Finite Dimensional Realization of a Guass-Newton Method for Ill-Posed Hammerstein Type Operator Equations

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Abstract. Finite dimensional realization of an iterative regularization method for approximately solving the non-linear ill-posed Hammerstein type operator equations $KF(x) = f$, is considered. The proposed method is a combination of the Tikhonov regularization and Guass-Newton method. The advantage of the proposed method is that, we use the Fréchet derivative of F only at one point in each iteration. We derive the error estimate under a general source condition and the regularization parameter is chosen according to balancing principle of Pereverzev and Schock (2005). The derived error estimate is of optimal order and the numerical example provided proves the efficiency of the proposed method.

Keywords: Newton's method, Tikhonov regularization, ill-posed Hammerstein operator, Balancing principle, Monotone operator, Regularization.

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1 Introduction

Let X be a real Hilbert space, $F : D(F) \subseteq X \to X$ be a monotone operator (i.e., $\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in D(F)$) and $K : X \to Y$ be a bounded linear operator between the Hilbert spaces X and Y. Consider the ill-posed operator equation

$$KF(x) = f. \tag{1.1}$$

Equation (1.1) is called ill-posed Hammerstein type([1], [2], [3], [4]) operator equation. Throughout the paper, the domain of F is denoted by $D(F)$, the Fréchet derivative of F is denoted by $F'(.)$ and the inner product and norm in X and Y are denoted by $\langle ., .\rangle$ and $\|.\|$ respectively.

It is assumed that the available data is f^δ with $\|f - f^\delta\| \leq \delta$ and hence one has to consider the equation

$$KF(x) = f^\delta \tag{1.2}$$

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instead of (1.1). Since (1.1) is ill-posed, its solution is not depending continuously on the given data. Thus one has to use regularization method (see [1], [2], [3], [4], [6], [7] and [10]) for obtaining an approximation for $\hat{x}$.

Observe that the solution x of (1.2) can be obtained by first solving

$$Kz = f^{\delta} \tag{1.3}$$

for z and then solving the non-linear problem

$$F(x) = z. \tag{1.4}$$

This was exploited in [1], [2], [3], [4] and [5]. As in [4], we assume that the solution $\hat{x}$ of (1.1) satisfies

$$\|F(\hat{x}) - F(x_0)\| = \min\{\|F(x) - F(x_0)\| : KF(x) = f, x \in D(F)\}.$$

The prime motive of this study is to develop an iterative regularization method to obtain an approximation for $\hat{x}$ in the finite dimensional subspace of X. Precisely we considered Discretized Tikhonov regularization for solving (1.3) and Discretized Newton's method for solving (1.4).

This paper is organized as follows. Preliminaries are given in Section 2, Section 3 deals with the convergence of the proposed method. A numerical example is given in Section 4 and finally the paper ends with a conclusion in section 5.

2 Preliminaries

Let $\{P_h\}_{h>0}$ be a family of orthogonal projections on X, let $\varepsilon_h := \|K(I - P_h)\|$, $\tau_h := \|F'(x)(I - P_h)\|$, $\quad \forall x \in D(F)$. Let $\{b_h : h > 0\}$ is such that $\lim_{h\to 0} \frac{\|(I-P_h)x_0\|}{b_h} = 0$, $\lim_{h\to 0} \frac{\|(I-P_h)F(x_0)\|}{b_h} = 0$ and $\lim_{h\to 0} b_h = 0$. We assume that $\varepsilon_h \to 0$ and $\tau_h \to 0$ as $h \to 0$. The above assumption is satisfied if, $P_h \to I$ pointwise and if K and $F'(x)$ are compact operators. Further we assume that $\varepsilon_h < \varepsilon_0$, $\tau_h \le \tau_0$, $b_h \le b_0$ and $\delta \in (0, \delta_0]$.

In [5], the authors studied a two step newton method defined iteratively by

$$y_{n,\alpha_k}^{h,\delta} = x_{n,\alpha_k}^{h,\delta} - (P_h F'(x_{n,\alpha_k}^{h,\delta})P_h + \frac{\alpha_k}{c}P_h)^{-1}[F(x_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c}(x_{n,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta})], \tag{2.5}$$

$$x_{n+1,\alpha_k}^{h,\delta} = y_{n,\alpha_k}^{h,\delta} - (P_h F'(x_{n,\alpha_k}^{h,\delta})P_h + \frac{\alpha_k}{c}P_h)^{-1}[F(y_{n,\alpha_k}^{h,\delta}) - z_{\alpha_k}^{h,\delta} + \frac{\alpha_k}{c}(y_{n,\alpha_k}^{h,\delta} - x_{0,\alpha_k}^{h,\delta})], \tag{2.6}$$

where $c \le \alpha_k$, $x_{0,\alpha_k}^{h,\delta} := P_h x_0$, the projection of initial guess x_0 and

$$z_{\alpha_k}^{h,\delta} = (P_h K^* K P_h + \alpha_k I)^{-1} P_h K^*[f^{\delta} - KF(x_0)] + P_h F(x_0), \tag{2.7}$$

for obtaining an approximation for $\hat{x}$ in the finite dimensional subspace $R(P_h)$, the range of P_h, in X.

The main draw back of this approach was that the iterations (2.5) and (2.6) requires Fréchet derivative of F at each iteration $x_{n,\alpha_k}^{h,\delta}$.