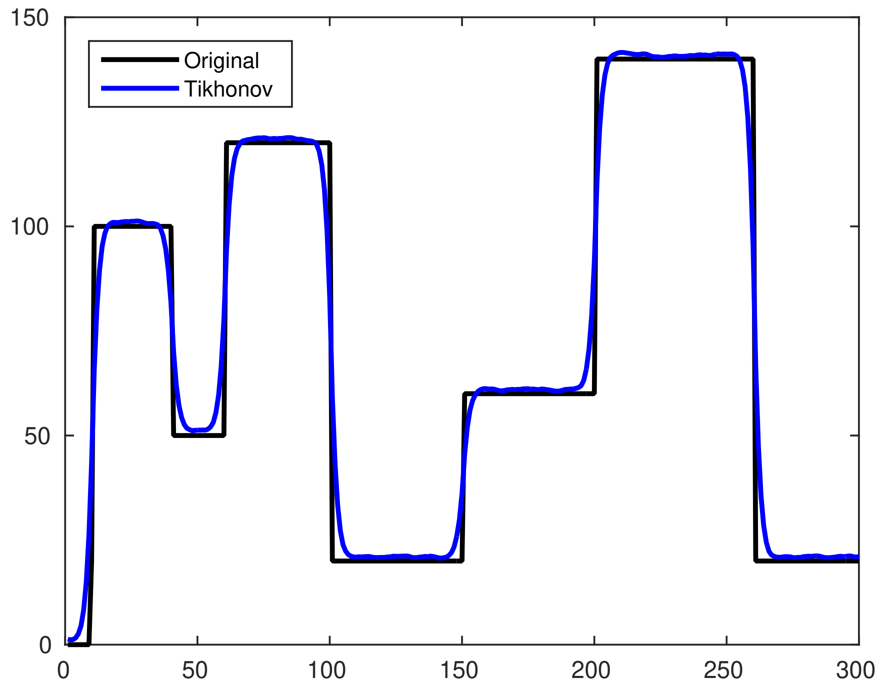

Figure 4.16: Lavrentiev regularization

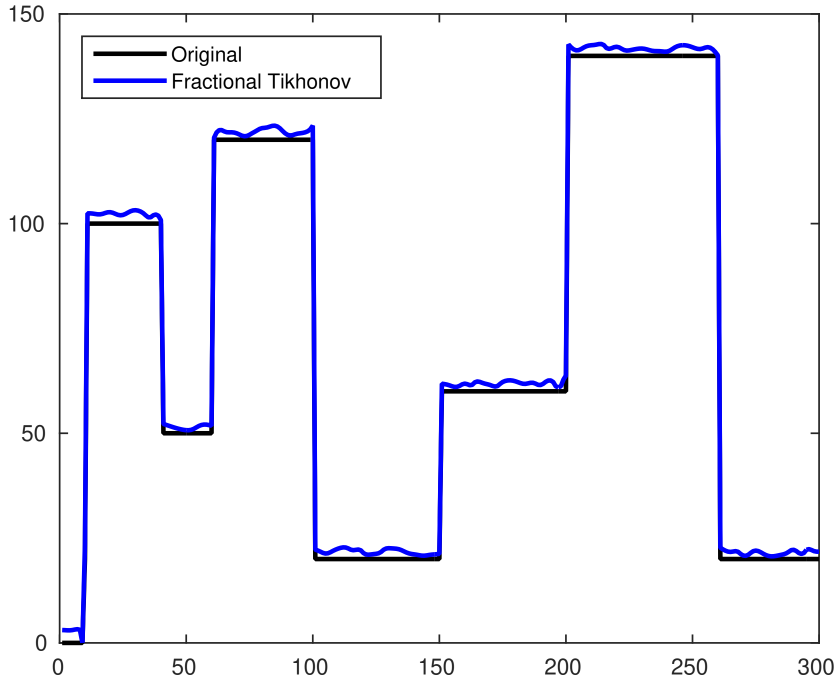

Figure 4.17: Fractional Lavrentiev: $\beta = 0.1$

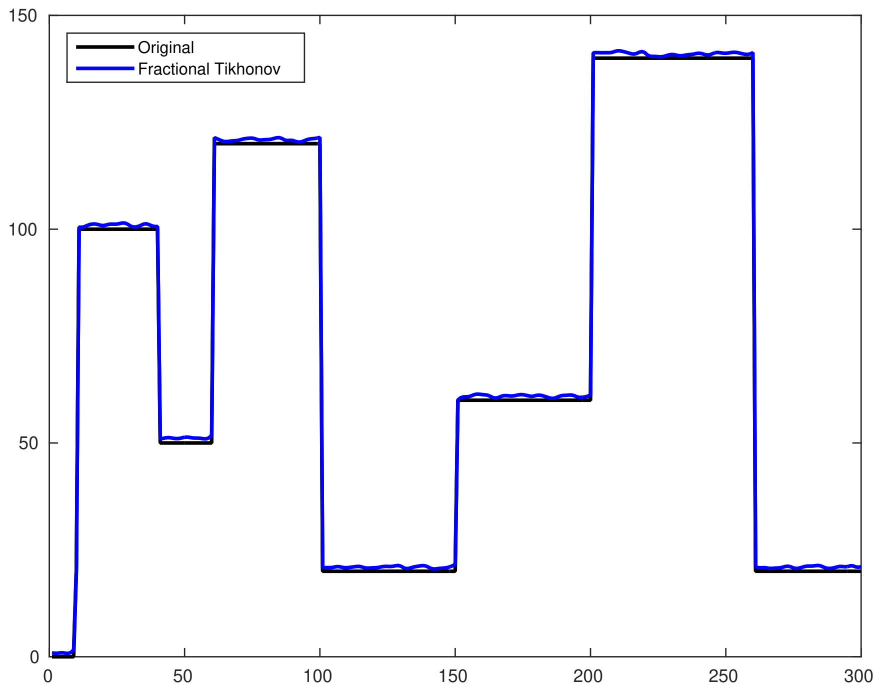

Figure 4.18: Fractional Lavrentiev: $\beta = 0.2$

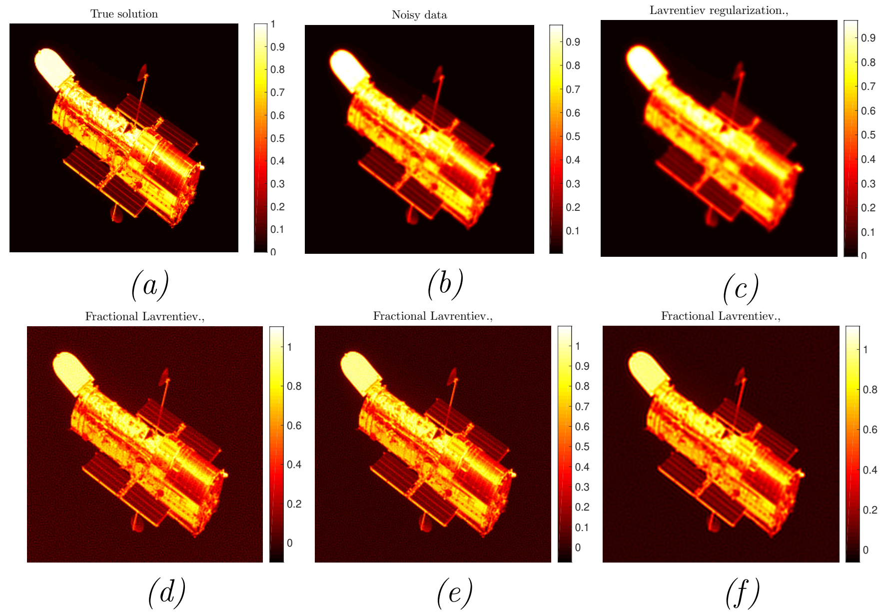

Figure 4.19: (a) Original image (b) blurred and noisy image, (c) restored using Lavrentiev regularization and (d, e, f) restored using the proposed model for $\beta = 0.1, 0.15, 0.2,$ *respectively*

compromising much on the details. The original, noisy, and restored images are shown in Figs. 4.19 and 4.20 for the two different input test images. The proposed restoration process is observed to denoise the data and preserve the details as observed from the results shown for different β *values. A statistical quantification has been performed using the well-known measure: Signal to Noise Ratio (SNR)* [1]*. The SNR of the noisy and restored versions of the test images for different* β *values are given in Table 4.5. The SNR measure being inversely proportional to the root mean square error, it increases with decrease in* β *value unlike the relative error.*

[1] $SNR = 20\log_{10}\frac{\sum_{i=0}^{N}\sum_{j=0}^{M}\hat{x}(i,j)^2}{\sum_{i=0}^{N}\sum_{j=0}^{M}[\hat{x}(i,j)^2 - x(i,j)]^2}$ *dB, where* x *and* $\hat{x}$ *are the original and restored images, respectively.*

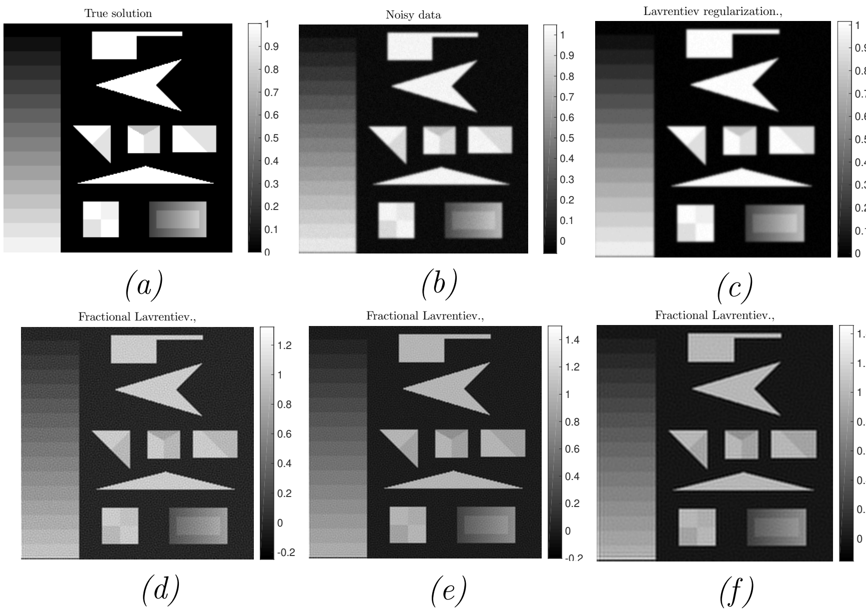

Figure 4.20: (a) Original image (b) blurred and noisy image, (c) restored using Lavrentiev regularization and (d, e, f) restored using the proposed model for $\beta = 0.1, 0.15, 0.2$, *respectively*

Table 4.5: SNR evaluated (in dB) for different β *values for two different images*

Image	β	*Noisy & Blurred Image*	*Restored by the proposed*
Satellite	*0*	*1.92*	*4.12*
	0.05		*5.32*
	0.1		*6.41*
	0.15		*7.22*
	0.2		*8.19*
Synthetic	*0*	*2.32*	*5.21*
	0.05		*6.23*
	0.1		*7.22*
	0.15		*8.12*
	0.2		*8.99*

Chapter 5

CONCLUSIONS AND SCOPE FOR FUTURE WORK

5.1 CONCLUDING REMARKS

We have mainly concentrated our work on solving ill-posed problems involving bounded, linear operators in a Hilbert scales setting. We have tried using various discrepancy principles for choosing the regularization parameter.

In Chapter 2, we study fractional Tikhonov regularization method for approximately solving a linear ill-posed operator equation $T(x) = y$ in the setting of Hilbert scales. We obtained optimal order error estimate under an a-priori and a new discrepancy principle. Using the adaptive parameter choice strategy, we obtained the optimal order error estimate for $\beta a \leq t \leq 2s + \frac{1+\beta}{2}a$.

As explained in the introduction of the Chapter, FTRM reduces the oversmoothing in the STRM in Hilbert space and Hilbert scales. The regularization saturation for Tikhonov regularization in Hilbert scales is $t = 2s + a$, (Engl et al. (1996); George and Nair (1997); Goldenshluger and Pereverzev (2000); Egger and Hofmann (2018); Jin (2000); Lu et al. (2010); Mathé and Pereverzev (2003); Natterer (1984); Neubauer (1988, 1992, 2000)) whereas that of FTRM is $t \leq 2s + \frac{1+\beta}{2}a$. The magnitude of regularization (smoothing) with reference to the values of β is observable from the examples illustrated in the Chapter.

We studied the finite dimensional realization of FTRM in Chapter 3. We also study the finite dimensional version of the parameter choice strategy introduced in Chapter 2, and obtained the optimal order error estimate for the finite dimensional FTRM in Hilbert scales . We have given a numerical example for validation of our results.

Finite dimensional fractional Lavrentiev regularization method for approximately solving a linear ill-posed operator equation in the setting of Hilbert scales is investigated in Chapter 4. We obtained optimal order error estimate under an a-priori and an a-posteriori parameter choice strategy. The method is applied to various well known examples in the literature.

We observe that fractional Lavrentiev regularization method reduces the oversmoothing in the standard Lavrentiev regularization method in Hilbert space and Hilbert scales. The regularization saturation for Fractional Lavrentiev regularization method is $t = s+(1-\beta)\frac{a}{2}$, whereas that of standard Lavrentiev regularization method is $t = s+a > s+(1-\beta)\frac{a}{2}$. So fractional Lavrentiev regularization method yields earlier saturation. The magnitude of regularization (smoothing) with reference to the values of β is observable from the example given for image restoration problem. As the noise variance increases, the value of β also needs to be decreased in order to obtain a proper restoration. Nevertheless the blurring artifacts start appearing in the resultant data as β increases. The value of β should provide a trade-off between smoothing and deblurring as these two are two complementary requirements.
The choice of optimal value of β is still an open problem.

5.2 FUTURE SCOPE OF THE RESEARCH

In the present study, we considered only linear ill-posed problems. To the best of our knowledge, there is no study on the fractional regularization method for non-linear ill-posed problems. So it is envisaged to study fractional regularization method for ill-posed non-linear operator equations. Similarly most of the methods for solving non-linear ill-posed equations are iterative and the Frechet derivative of the operator is involved in the iterative methods. We are interested in studying iterative methods involving fractional powers of the Frechet derivative. Also, we intend to study higher order iterative methods to obtain fast converging iterative methods, using assumptions only on the first derivative of the operator involved.

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1. Mekoth, C., George, S., and Jidesh, P. (2021). Fractional Tikhonov regularization method in Hilbert scales. Appl. Math. Comput., 392:125701, 26.(SCI)(I.F. 4.091). https://doi.org/10.1016/j.amc.2020.125701

2. Mekoth, C., George, S., Jidesh, P., and Erappa, M. S. (2022). Finite dimensional realization of fractional Tikhonov regularization method in Hilbert scales. Partial Differential Equations in Applied Mathematics. 5, 100246. https://doi.org/10.1016/j.padiff.2021.100246

3. Mekoth, C., George, S., and Jidesh, P., Finite dimensional realization of a parameter choice strategy for Fractional Tikhonov regularization method in Hilbert scales. (Communicated)

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