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Shock Coupled Coherence Enhancing Diffusion for Robust Core-point Detection in Fingerprints

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Abstract— **Enhancing the flow-like structures is important in forensic applications especially in fingerprint analysis. In most of the practical scenarios the poor quality of the off-line prints collected, adversely affect the verification process. Though there has been a plethora of methods proposed in literature for enhancing the degraded images, very few of them are suitable for enhancing the flow-like structures because they are ignorant of the coherence features present in images with dominant flow-like structures. In this paper we propose a method which enhances the fingerprint images with utmost consideration to the coherence features of the fingerprints. This method provides a shock at the inflection points while retaining the flow-like nature of the fingerprints. In other words, it enhances the coherence of features along with the edges. The experimental results shown endorses on the capability of the method to enhance the fingerprints which in turn will result in identification of the core-points in the fingerprints with a better accuracy. The core-point identification is a crucial step in fingerprint verification.**

Keywords— **Shock; Coherence Enhancement; Fingerprint Core-point Detection;**

I. Introduction

Oriented patterns arise in many computer vision and image processing applications. Though many enhancing methods have been proposed in literature during the last couple of decades [6], [8], [13] most of them pay less attention on how to enhance the flow-like patterns. In case of forensic applications, it is quite common to have the fingerprints with poor quality. In such cases it would be desirable to have a method which improves the quality of flow-like patterns without destroying the semantically important singularities like the minutiae in fingerprints.

Partial Differential Equations (PDEs) and variational methods are used for image denoising and enhancement for last few decades [1], [5], [7], [8], [13]. These methods allow the image to evolve in a scale space with respect to time and denoise the image in the course of evolution. This may eventually converge to the desired solution in a finite amount of time. The image enhancing properties of the diffusion filters were analysed thoroughly in the literature [13]. Even though the non-linear diffusion behaves like a backward parabolic PDE conditionally, which will eventually enhance the edges; it is highly unstable and may result in an ill-posed problem in the sense of Hardmard. Many regularization methods were employed to solve this problem see [4], [8]. However the non-linear diffusion methods are conditionally anisotropic, in other words they diffuse linearly in the homogeneous areas where the gradient magnitudes are negligible. Hence the non-linear diffusion is not recommended for enhancing images with flow-like structures.

Another kind of enhancing filter namely “shock” filter was introduced in literature for image enhancement, see [1] and [12]. This filter belongs to a class of hyperbolic PDE’s which can enhance the images without causing any instability. This filter was first introduced for image enhancement by Rudin and Osher in [12] and they called it a “classical” shock filter. Though this “classical” shock filter is capable of image enhancement it enhances the noise along with the image features. In fact this filter cannot be used in its original form. There has been many modification proposed for this filter the details can be found in [1], [3] and [5]. One notable improvement was suggested by Alvarez et. al. in [1], in this they propose to couple shock along with the diffusion namely shock coupled diffusion. This filter will act like a shock filter in the direction of the gradient and a diffusion filter along the direction of the isophotes (level-lines). The diffusion term in this filter is the Mean Curvature Motion (MCM) [9]. The MCM is a fully anisotropic diffusion method which does not diffuse at all in the direction along the gradient. This way the edges will not get diffused, whereas they will be enhanced by the shock component. There were further modifications to this filter in the literature see [5] and [15] for details.

The shock coupled diffusion filter with MCM as the diffusion term will denoise the images while giving a sharp shock at the inflection points, but this filter does not have the capability to enhance the coherence features in the images. An alternative strategy was proposed by Wickert in [14], in this the author proposes to enhance the coherence features of the image with the help of structure tensor in place of the coefficient of diffusion. In fact these tensors would give an anisotropic property to the filter which will force the filter to diffuse in the direction normal to the gradient with a high magnitude as compared to the direction along the gradient. This in turn will retain the flow-like patterns while denoising the image.

In this paper we propose to replace the diffusion term in the shock filter in [1] with the diffusion term in [14]. By using the structure tensor in the diffusion term of the shock filter the image will be enhanced, while retaining the flow-like patterns. This finds a good application in the area of forensic science especially in fingerprint analysis. This will also help to locate

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the core-points in the fingerprint [2] with good accuracy. We use the method proposed in [10] to locate the core-point in the fingerprint image.

The paper is organized into five sections. Section II explains about the background of the image enhancement methods. Section III highlights on the proposed method and the numerical schemes employed for solving the PDE's. In section IV the experimental results are demonstrated and compared. Section V concludes the work.

II. Image Enhancement Using Shock Filters

The classical shock filter was introduced by Rudin and Osher in [12] for image enhancement. The shocks are developed at the zero crossings of the second derivative, while the local extrema remain unchanged in each evolution. New local extrema is not introduced as a part of evolution and the steady state solution will have discontinuities at the zero crossing points of the *Laplacian* of the image. These properties make this filter behave like a deconvolution filter which can sharpen the image edges. One of the major issues with classical shock filter is; it enhances the noise along with the image features. The "classical" shock filter is:

$$I_t = -sign(I_{\eta\eta})\,|\nabla I| \tag{1}$$

where I_t stands for $\frac{\partial I}{\partial t}$, η is in the direction of the gradient ∇I of image function I, $|.|$ denotes the Euclidean norm and $sign$ function is defined as:

$$sign(x) = \begin{cases} -1 & if\ x<0 \\ 0 & if\ x=0 \\ +1 & if\ x>0 \end{cases}$$

Note that the PDE in (1) is a hyperbolic PDE, with the Neumann Boundary condition:

$$\frac{\partial I}{\partial \vec{n}} = 0 \tag{2}$$

where $\vec{n}$ is the outward normal to the level curve, and

$$I(x,y,0) = I_0(x,y) \tag{3}$$

is the initial image. Throughout this paper we assume the boundary condition (2) and the initial condition (3) for all the PDEs, unless stated otherwise. The hyperbolic PDE in (1) is an anisotropic one, which will give a "shock" in the direction of gradient and will not have any effect on the direction of isophotes, making the edges more prominent while enhancing the noise as well. This property makes them unsuitable for image enhancement, because in many practical scenarios the images are found to be corrupted by noise.

A. Shock Coupled Diffusion

Alvarez et. al. [1] proposed to use curvature based diffusion along with the shock term in order to avoid the noise enhancing property of the "classical" shock filter in (1). This filter is formulated as:

$$I_t = -sign(G_\sigma * I_{\eta\eta})\,|\nabla I| + \lambda I_{\xi\xi} \tag{4}$$

where

$$I_{\xi\xi} = |\nabla I|\, div\left(\frac{\nabla I}{|\nabla I|}\right) \tag{5}$$

denotes the MCM term, λ denotes the control parameter which is a positive scalar value, η is the direction along the gradient and ξ the direction perpendicular to the gradient. This anisotropic filter will act like, a "shock" filter in the direction of gradient and a diffusion filter in the direction of the isophotes (level-lines). The term $I_{\xi\xi}$ represents MCM, where each of the level curves in the image moves in the normal direction in the speed proportional to their mean curvature. The parameter λ controls the amount of diffusion. This diffusion does not work well with structures having high coherence especially in fingerprint images. In these images there are many parallel lines, so the edge descriptor used in MCM will cancel out the orientations with opposite signs. The gradient smoothing will average directions instead of orientations. This will adversely affect the parallel edges. To make the edge descriptor neutral to sign changes the structure tensor can be used as a descriptor.

In MCM each of the level curves continues to evolve in the normal direction until they converge to a point. During the course of evolution the convex level-lines remain convex and non-convex level lines become convex after finite number of iterations. This property makes the MCM a non-suitable choice for denoising the images with structures having non-zero curvature value like curvy lines, which are common in fingerprint images. The structure descriptor used in most of the non-linear diffusion methods including MCM is ∇I which is not very suitable for finding parallel edges. A Gaussian convolved gradient image ∇I_σ (σ is the spread of the Gaussian Kernel) is used instead of gradient image ∇I as a structure descriptor in [4]. Here σ denotes the noise scale and it makes the detector ignorant of the details having frequency smaller than $O(\sigma)$. Even though ∇I is a useful edge detector, it fails to find the parallel structures. This is because for larger values of σ the gradient smoothing or averaging will cancel out the gradient with same orientation but different signs giving a low response value at the parallel edges.

B. Coherence Enhancing Diffusion

To make the descriptor invariant under sign changes a structure tensor was proposed as an edge detector instead of the gradient image, by Weickert in [14], [15]. Here the descriptor ∇I_σ was replaced with its tensor product

$$\begin{aligned} J_0(\nabla I_\sigma) &:= \nabla I_\sigma \otimes \nabla I_\sigma \\ &:= \nabla I_\sigma\, \nabla I_\sigma^T \end{aligned} \tag{6}$$

where J_0 is a symmetric positive definite matrix and its eigenvectors are parallel and orthogonal to ∇I_σ respectively. The corresponding eigenvalues are $|\nabla I_\sigma|^2$ and 0. Now replacing the direction by orientation and averaging the orientation we get the structure tensor as:

$$J_\rho(\nabla_\sigma I) := G * \nabla I_\sigma \otimes \nabla I_\sigma \quad \rho > 0 \tag{7}$$

where G is denotes the Gaussian function with standard deviation ρ and I_σ is the regularized version of the image I. The eigenvalues (λ_i) of the symmetric matrix J_ρ integrate the gray-level changes in the neighbourhood of $O(\rho)$ and they represents the average contrast in the direction of corresponding eigenvectors (v_i). These eigenvalues reveal important information regarding the coherence of the structures in the neighbourhood of $O(\rho)$. The measure of coherence can be written as:

$$c = \sum_{i=1}^{m-1} \sum_{j=i+1}^{m} (\lambda_i - \lambda_j)^2 \tag{8}$$

The value c determines the magnitude of coherence i.e. if $c \to 0$ then that area is having less coherence features and is a homogeneous smooth area. Now the anisotropic diffusion with the diffusion tensor evolves under the equation:

$$I_t = \nabla.(D \nabla I) \tag{9}$$

where $I(x, y, t)$ is the evolving image in time t and D is the diffusion tensor which is constructed in such a way that it has the same eigenvalues as J_ρ.

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix} = R^T \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} R$$

where R is the rotation matrix:

$$R = \frac{1}{\sqrt{(\partial I^\rho / \partial x)^2 + (\partial I^\rho / \partial y)^2}} \begin{pmatrix} \partial I^\rho / \partial x & -\partial I^\rho / \partial y \\ \partial I^\rho / \partial y & \partial I^\rho / \partial x \end{pmatrix}$$

whose columns are the eigenvalues of the structure tensor J_σ, c_1 and c_2 are the conductivity coefficients along the principal directions and ρ is the scale. The eigenvectors of R and J_σ can be calculated analytically this will lead to the diffusion tensor D with the components:

$$d_{11} = \frac{1}{2}\left(c_1 + c_2 + \frac{(c_2 - c_1)(s_{11} - s_{22})}{\alpha} \right) \tag{10}$$

$$d_{12} = \frac{(c_2 - c_1) s_{12}}{\alpha} \tag{11}$$

$$d_{22} = \frac{1}{2}\left(c_1 + c_2 - \frac{(c_2 - c_1)(s_{11} - s_{22})}{\alpha} \right) \tag{12}$$

where $s_{11} = G_\sigma R_x^2,\ s_{12} = G_\sigma R_x R_y,\ s_{22} = G_\sigma R_y^2$ with $R_x = \partial R / \partial x$, $R_y = \partial R / \partial y$ and $G_\sigma R$ is the Gaussian convolved version of R with the scale parameter σ and the parameter $\alpha = \sqrt{(s_{11} - s_{22})^2 + 4 s_{12}^2}$. The eigenvalues of the structure tensor are given by $\lambda_{1,2} = \frac{1}{2}(s_{11} + s_{22} \pm \alpha)$ these eigenvalues determine the speed of diffusion c_1 and c_2. Here $c_1 = max\left(0.01, 1 - e^{-(\lambda_1 - \lambda_2)^2 / k^2}\right)$, $c_2 = 0.01$. Further details can be found in [14] and [15].

C. Core-point Detection in Fingerprints

Core-point detection is an important activity in fingerprint analysis, see [2]. Most of the fingerprint detection algorithms are developed based on the core-point detection. Once the core-point is located then the feature vector is generated from this core-point and the detection process is initiated. In most of the practical cases, the fingerprints that are subjected to verification may be of poor quality. So a pre-processing step is essential to make the core-point detection more reliable. Generally image enhancement is the pre-processing step followed in most of the methods proposed in literature. One of the remarkable issues with ordinary enhancement methods is that they give less concern to the coherence structure of the fingerprints. Hence most of the enhancement techniques will first diffuse in order to remove the noise while enhancing the flow-like structures. This may not be adequate for the images abundant in flow-like structures, especially in fingerprint images.

All these facts motivated us to combine the shock as well as the coherence enhancing diffusion to devise a new filter which is capable of denoising as well as enhancing the flow-like structures with utmost care to the coherence of image features. The details are given in next section.

There are many methods available in literature for detecting the core-points in the fingerprint image see [2], [10]. The commonly used methods are based on the orientation tensor field [2] and the complex patterns. In this work we use the complex filtering technique in [10], due to its efficiency, high accuracy and its multi-resolution detection capability. The core-point in a fingerprint can be represented with complex pattern. When a complex image is filtered using a complex filter the corresponding complex patterns can be identified [10]. A symmetric Gaussian filter is used to create the complex filter.

Multi-resolution techniques are more robust for core-point detection, and noise removal. In this method the core-point is detected at the lowest resolution level or at the top of the Gaussian pyramid and it is projected to the lower levels to refine it-up. The same filter can be applied in different resolution levels to get the abstract position of the core-point at the corresponding level.

The first derivative of Gaussian is defined as a complex partial derivative defined as: $D_x + i\,D_y = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$. When it is applied to a function $f(x,y)$ this will result in a complex field than a vector field. Complex filters of order m can be used for identifying the pattern (core-point) in the fingerprints [10]. An approximation of this filter in a Gaussian window will yield $(x+iy)^m\, g(x,y)$ *where* $g(x,y)$ is the Gaussian window defined as: $g(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$. The choice of Gaussian window can be well justified by its isotropic properties. The first order filter is sufficient to detect the core-points in the fingerprints, hence we take $m = 1$. The core-point detection filter can be easily modelled as: $h = (x+iy)\, g(x,y)$ and this filter is applied on the complex valued orientation tensor field image: $z(x,y) = (f_x + i\, f_y)^2$ to localize the core-point in the image. Here f_x and f_y denotes the gradient along x and y directions respectively.

III. Proposed Method

In this paper we propose to use the shock filter proposed in [1] with a modification in shock and diffusion term. The MCM term in the filter (4) is replaced with a structure tensor proposed in [14], so the diffusion term will be as in (9). The modified filter can be formulated as:

$$I_t = -sign(G_\sigma * I_{\eta\eta} t)\,|\nabla I| + \nabla.(D\nabla I) \qquad (13)$$

where D is as in (9). It is possible to replace the hard $sign$ with a "*soft-* $sign$" function for getting more natural results, so we replace the sign with arctan function, therefore equation in (13) can be written as:

$$I_t = -\frac{2}{\pi} arctan(G_\sigma * I_{\eta\eta} t)\,|\nabla I| + \nabla.(D\nabla I) \qquad (14)$$

The parameter t represents the time of evolution. Since during the early stages of evolution the noise will be dominant in the signal, so the diffusion should be prominent as compared to shock, this can be handled by incorporating the time function with the shock term see [5]. Hence the shock component will have less response during the early stages of evolution and will dominate as the evolution progresses. By using this diffusion tensor as a coefficient of diffusion the filter will get an anisotropic nature. This will enhance the coherence of the image features by diffusing only in the direction along the isophotes.

A. Numerical Implementation

We use the explicit Euler numerical schemes for solving the PDEs given in the above equations. Since the shock filters are hyperbolic PDEs, the usual central difference scheme may not help in getting a proper result. This scheme will be highly unstable and may not converge. So we use the upwind scheme proposed in [11]. The scale space parameter h is assumed to be 1 and Δt is the time step. Using the upwind for solving ∇I in (14) will result in the following expressions:

$$|\nabla I| = \sqrt{D_x^2 + D_y^2}$$

where

$$D_x = minmod(I_x^+(x,y), I_x^-(x,y))$$

$$D_y = minmod(I_y^+(x,y), I_y^-(x,y))$$

The $minmod$ operator is defined as:

$$minmod(x,y) = \begin{cases} min(|x|,|y|) & if\ \ xy > 0 \\ 0 & \text{Otherwise} \end{cases}$$

$$\begin{aligned} I_x^+(x,y) &= I(x+1,y) - I(x,y), \\ I_x^-(x,y) &= I(x,y) - I(x-1,y), \\ I_x(x,y) &= (I_x^+ + I_x^-)/2, \\ I_y^+(x,y) &= I(x,y+1) - I(x,y), \\ I_y^-(x,y) &= I(x,y) - I(x,y-1), \\ I_y(x,y) &= (I_y^+ + I_y^-)/2 \end{aligned} \qquad (15)$$

The diffusion equation $\nabla.(D\nabla I)$ is implemented using finite central difference formula.

IV. Results and Discussions

We carried out the experiments with different fingerprints which are degraded by noise and blurred by out-of-focus blur. The offline fingerprint database used for the test purpose contains 200 fingerprint images of dimension 512×512 pixels. The results of the experiments are shown in the figures below. The method is tested for naturally degraded image and the image which is corrupted by the Gaussian noise making the SNR 10dB. The image is blurred (out-of-focus) by convolving it with a Gaussian kernel of standard deviation $\sigma = 17$. We apply the core-point detection on the enhanced image. It can be easily found that the core-point detection is robust in the proposed method than the normal enhancing method available in literature. Figure 1 and Fig. 3 show the results of different enhancement methods including the proposed one. The results are taken after 15 iterations with time step Δt=0.1and scale space h=1. Figure 2 and Fig. 4 show the result of the core-point detection algorithm applied to the enhanced and non-enhanced images. It can be easily verified that after enhancing with the proposed method the core-point detection is more accurate.

We quantify the accuracy of core-point detection using the absolute distance of the core-point detected by various methods from the actual one which is manually detected. Table I shows the *absolute distance*, calculated using the following expression:

$$dist = \left(\left(x_{man} - x_{det} \right)^2 + \left(y_{man} - y_{det} \right)^2 \right)^{\frac{1}{2}}, \qquad (16)$$

where x_{man} and y_{man} denote the x and y coordinates of manually detected core-point , x_{det} and y_{det} denote the x and y coordinates of the core-points detected using the automated methods.

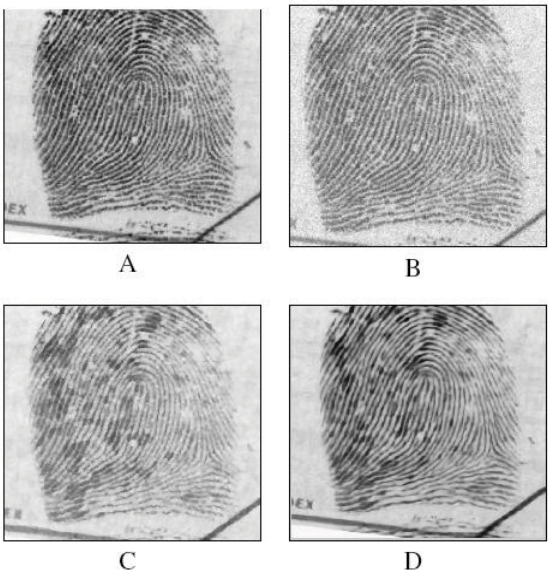

Fig. 1 Fingerprint images (before and after processing) :(A) Original image(B) Blur and noisy image (out of focus blur) (C) After applying filter in [1](D) After applying the proposed method.

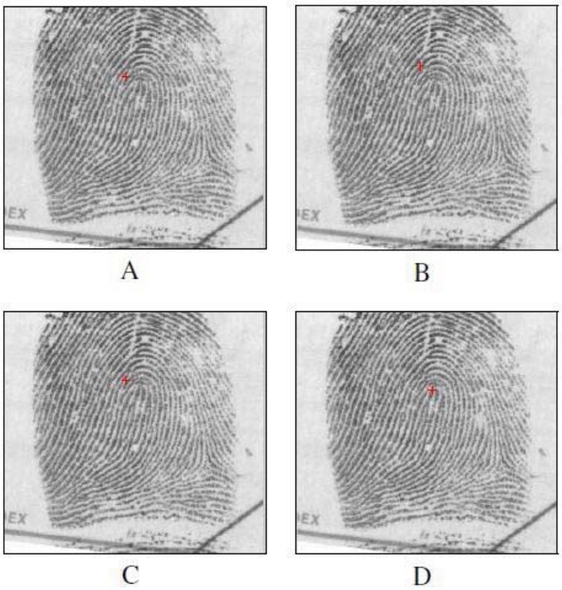

Fig. 2 Core-point detected in the fingerprint: (A) Without any enhancement (B) Using the enhancement with the model in [1] (C) After applying coherence enhancement in [14] (D) After enhancing with the proposed method.

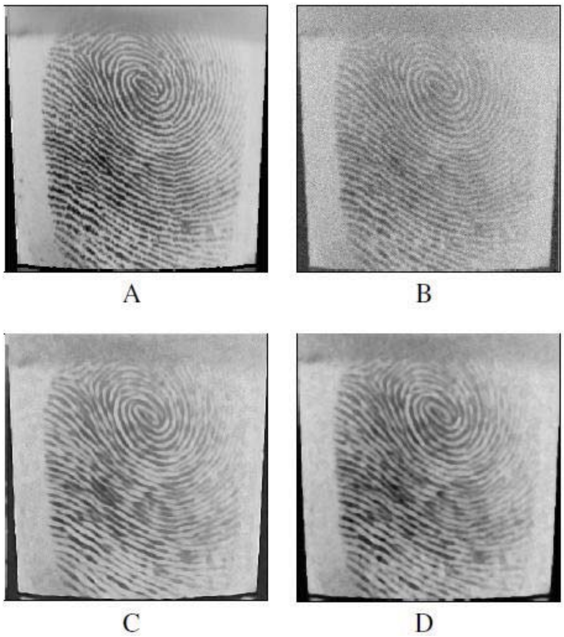

Fig. 3 Fingerprint images (before and after processing) :(A) Original image(B) Blur and noisy image (out of focus blur) (C) After applying filter in [1](D) After applying the proposed method.

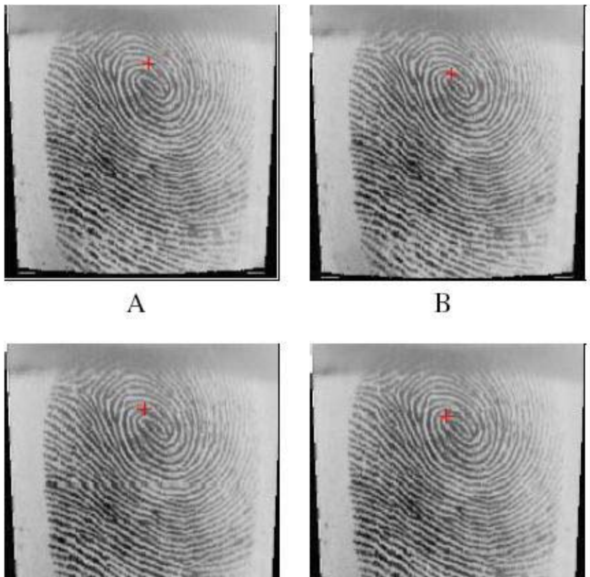

Fig. 4 Core-point detected in the fingerprint: (A) Without any enhancement (B) Using the enhancement with the model in [1] (C) After applying coherence enhancement in [14] (D) After enhancing with the proposed method.

TABLE I. THE CORE-POINT DETECTION ACCURACY FOR VARIOUS ENHENCEMENT METHODS.

Methods	Absolute distance in pixels calculated using (16)	
	With noise variance σ=0.4	*With noise variance σ=0.8*
Without enhancement	15.2	18.2
Alvarez model [1]	9.6	12.8
Weickert model [14]	8.7	10.4
Proposed method	5.7	8.7

It is quite evident from Table I that the proposed method has the least value for the *absolute distance* (measured in pixels), in other words the proposed enhancement method helps in locating core-points with a better accuracy as compared to the other enhancement methods in the literature. The values shown in Table I are the average values obtained after applying different methods on 30 different fingerprints in the offline fingerprint database.

The contrast enhancing capability of the method is tested using a local contrast measure. Let $I(x, y)$ be the intensity of the pixel at (x, y) coordinates of the image and let us consider a $2n+1$ neighbourhood around the pixel, then the local contrast at the pixel (x, y) is defined as:

$$C(x,y) = \frac{I_{max}(x,y) - I_{min}(x,y)}{I_{max}(x,y) + I_{min}(x,y)} \quad (17)$$

where $I_{min}(x,y)$ and $I_{max}(x,y)$ are the minimum and maximum values of the pixels in the neighbourhood respectively. The metric used to measure the performance of the method in a region is defined by the average contrast of the region defined as:

$$C = \frac{\sum_{(x,y)\in R} C(x,y)}{N} \quad (18)$$

where R is the region selected and N is the number of pixels in the selected region. We have chosen a neighbourhood of size 10×10 and a region (R) of size 10×10 for testing the local contrast of the images, the region is selected from the foreground of the image where the intensity values changes frequently. The local contrast measure of different methods for the image in Fig. 1 is shown in Table II. One can observe from this table that, the proposed method enhances the local contrast much better compared to the other methods.

TABLE II. THE CONTRAST MEASURE FOR VARIOUS ENHENCEMENT METHODS.

Methods	Contrast measure defined in (18)	
	With noise variance σ=0.4	*With noise variance σ=0.8*
Noisy Image	0.324	0.245
Alvarez model [1]	0.374	0.321
Weickert model [14]	0.356	0.314
Proposed method	0.441	0.391

V. CONCLUSION

In this paper we have proposed an enhancement method tailored to the fingerprint images by modifying the shock filter proposed in [1] and the diffusion filter proposed in [14] and we have shown that coupling the shock filter and diffusion filter in [1] and [14] respectively will yield a good enhancement to the flow-like structures. Further we have demonstrated that the core-point detection process in fingerprint identification will be accurate and robust with the proposed enhancement.

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