Generalized Voronoi partition: A new tool for optimal placement of base stations

K.R. Guruprasad
Department of Mechanical Engineering
National Institute of Technology Karnataka, Surathkal, 57025, India
krgprao@gmail.com

***Abstract*—This paper presents a novel framework for placement of base stations (BS)s. A generalization of Voronoi partition is used, where, a set of node functions are used in place of the usual distance measure. Functions modeling the effectiveness of BSs at different locations in the region are used as node functions, and the locations of BSs as nodes or sites. Further the base stations are assumed to be heterogeneous and anisotropic.**

I. Introduction

Cellular network planning is a multi-objective optimization problem, which involves deciding on the number of BSs, their configuration such as power, type of antennae, height of the tower, etc, and locating BSs in the geographical area [1], [2], [3], [4], [5]. This problem is known to be NP-hard [6]. In order to solve this problem in a computationally feasible manner, in majority of works in the literature, a (finite) set of candidate sites is selected, where BSs could potentially be located, along with a finite set of service test points [7], [8]. In [9], authors consider combinatorial problem which has two components, first, a continuous component, a site placement problem, and second, an integer component, referred to as site selection problem. The continuous component solves the problem of optimally locating a fixed number of BSs given a user distribution, while the integer component finds the minimum number of BSs. The BSs are assumed to be homogeneous and isotropic in their service capabilities.

In this paper, we address the site placement problem. This problem belongs to a class of problems known as facility location [10] or locational optimization problems. Centrodal Voronoi configuration [11], [12], where each of the sites is located at the centroid of corresponding Voronoi cell, is one of the well known solution to these problems. The concept of centroidal Voronoi configuration has been used in sensor coverage [13], multi-agent systems [14], and several other applications. In fact, the problem of BS placement is similar to a sensor coverage optimization problem. However, the standard Voronoi partition can not be used in a scenario where the BSs have heterogeneous and anisotropic (directional antennae) characteristics. Further, any geographical terrain is not a planar 2-dimensional area. In this paper, we use a generalization the standard Voronoi partition [15], [16] replacing the usual distance measure with concept of node functions associated with each sites or nodes. Using this generalization, we provide a framework for formulating and solving optimal site placement problem, taking into account heterogeneity and anisotropy of the BSs, and non-planar terrain.

K.R. Guruprasad is currently visiting the Department of Computer Science, University of Nebraska at Omaha, Omaha, NE, USA, 68182.

II. Generalized Voronoi partition

Voronoi partition [12] (named after Georgy Voronoi), also called *Dirichlet tessellation* (named after Gustav Lejeune Dirichlet), is a widely used scheme of partitioning a given space based on the concept of "nearness" of points in a set to some finite number of pre-defined locations in the set. By a *partition* of a set X we mean a collection of subsets W_i of X with disjoint interiors such that their union is X itself. Let $Q \subset \mathbb{R}^d$, be a convex polytope. Let $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$ be a finite set of nodes, or generators, or sites, $p_i \in Q$, and $I_N = \{1, 2, \ldots, N\}$. The *Voronoi partition* generated by $\mathcal{P}$ is the collection $\{V_i(\mathcal{P})\}_{i \in I_N}$ defined as,

$$V_i(\mathcal{P}) = \{q \in Q | \parallel q - p_i \parallel \leq \parallel q - p_j \parallel, \forall j \in I_N \setminus \{i\}\} \quad (1)$$

The Voronoi cell V_i is the collection of those points which are closest (with respect to the Euclidean metric) to p_i compared to any other point in $\mathcal{P}$. Each of the Voronoi cells is a topologically connected non-null set.

Let us consider the problem of deploying $N \in \mathbb{N}$ BSs which are homogeneous in their characteristics (such as their power), and each of them having isotropic effectiveness (omnidirectional antennae). The locations of BSs as node (or site) set induces a Voronoi partition, where each BS is effective within the corresponding Voronoi cell. This is due to the fact that effectiveness of a BS located at $p_i \in Q$ at a point $q \in Q$, is naturally assumed to be a strictly decreasing function of $\|p_i - q\|$. However, it is possible that the BSs are heterogenous in nature (such as power, tower height, etc.), and their effectiveness may also be anisotropic (directional antennae). Even if the BSs are homogeneous and isotropic, variation of altitude of $q \in Q$ (non-planar geographical area), or any other issues such as interference or absorption of signals in some regions in Q, make the effectiveness of BS heterogeneous and anisotropic. Known heterogeneity and anisotropy can be still be incorporated by using some of the well known generalizations of the standard Voronoi partition such as weighted Voronoi partition, Power diagram, and Voronoi partition using other non-Euclidean distances. However, each generalization handles only a particular situation. In the following, we discuss a generalization of the standard Voronoi partition to incorporate the heterogeneity and anisotropy of the BSs' effectiveness for their optimal deployment.

Consider a space $Q \subset \mathbb{R}^d$, a set of points called *nodes* or *generators* or *sites* $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$, $p_i \in Q$, with $p_i \neq p_j$, whenever $i \neq j$, and functions $f_i : \mathcal{P} \times Q \mapsto \mathbb{R}$, where f_i is called a *node function* for the i-th node. Define a collection $\{V_i^g\}$, $i \in I_N$, such that $Q = \cup_{i \in I_N} V_i^g$, where V_i^g is defined as

$$V_i^g = \{q \in Q | f_i(p_i, q) \geq f_j(p_j, q), \forall j \in I_N \setminus \{i\}\} \quad (2)$$

For every $i, j \in N$, such that $V_i^g \cap V_j^g \neq \emptyset$, define the boundary between Voronoi cells V_i and V_j as

$$\partial V_{ij}^g = \{q \in Q | f_i(p_i, q) = f_j(p_j, q)\} \quad (3)$$

If the collection $\{V_i^g | i \in I_N\}$, has mutually disjoint interiors, we call such a collection partitions Q, and we call $\mathcal{V}^g(\mathcal{P}) = \{V_i^g\}$, $i \in I_N$, as a *generalized Voronoi partition* of Q with nodes $\mathcal{P}$ and node functions f_i.

For $\{V_i^g\}$, $i \in I_N$ to qualify as a generalized Voronoi partition, the boundaries of each of the cells V_i^g need to be sets of measure zero. Thus, the node functions should be selected to ensure that ∂V_{ij}^g should be a set of measure zero. It can be noted that, the standard Voronoi configuration and its known variations can be extracted from the generalized Voronoi partition (2) by suitably selecting i) space to be partitioned, ii) site set, and iii) the node functions. For example setting $f_i(p_i, q) = -\|p_i - q\|$, in (2) gives the standard Voronoi partition (1).

III. Locational optimization of BSs

Consider a set of BSs, $\mathcal{B} = \{B_1, B_2, \ldots, B_N\}$ to be deployed in $Q \subset \mathbb{R}^2$, the identified geographic area. Let $h(q), q \in Q$ represent the hight of point $q \in Q$ above sea level. Let $p_i \in Q$ be the position of i-th BS with a tower height h_i. Let $\phi : Q \to [0, 1)$ represent the (relative) user distribution. Higher the $\phi(q)$, higher should be the signal strength available (or effectiveness) at $q \in Q$. Let $f_i(p_i, q)$ be the signal strength available at $q \in Q$ from the BS B_i. In fact, f_i depends on several parameters such as, the transmitting power, antenna type, antenna orientation, the tower height h_i, etc. of the BS B_i, and the terrain $h(q)$. In fact designing these parameters, and modeling f_i itself is very important component of the cellular network planing. However, these issues are beyond the scope of this paper. For the purpose of formulating the optimal deployment problems, we assume that f_is are known . We call f_i as effectiveness function of BS B_i, and assume they are sufficiently smooth. Unlike in several locational optimization problems in the literature including BS placement problems [9], we let the BS effectiveness functions to be heterogenous and anisotropic. Because of this heterogeneity and anisotropy, the standard Voronoi partition and its known variations (such as weighted Voronoi partition, Power diagrams, Voronoi partition with other non-Euclidean distances, etc.) cannot be used here.

Consider the objective function,

$$\mathcal{H}(\mathcal{P}, \mathcal{W}) = \sum_{i \in I_N} \mathcal{H}(\mathcal{P}, W_i) = \sum_{i \in I_N} \int_{W_i} f_i(q, p_i) \phi(q) dQ \quad (4)$$

Where $\{W_i | i \in I_N\}$ is a partition of Q. Each BS B_i serves $W_i \subset Q$. The configuration $\mathcal{P}$ of locations of the BSs, maximizing (4), gives optimal locations of BSs in Q. There are two aspects in (4): first being the partition $\{W_i\}$, and second being $p_i \in Q$, locating BSs within the cell. It is intuitive and easy to show that the generalized Voronoi partition with $\mathcal{P}$ as site set and $\{f_i\}, i \in I_N$ as node functions is optimal partitioning. Thus, the objective function (4) now becomes

$$\mathcal{H}(\mathcal{P}) = \sum_{i \in I_N} \mathcal{H}(\mathcal{P}) = \sum_{i \in I_N} \int_{V_i^g} f_i(q, p_i) \phi(q) dQ \quad (5)$$

A very important and useful property of the objective function (5) is that it is sum of contribution from components corresponding to each BS.

Lemma 1: The gradient of the objective function (5) with respect to p_i is given by

$$\frac{\partial \mathcal{H}}{\partial p_i} = \int_{V_i} \phi(q) \frac{\partial f_i(p_i, q)}{\partial p_i} dQ \quad (6)$$

Proof. Applying the general form of the Leibniz theorem [17]

$$\begin{aligned} \frac{\partial \mathcal{H}}{\partial p_i} = & \int_{V_i^g} \phi(q) \frac{\partial f_i}{\partial p_i}(p_i, q) dQ \\ & + \sum_{j \in N_i} \int_{\partial V_{ij}^g} \mathbf{n}_{ij}(q).\mathbf{u}_{ij}(q) \phi(q) f_i(p_i, q) dQ \quad (7) \\ & + \sum_{j \in N_i} \int_{\partial V_{ji}^g} \mathbf{n}_{ji}(q).\mathbf{u}_{ji}(q) \phi(q) f_j(p_j, q) dQ \end{aligned}$$

where, $N_i = \{j | V_i^g \cap V_j^g \neq \emptyset\}$; $\mathbf{n}_{ij}(q)$ is the unit outward normal to ∂V_{ij}^g at $q \in \partial V_{ij}^g$; and $\mathbf{u}_{ij}(q) = \frac{d(\partial V_{ij}^g)}{dp_i}(q)$, the rate of movement of the boundary at $q \in \partial V_{ij}^g$ with respect to p_i. Note that $\mathbf{n}_{ij}(q) = -\mathbf{n}_{ji}(q)$, $\mathbf{u}_{ij}(q) = \mathbf{u}_{ji}(q)$, $f_i(p_i, q) = f_j(p_j, q)$, $\forall q \in \partial V_{ij}^g$, by definition of the generalized Voronoi partition, and ϕ is bounded. Thus, last two terms on the righthand side of Eqn. (7) cancel each other. □

The height h_i of the tower and the height $h(q)$ can also be incorporated into the optimization problem by using $\hat{f}_i(\hat{p}_i, \hat{q})$ as node functions with $\hat{p}_i = (p_i(1), p_i(2), h(p_i) + h_i)'$ and $\hat{q} = (q(1), q(2), h(q))'$. In such a scenario, the constraints to be imposed on $\hat{p}_i, i \in I_N$ are $p_i \in Q$ and $0 \leq h_i \leq h_{\max}$, where $h_{\max}$ is the maximum allowed height of the tower.

Now, the optimization problem can be solved by one of the several methods such as steepest gradient, using

$$\dot{\hat{p}}_i = \eta_i \frac{\partial \mathcal{H}}{\partial \hat{p}_i} \quad (8)$$

or a discrete update rule

$$\hat{p}_i(t + 1) = \hat{p}_i(t) + \eta_i \frac{\partial \mathcal{H}}{\partial \hat{p}_i} \quad (9)$$

Where, $\eta_i > 0$ is chosen step-size.

The gradient based approaches are known to guarantee only a local optimal solutions. There are several methods reported in the literature which ensure the global optimal solution for the objective function (5). The exact solution to the optimization problem depends on f_i, and is beyond the scope of this paper. However, in the following, we provide a brief discussion on some known special cases of the objective function (5) presented as the general framework.

Homogeneous and isotropic f_i: Consider $f_i(p_i, q) = f(\|p_i - q\|), \forall i \in I_N$, with $f(\cdot)$ a strictly decreasing function. This situation is related to homogeneous and isotropic BSs and a flat terrain Q. It is easy to see that the objective function (5) now takes the form

$$\mathcal{H}(\mathcal{P}) = \sum_{i \in I_N} \mathcal{H}(\mathcal{P}) = \sum_{i \in I_N} \int_{V_i} f(\|p_i - q\|)\phi(q)dQ \quad (10)$$

A local optimal solution to the objective function (10) is when the BS are located at the centroid of their (standard) Voronoi cells with $\tilde{\phi}(q) = -2\phi(q)\frac{\partial f(r_i)}{\partial({r_i}^2)} \in [0, 1)$ as density (note that f is strictly decreasing and hence its derivative is negative), where $r_i = \|p_i - q\|$ [14]. (In most homogeneous locational optimization problems, $f(r_i) = -r_i^2$ is used, and hence $\tilde{\phi}(q) = \phi(q)$ [9].)

Heterogenous and isotropic f_i: Consider $f_i(p_i, q) = f_i(\|p_i - q\|), \forall i \in I_N$, with strictly decreasing functions $f(\cdot)$s. This situation incorporates BS with different power and tower heights. The objective function (5) now takes the form

$$\mathcal{H}(\mathcal{P}) = \sum_{i \in I_N} \mathcal{H}(\mathcal{P}) = \sum_{i \in I_N} \int_{V_i^g} f_i(\|p_i - q\|)\phi(q)dQ \quad (11)$$

where V_i^g is a generalized Voronoi cell with $f_i(\|p_i - q\|), i \in I_N$ as node functions. A local optimal solution to the objective function (11) is when the BSs are located at the centroid of their generalized Voronoi cells with $\tilde{\phi}(p_i, q) = -2\phi(q)\frac{\partial f_i(r_i)}{\partial({r_i}^2)} \in [0, 1)$ as density [15]. By suitable choice of a distance measure $d(p, q), p, q \in Q$, and node functions $f_i(d(p_i, q))$, a similar solution is obtained, where the (local) optimal solution is when BSs are located at the centroid of corresponding variation of Voronoi cell (such as weighted Voronoi cell, power diagram, etc). However, in a realistic situation, it may not be possible to model the effectiveness of a BS as a function of a distance measure, and the general objective function (5) should be maximized.

IV. Conclusions

This paper discussed a generalization of Voronoi partition, where in place of usual distance measures used in the standard Voronoi partition and its variations, a set of node functions were used. A framework for formulating and solving the problem of optimal placement of base stations for a cellular network, using the generalized Voronoi partition, was presented. Here, the locations of the BSs are used as nodes or sites and the functions modeling the effectiveness of BSs are used as node functions. The proposed optimal deployment framework can be combined with other aspects of the cellular network planning problem such as optimizing the number of base stations and choosing the operational parameters of base stations, such as, transmitting power, antenna type, antenna orientation, etc.

This paper attempts only to provide a new framework for locational optimization problems in general, and BSs site placement problem in particular. Before the concepts developed in this paper can be successfully applied to a practical problem, several important issues such as existence of optimal solutions/critical points, convergence of the solution to local/global optimal configurations, etc. have to be addressed. Further, it is also interesting to investigate if the concept of centroid can be generalized for the generalized Voronoi cell, and a generalized centroidal Voronoi configuration can be obtained as a local optimal solution.

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