An iterative regularization method for ill-posed Hammerstein type operator equation

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2 authors:

Santhosh George
National Institute of Technology Karnataka

M. Kunhanandan
Goa University

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An iterative regularization method for ill-posed Hammerstein type operator equation

S. George and M. Kunhanandan

Abstract. A combination of Newton's method and a regularization method has been considered for obtaining a stable approximate solution for ill-posed Hammerstein type operator equation. By choosing the regularization parameter according to an adaptive scheme considered by Pereverzev and Schock (2005) an order optimal error estimate has been obtained. Moreover the method that we consider gives quadratic convergence compared to the linear convergence obtained by George and Nair (2008).

Key words. Nonlinear ill-posed equations, Hammerstein type equations, iterative regularization, adaptive choice.

AMS classification. 65J20, 65J10, 65R10.

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1. Introduction

Regularization methods used for obtaining approximate solution of nonlinear ill-posed operator equation

$$Ax = y, \tag{1.1}$$

where A is a nonlinear operator with domain $D(A)$ in a Hilbert space X, and with its range $R(A)$ in a Hilbert space Y, include Tikhonov regularization (see [6, 7, 17, 20, 22, 25]), Landweber iteration [15], iteratively regularized Gauss–Newton method [1] and Marti's method [16]. Here the equation (1.1) is ill-posed in the sense that the solution of (1.1) does not depend continuously on the data y.

The optimality of these methods are usually obtained under a number of restrictive conditions on the operator A (see for example assumptions (10)–(14) and (93)–(98) in [23]). For the special case where A is a Hammerstein type operator, George [10, 11] and George and Nair [14] studied a new iterative regularization method and had obtained optimality under weaker conditions on A (that are more easy to verify in concrete problems).

Recall that a Hammerstein type operator is an operator of the form $A = KF$, where $F : D(F) \subset X \to Z$ is nonlinear and $K : Z \to Y$ is a bounded linear operator where we take X, Y, Z to be Hilbert spaces.

So we consider an equation of form

$$KF(x) = y. \tag{1.2}$$

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In [14] George and Nair, studied a modified form of NLR method for obtaining approximations for a solution $\hat{x} \in D(F)$ of (1.2), which satisfies

$$\|F(\hat{x}) - F(x_0)\| = \min\{\|F(x) - F(x_0)\| : KF(x) = y,\ x \in D(F)\}. \tag{1.3}$$

We assume throughout that the solution $\hat{x}$ satisfies (1.3) and that $y^\delta \in Y$ are the available noisy data with

$$\|y - y^\delta\| \le \delta. \tag{1.4}$$

The method considered in [14] gives only linear convergence. This paper is an attempt to obtain quadratic convergence.

Recall that a sequence (x_n) is X with $\lim x_n = x^*$ is said to be convergent of order $p > 1$, if there exist positive reals β, γ, such that for all $n \in \mathbb{N}$

$$\|x_n - x^*\| \le \beta\, \mathrm{e}^{-\gamma p^n}. \tag{1.5}$$

If the sequence (x_n) has the property that

$$\|x_n - x^*\| \le \beta q^n, \qquad 0 < q < 1,$$

then (x_n) is said to be linearly convergent. For an extensive discussion of convergence rate see Kelley [18].

Organization of this paper is as follows. In Section 2, we introduce the iterated regularization method. In Section 3 we give error analysis and in section 4 we derive error bounds under general source conditions by choosing the regularization parameter by an a priori manner as well as by an adaptive scheme proposed by Pereverzev and Schock in [21]. In Section 5 we consider the stopping rule and the algorithm for implementing the iterated regularization method.

2. Iterated regularization method

Assume that the function F in (1.2) satisfies the following:

1. F possesses a uniformly bounded Fréchet derivative $F'(\cdot)$ in a ball $B_r(x_0)$ of radius $r > 0$ around $x_0 \in X$, where x_0 is an initial approximation for a solution $\hat{x}$ of (1.2).

2. There exists a constant κ_0 such that

$$\|F'(x) - F'(y)\| \le \kappa_0\|x - y\|, \qquad \forall x, y \in B_r(x_0). \tag{2.1}$$

3. $F'(x)^{-1}$ exists and is a bounded operator for all $x \in B_r(x_0)$.

Consider e.g., (cf. [23]) the nonlinear Hammerstein operator equation

$$(KFx)(t) = \int_0^1 k(s,t)h(s,x(s))x(s)\,ds$$

with k continuous and h is differentiable with respect to the second variable. Here $F : D(F) = H^1(]0,1[) \to L^2(]0,1[)$ is given by

$$F(x)(s) = h(s, x(s)), \qquad s \in [0,1],$$

and $K : L^2(]0,1[) \to L^2(]0,1[)$ is given by

$$Ku(t) = \int_0^1 k(s,t)u(s)\,ds, \qquad t \in [0,1].$$

Then F is Fréchet differentiable and we have

$$[F'(x)]u(t) = \partial_2 h(t, x(t))u(t), \qquad t \in [0,1].$$

Assume that $N : H^1(]0,1[) \to H^1(]0,1[)$ defined by $(Nx)(t) := \partial_2 h(t, x(t))$ is locally Lipschitz continuous, i.e., for all bounded subsets $U \subseteq H^1$ there exists $\kappa_0 := \kappa_0(U)$ such that

$$\|\partial_2 h(\cdot, x(\cdot)) - \partial_2 h(\cdot, y(\cdot))\|_{H^1} \leq \kappa_0 \|x - y\| \tag{2.2}$$

for all $x, y \in H^1$. Further if we assume that there exists κ_1 such that

$$\partial_2 h(t, x_0(t)) \geq \kappa_1 \qquad t \in [0,1], \tag{2.3}$$

then by (2.2) and (2.3), there exists a neighborhood $U(x_0)$ of x_0 in H^1 such that

$$\partial_2 h(t, x(t)) \geq \kappa_1/2$$

for all $t \in [0,1]$ and for all $x \in U(x_0)$. So $F'(x)^{-1}$ exists and is a bounded operator for all $x \in U(x_0)$.

Observe that (cf. [14]) equation (1.2) is equivalent to

$$K[F(x) - F(x_0)] = y - KF(x_0) \tag{2.4}$$

for a given x_0, so that the solution x of (1.2) is obtained by first solving

$$Kz = y - KF(x_0) \tag{2.5}$$

for z and then solving the nonlinear equation

$$F(x) = z + F(x_0). \tag{2.6}$$

For fixed $\alpha > 0$, $\delta > 0$ we consider the regularized solution of (2.5) with y^δ in place of y as

$$z_\alpha^\delta = (K + \alpha I)^{-1}(y^\delta - KF(x_0)) + F(x_0) \tag{2.7}$$

if the operator K in (2.5) is positive self adjoint and $Z = Y$, otherwise we consider

$$z_\alpha^\delta = (K^*K + \alpha I)^{-1}K^*(y^\delta - KF(x_0)) + F(x_0). \tag{2.8}$$

Note that (2.7) is the simplified or Lavrentiev regularization of equation (2.5) and (2.8) is the Tikhonov regularization of (2.5).

Now for obtaining approximate solutions for the equation (1.2), for $n \in \mathbb{N}$ we consider $x_{n,\alpha}^\delta$, defined iteratively as

$$x_{n+1,\alpha}^\delta = x_{n,\alpha}^\delta - F'(x_{n,\alpha}^\delta)^{-1}(F(x_{n,\alpha}^\delta) - z_\alpha^\delta), \tag{2.9}$$

with $x_{0,\alpha}^\delta = x_0$.

Note that the iteration (2.9) is the Newton's method for the nonlinear problem

$$F(x) - z_\alpha^\delta = 0.$$

We shall make use of the adaptive parameter selection procedure suggested by Pereverzev and Schock [21] for choosing the regularization parameter α, depending on the inexact data y^δ and the error δ satisfying (1.4).

We shall need the following lemma which can be found in [14].

Lemma 2.1. *Let* $0 < \rho < r$ *and* $x, u \in \overline{B_\rho(x_0)}$. *Then*

$$\|F'(x_0)(x - x_0) - [F(x) - F(x_0)]\| \leq \kappa_0 \|x - x_0\|^2/2,$$

and

$$\|F'(x_0)(x - u) - [F(x) - F(u)]\| \leq \kappa_0 \rho \|x - u\|.$$

3. Error analysis

For investigating the convergence of the iterate $(x_{n,\alpha}^\delta)$ defined in (2.9) to an element $x_\alpha^\delta \in B_r(x_0)$ we introduce the following notations: Let for $n = 1, 2, 3, \dots,$

$$\begin{aligned} \beta_n &:= \|F'(x_{n,\alpha}^\delta)^{-1}\|, \qquad e_n := \|x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta\|, \\ \gamma_n &:= \kappa_0 \beta_n e_n, \qquad d_n := 3\gamma_n(1 - \gamma_n)^{-1}, \\ \omega &:= \|F(\hat{x}) - F(x_0)\|. \end{aligned} \tag{3.1}$$

Further we assume that

$$\gamma_0 := \kappa_0 e_0 \beta_0 < 1/4 \tag{3.2}$$

and

$$\eta := 2e_0 < r. \tag{3.3}$$

Theorem 3.1. *Suppose that* (2.1), (3.2) *and* (3.3) *hold. Then* $x_{n,\alpha}^\delta$ *defined in* (2.9) *belongs to* $B_\eta(x_0)$ *and is a Cauchy sequence with* $\lim_{n\to\infty} x_{n,\alpha}^\delta = x_\alpha^\delta \in \overline{B_\eta(x_0)} \subset B_r(x_0)$. *Further we have the following:*

$$\|x_{n,\alpha}^\delta - x_\alpha^\delta\| \leq \eta d_0^{2^n-1}/2^n \leq \beta\, \mathrm{e}^{-\gamma 2^n}, \tag{3.4}$$

where $\beta = \eta/d_0$ *and* $\gamma = -\log d_0$.

Proof. First we shall prove that

$$\|x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta\| \leq (3/2)\beta_n \kappa_0 \|x_{n,\alpha}^\delta - x_{n-1,\alpha}^\delta\|^2, \tag{3.5}$$

and then by induction we prove $x_{n,\alpha}^\delta \in B_\eta(x_0)$.

Let $G(x) = x - F'(x)^{-1}[F(x) - z_\alpha^\delta]$. Then

$$\begin{aligned} G(x) - G(y) &= x - y - F'(x)^{-1}[F(x) - z_\alpha^\delta] + F'(y)^{-1}[F(y) - z_\alpha^\delta] \\ &= x - y + [F'(x)^{-1} - F'(y)^{-1}]z_\alpha^\delta - F'(x)^{-1}F(x) + F'(y)^{-1}F(y) \\ &= x - y + [F'(x)^{-1} - F'(y)^{-1}](z_\alpha^\delta - F(y)) \\ &\quad - F'(x)^{-1}[F(x) - F(y)] \\ &= F'(x)^{-1}[F'(x)(x-y) - (F(x) - F(y))] \\ &\quad + F'(x)^{-1}[F'(y) - F'(x)]F'(y)^{-1}(z_\alpha^\delta - F(y)) \\ &= F'(x)^{-1}[F'(x)(x-y) - (F(x) - F(y))] \\ &\quad + F'(x)^{-1}[F'(y) - F'(x)](G(y) - y). \end{aligned} \tag{3.6}$$

Now observe that $G(x_{n,\alpha}^\delta) = x_{n+1,\alpha}^\delta$, so by putting $x = x_{n,\alpha}^\delta$ and $y = x_{n-1,\alpha}^\delta$ in (3.6), we obtain

$$\begin{aligned} x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta &= F'(x_{n,\alpha}^\delta)^{-1}[F'(x_{n,\alpha}^\delta)(x_{n,\alpha}^\delta - x_{n-1,\alpha}^\delta) - (F(x_{n,\alpha}^\delta) - F(x_{n-1,\alpha}^\delta))] \\ &\quad + F'(x_{n,\alpha}^\delta)^{-1}[F'(x_{n-1,\alpha}^\delta) - F'(x_{n,\alpha}^\delta)](x_{n,\alpha}^\delta - x_{n-1,\alpha}^\delta). \end{aligned} \tag{3.7}$$

Thus by Lemma 2.1 and (2.1),

$$\|x_{n+1,\alpha}^\delta - x_{n,\alpha}^\delta\| \le \beta_n\kappa_0\|x_{n,\alpha}^\delta - x_{n-1,\alpha}^\delta\|^2/2 + \beta_n\kappa_0\|x_{n,\alpha}^\delta - x_{n-1,\alpha}^\delta\|^2. \tag{3.8}$$

This proves (3.5). Again since

$$\begin{aligned} F'(x_{n,\alpha}^\delta) &= F'(x_{n-1,\alpha}^\delta) + F'(x_{n,\alpha}^\delta) - F'(x_{n-1,\alpha}^\delta) \\ &= F'(x_{n-1,\alpha}^\delta)[I + F'(x_{n-1,\alpha}^\delta)^{-1}(F'(x_{n,\alpha}^\delta) - F'(x_{n-1,\alpha}^\delta))], \end{aligned} \tag{3.9}$$

$$F'(x_{n,\alpha}^\delta)^{-1} = [I + F'(x_{n-1,\alpha}^\delta)^{-1}(F'(x_{n,\alpha}^\delta) - F'(x_{n-1,\alpha}^\delta))]^{-1}F'(x_{n-1,\alpha}^\delta)^{-1}. \tag{3.10}$$

So if

$$\|F'(x_{n-1,\alpha}^\delta)^{-1}(F'(x_{n,\alpha}^\delta) - F'(x_{n-1,\alpha}^\delta))\| \le \beta_{n-1}\kappa_0 e_{n-1} = \gamma_{n-1} < 1,$$

then

$$\beta_n \le \beta_{n-1}(1 - \gamma_{n-1})^{-1} \tag{3.11}$$

and by (3.5)

$$e_n \le 3\kappa_0\beta_{n-1}(1 - \gamma_{n-1})^{-1}e_{n-1}^2/2 \tag{3.12}$$

$$= 3\gamma_{n-1}(1 - \gamma_{n-1})^{-1}e_{n-1}/2 \tag{3.13}$$

$$= d_{n-1}e_{n-1}/2. \tag{3.14}$$

Again by (3.11) and (3.13),

$$\gamma_n = \kappa_0 e_n \beta_n \le (3/2)\kappa_0 \gamma_{n-1}(1-\gamma_{n-1})^{-1} e_{n-1}\cdot \beta_{n-1}(1-\gamma_{n-1})^{-1}$$
$$= (3/2)\gamma_{n-1}^2 (1-\gamma_{n-1})^{-2}. \tag{3.15}$$

The above relation together with $\gamma_0 = \kappa_0 e_0 \beta_0 < 1/4$ implies $\gamma_n < 1/4$. Consequently by (3.13),

$$e_n < e_{n-1}/2, \tag{3.16}$$

for all $n \ge 1$. So $e_n \le 2^{-n} e_0$, and hence

$$\|x^\delta_{n+1,\alpha} - x_0\| \le \sum_{j=0}^{n} \|x^\delta_{j+1,\alpha} - x^\delta_{j,\alpha}\| \le \sum_{j=0}^{n} 2^{-j} e_0 \le 2e_0 < r.$$

Thus $(x^\delta_{n,\alpha})$ is well defined and is a Cauchy sequence with $x^\delta_\alpha = \lim_{n\to\infty} x^\delta_{n,\alpha} \in \overline{B_\eta(x_0)} \subset B_r(x_0)$. So from (2.9), it follows that $F(x^\delta_\alpha) = z^\delta_\alpha$.

Further note that since $\gamma_n \le 1/4$, and by (3.15) we have

$$d_n = 3\gamma_n(1-\gamma_n)^{-1} < 4\gamma_n < 4\cdot(3/2)\gamma_{n-1}^2(1-\gamma_{n-1})^{-2} < d_{n-1}^2.$$

Hence

$$d_n \le d_0^{2^n}, \tag{3.17}$$

consequently, by (3.14), (3.16) and (3.17)

$$e_n \le d_{n-1} e_{n-1}/2 \le 2^{-n} d_0^{2^{n-1}} e_0.$$

Therefore

$$\|x^\delta_{n,\alpha} - x^\delta_\alpha\| = \lim_i \|x^\delta_{n,\alpha} - x^\delta_{n+i,\alpha}\| \le \sum_{j=n}^{\infty} e_j$$
$$\le \sum_{j=n}^{\infty} 2^{-j} d_0^{2^{j-1}} e_0 \le 2\cdot 2^{-n} d_0^{2^{n-1}} e_0 = \frac{2e_0 d_0^{2^{n-1}}}{2^n} \tag{3.18}$$
$$\le \frac{\eta d_0^{2^{n-1}}}{2^n} = \frac{\eta}{d_0 2^n}\,\mathrm{e}^{-\gamma 2^n} \le \frac{\eta}{d_0}\,\mathrm{e}^{-\gamma 2^n} = \beta\,\mathrm{e}^{-\gamma 2^n}.$$

This completes the proof. □

Remark 3.2. Note that $\gamma > 0$ because $\gamma_0 < 1/4 \implies d_0 < 1$. So by (1.5), sequence $(x^\delta_{n,\alpha})$ converges quadratically to x^δ_α.

Theorem 3.3. *Suppose that* (2.1), (3.2) *and* (3.3) *hold. If, in addition,* $\|x_0 - \hat{x}\| \le \eta < r < 1/(\beta_0\kappa_0)$, *then*

$$\|\hat{x} - x^\delta_\alpha\| \le \frac{\beta_0}{1-\beta_0\kappa_0 r}\|F(\hat{x}) - z^\delta_\alpha\|.$$

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Proof. Observe that

$$\begin{aligned}\|\hat{x}-x_\alpha^\delta\| &= \|\hat{x}-x_\alpha^\delta + F'(x_0)^{-1}[F(x_\alpha^\delta)-F(\hat{x})+F(\hat{x})-z_\alpha^\delta]\| \\ &\le \|F'(x_0)^{-1}[F'(x_0)(\hat{x}-x_\alpha^\delta)-(F(\hat{x})-F(x_\alpha^\delta))]\| \\ &\quad + \|F'(x_0)^{-1}[F(\hat{x})-z_\alpha^\delta]\| \\ &\le \beta_0\kappa_0 r\|\hat{x}-x_\alpha^\delta\| + \beta_0\|F(\hat{x})-z_\alpha^\delta\|.\end{aligned}$$

Thus

$$(1-\beta_0\kappa_0 r)\|\hat{x}-x_\alpha^\delta\| \le \beta_0\|F(\hat{x})-z_\alpha^\delta\|.$$

This completes the proof. □

Remark 3.4. If z_α^δ is as in (2.8) and if

$$\|F(x_0)-F(\hat{x})\| + \frac{\delta}{\sqrt{\alpha}} < \frac{r}{2\beta_0} < \frac{1}{2\beta_0^2\kappa_0}$$

then

$$\|x_0-\hat{x}\| \le \eta < r < \frac{1}{\beta_0\kappa_0}$$

holds (see Section 5).

The following theorem is a consequence of Theorem 3.1 and Theorem 3.3.

Theorem 3.5. *Suppose that* (2.1), (3.2) *and* (3.3) *hold. If in addition* $\beta_0\kappa_0 r < 1$, *then*

$$\|\hat{x}-x_{n,\alpha}^\delta\| \le \frac{\beta_0}{1-\beta_0\kappa_0 r}\|F(\hat{x})-z_\alpha^\delta\| + \frac{\eta d_0^{2^{n-1}}}{2^n}.$$

Remark 3.6. Hereafter we consider z_α^δ as the Tikhonov regularization of (2.5) given in (2.8). All results in the forthcoming sections are valid for the simplified regularization of (2.5).

In view of the estimate in Theorem 3.5, the next task is to find an estimate $\|F(\hat{x})-z_\alpha^\delta\|$. For this, let us introduce the notation

$$z_\alpha := F(x_0) + (K^*K+\alpha I)^{-1}K^*(y-KF(x_0)).$$

We may observe that

$$\|F(\hat{x})-z_\alpha^\delta\| \le \|F(\hat{x})-z_\alpha\| + \|z_\alpha - z_\alpha^\delta\| \le \|F(\hat{x})-z_\alpha\| + \delta/\sqrt{\alpha}, \tag{3.19}$$

and

$$\begin{aligned}F(\hat{x})-z_\alpha &= F(\hat{x})-F(x_0)-(K^*K+\alpha I)^{-1}K^*K[F(\hat{x})-F(x_0)] \\ &= [I-(K^*K+\alpha I)^{-1}K^*K][F(\hat{x})-F(x_0)] \\ &= \alpha(K^*K+\alpha I)^{-1}[F(\hat{x})-F(x_0)].\end{aligned} \tag{3.20}$$

Note that for $u \in R(K^*K)$ with $u = K^*Kz$ for some $z \in Z$,

$$\|\alpha(K^*K + \alpha I)^{-1}u\| = \|\alpha(K^*K + \alpha I)^{-1}K^*Kz\| \leq \alpha\|z\| \to 0$$

as $\alpha \to 0$. Now since $\|\alpha(K^*K + \alpha I)^{-1}\| \leq 1$ for all $\alpha > 0$, it follows that for every $u \in \overline{R(K^*K)}$, we have $\|\alpha(K^*K + \alpha I)^{-1}u\| \to 0$ as $\alpha \to 0$. Thus we achieve the following theorem.

Theorem 3.7. *If* $F(\hat{x}) - F(x_0) \in \overline{R(K^*K)}$, *then* $\|F(\hat{x}) - z_\alpha\| \to 0$ *as* $\alpha \to 0$.

4. Error bounds under source conditions

In view of the above theorem, we assume that

$$\|F(\hat{x}) - z_\alpha\| \leq \varphi(\alpha) \tag{4.1}$$

for some positive monotonic increasing function φ defined on $(0, \|K\|^2]$ such that

$$\lim_{\lambda \to 0} \varphi(\lambda) = 0.$$

Suppose φ is a source function in the sense that $\hat{x}$ satisfies a source condition of the form

$$F(\hat{x}) - F(x_0) = \varphi(K^*K)w, \qquad \|w\| \leq 1,$$

such that

$$\sup_{0<\lambda<\|K\|^2} \frac{\alpha\varphi(\lambda)}{\lambda + \alpha} \leq \varphi(\alpha), \tag{4.2}$$

then the assumption (4.1) is satisfied. Note that if $F(\hat{x}) - F(x_0) \in R((K^*K)^\nu)$, for some ν with, $0 < \nu \leq 1$, then by (3.20)

$$\begin{aligned} \|F(\hat{x}) - z_\alpha\| &\leq \|\alpha(K^*K + \alpha I)^{-1}(K^*K)^\nu\omega\| \\ &\leq \sup_{0<\lambda\leq\|K\|^2} \frac{\alpha\lambda^\nu}{\lambda + \alpha}\|\omega\| \leq \alpha^\nu\|\omega\|. \end{aligned}$$

Thus in this case $\varphi(\lambda) = \lambda^\nu/\|\omega\|$ satisfies the assumption (4.1). Therefore by (3.19) and by the assumption (4.1), we have

$$\|F(\hat{x}) - z_\alpha^\delta\| \leq \varphi(\alpha) + \delta/\sqrt{\alpha}. \tag{4.3}$$

So, we have the following theorem.

Theorem 4.1. *Under the assumptions of Theorem 3.5 and* (4.3)*,*

$$\|\hat{x} - x_{n,\alpha}^\delta\| \leq \frac{\beta_0}{1 - \beta_0\kappa_0 r}\Big(\varphi(\alpha) + \frac{\delta}{\sqrt{\alpha}}\Big) + \frac{\eta d_0^{2^{n-1}}}{2^n}.$$

4.1. A priori choice of the parameter

Note that the estimate $\varphi(\alpha) + \delta/\sqrt{\alpha}$ in (4.2) attains minimum for the choice $\alpha := \alpha_\delta$ which satisfies $\varphi(\alpha_\delta) = \delta/\sqrt{\alpha_\delta}$. Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \le \|K\|^2$. Then we have $\delta = \sqrt{\alpha_\delta}\varphi(\alpha_\delta) = \psi(\varphi(\alpha_\delta))$, and

$$\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta)). \tag{4.4}$$

So the relation (4.3) leads to

$$\|F(\hat{x}) - z_\alpha^\delta\| \le 2\psi^{-1}(\delta).$$

Theorem 4.1 and the above observation leads to the following.

Theorem 4.2. *Let $\psi(\lambda) := \lambda\sqrt{\varphi^{-1}(\lambda)}$, $0 < \lambda \le \|K\|^2$, and the assumptions of Theorem 3.5 and 4.1 be satisfied. For $\delta > 0$, let $\alpha_\delta = \varphi^{-1}(\psi^{-1}(\delta))$. If*

$$n_\delta := \min\{n : rd_0^{2^{n-1}}/2^n < \delta/\sqrt{\alpha_\delta}\},$$

then

$$\|\hat{x} - x_{\alpha_\delta, n_\delta}^\delta\| = O(\psi^{-1}(\delta)).$$

4.2. An adaptive choice of the parameter

The error estimate in the above Theorem has optimal order with respect to δ. Unfortunately, an a priori parameter choice (4.4) cannot be used in practice since the smoothness properties of the unknown solution $\hat{x}$ reflected in the function φ are generally unknown. There exist many parameter choice strategies in the literature, for example see [2, 8, 9, 12, 13, 24, 26].

In [21], Pereverzev and Schock considered an adaptive selection of the parameter which does not involve even the regularization method in an explicit manner. In this method the regularization parameter α_i are selected from some finite set $\{\alpha_i : 0 < \alpha_0 < \alpha_1 < \cdots < \alpha_N\}$ and the corresponding regularized solution, say $u_{\alpha_i}^\delta$ are studied on-line. Later George and Nair [14] considered the adaptive selection of the parameter for choosing the regularization parameter in Newton–Lavrentiev regularization method for solving Hammerstein-type operator equation. In this paper also, we consider the adaptive method for selecting the parameter α in $x_{\alpha,n}^\delta$. Rest of this section is essentially a reformulation of the adaptive method considered in [21] in a special context.

Let $i \in \{0, 1, 2, \dots, N\}$ and $\alpha_i = \mu^{2i}\alpha_0$ where $\mu > 1$ and $\alpha_0 = \delta^2$. Let

$$l := \max\{i : \varphi(\alpha_i) \le \delta/\sqrt{\alpha_i}\} \tag{4.5}$$

and

$$k := \max\{i : \|z_{\alpha_i}^\delta - z_{\alpha_j}^\delta\| \le 4\delta/\sqrt{\alpha_j},\ j = 0, 1, 2, \dots, i\}. \tag{4.6}$$

The proof of the next theorem is analogous to the proof of Theorem 1.2 in [21], but for the sake of completeness, we supply its proof as well.

Theorem 4.3. *Let l be as in* (4.5)*, k be as in* (4.6) *and $z^\delta_{\alpha_k}$ be as in* (2.8) *with $\alpha = \alpha_k$. Then $l \le k$ and*

$$\|F(\hat{x}) - z^\delta_{\alpha_k}\| \le \Big(2 + \frac{4\mu}{\mu - 1}\Big)\mu\psi^{-1}(\delta).$$

Proof. Note that, to prove $l \le k$, it is enough to prove that, for $i = 1, 2, \ldots, N$

$$\varphi(\alpha_i) \le \frac{\delta}{\sqrt{\alpha_i}} \quad \Longrightarrow \quad \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \le \frac{4\delta}{\sqrt{\alpha_j}}, \quad \forall j = 0, 1, 2, \ldots, i.$$

For $j \le i$,

$$\begin{aligned}\|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| &\le \|z^\delta_{\alpha_i} - F(\hat{x})\| + \|F(\hat{x}) - z^\delta_{\alpha_j}\| \\ &\le \varphi(\alpha_i) + \frac{\delta}{\sqrt{\alpha_i}} + \varphi(\alpha_j) + \frac{\delta}{\sqrt{\alpha_j}} \le \frac{2\delta}{\sqrt{\alpha_i}} + \frac{2\delta}{\sqrt{\alpha_j}} \le \frac{4\delta}{\sqrt{\alpha_j}}.\end{aligned}$$

This proves the relation $l \le k$. Now since $\sqrt{\alpha_{l+m}} = \mu^m\sqrt{\alpha_l}$, by using triangle inequality successively, we obtain

$$\begin{aligned}\|F(\hat{x}) - z^\delta_{\alpha_k}\| &\le \|F(\hat{x}) - z^\delta_{\alpha_l}\| + \sum_{j=l+1}^{k} \frac{4\delta}{\sqrt{\alpha_{j-1}}} \\ &\le \|F(\hat{x}) - z^\delta_{\alpha_l}\| + \sum_{m=0}^{k-l-1} \frac{4\delta}{\sqrt{\alpha_l}\mu^m} \le \|F(\hat{x}) - z^\delta_{\alpha_l}\| + \Big(\frac{\mu}{\mu - 1}\Big)\frac{4\delta}{\sqrt{\alpha_l}}.\end{aligned}$$

Therefore by (4.2) and (4.5) we have

$$\|F(\hat{x}) - z^\delta_{\alpha_k}\| \le \varphi(\alpha_l) + \frac{\delta}{\sqrt{\alpha_l}} + \Big(\frac{\mu}{\mu - 1}\Big)\frac{4\delta}{\sqrt{\alpha_l}} \le \Big(2 + \frac{4\mu}{\mu - 1}\Big)\mu\psi^{-1}(\delta).$$

The last step follows from the inequality $\sqrt{\alpha_\delta} \le \sqrt{\alpha_{l+1}} \le \mu\sqrt{\alpha_l}$ and $\delta/\sqrt{\alpha_\delta} = \psi^{-1}(\delta)$. This completes the proof. □

5. Stopping rule

Note that

$$\begin{aligned}e_0 = \|x^\delta_{1,\alpha} - x_0\| &= \|F'(x_0)^{-1}(K^*K + \alpha I)^{-1}K^*(y^\delta - KF(x_0))\| \\ &= \|F'(x_0)^{-1}(K^*K + \alpha I)^{-1}K^*(y^\delta - y + y - KF(x_0))\| \\ &\le \beta_0(\|(K^*K + \alpha I)^{-1}K^*(y^\delta - y)\| \\ &\quad + \|(K^*K + \alpha I)^{-1}K^*K(F(\hat{x}) - F(x_0))\|) \\ &\le \beta_0(\omega + \delta/\sqrt{\alpha}\,),\end{aligned}$$

so if

$$\omega + \frac{\delta}{\sqrt{\alpha}} < \frac{1}{2\beta_0} \min\Big\{r, \frac{1}{2\beta_0\kappa_0}\Big\}, \tag{5.1}$$

then $2e_0 \le 2\beta_0(\omega + \delta/\sqrt{\alpha}\,) < r$, and

$$\gamma_0 = e_0\beta_0\kappa_0 < 1/4.$$

Again since $\alpha_j = \mu^{2j}\delta^2$, $\delta/\sqrt{\alpha_k} = \mu^{-k}$; the condition (5.1) with $\alpha = \alpha_k$ takes the form

$$\omega + \frac{1}{\mu^k} < \frac{1}{2\beta_0} \min\Big\{r, \frac{1}{2\beta_0\kappa_0}\Big\}. \tag{5.2}$$

Note that if we assume that $2\beta_0\kappa_0 r < 1$. Then condition (5.2) takes the form $\omega + 1/\mu^k < r/(2\beta_0)$. So if we assume

$$r < 2\beta_0(1+\omega), \qquad \frac{1}{\mu} + \omega < \frac{r}{2\beta_0}$$

then $\mu > 1$ and (3.2) and (3.3) hold. The above discussion leads to the following theorem.

Theorem 5.1. *Assume that* $\mu > 2\beta_0/(r - 2\beta_0\omega)$, $2\beta_0\omega < r < \min\{2\beta_0(1+\omega), 1/(2\beta_0\kappa_0)\}$. *Let* $\alpha_0 = \delta^2$, $\alpha_j = \mu^{2j}\delta^2$ *for* $j = 1, 2, \ldots, N$ *and* $k := \max\{i\colon \|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| \le 4\mu^{-j},\ j = 0, 1, 2, \ldots, i\}$. *Then*

$$\|F(\hat{x}) - z^\delta_{\alpha_k}\| \le \Big(2 + \frac{4\mu}{\mu - 1}\Big)\mu\psi^{-1}(\delta)$$

where $\psi(t) = t\sqrt{\varphi^{-1}(t)}$ *for* $0 < t < \|K\|^2$. *Further* $\gamma_k := \beta_k e_k \kappa_0 < 1/4$ *and if*

$$n_k := \min\Big\{n : \frac{r d_0^{2^{n-1}}}{2^n} < \frac{1}{\mu^k}\Big\},$$

then

$$\|\hat{x} - x^\delta_{n_k,\alpha_k}\| = O(\psi^{-1}(\delta)).$$

Algorithm: Note that for $i, j \in \{0, 1, 2, \ldots, n\}$

$$\|z^\delta_{\alpha_i} - z^\delta_{\alpha_j}\| = (\alpha_j - \alpha_i)(K^*K + \alpha_j I)^{-1}(K^*K + \alpha_i I)^{-1}K^*(y^\delta - KF(x_0)).$$

Therefore the adaptive algorithm associated with the choice of the parameter specified in the above theorem is as follows.

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```
begin
  i = 0
    repeat
      i = i + 1
      solve for w_i:  (K*K + α_i I) w_i = K*(y^δ − KF(x_0))
      j = −1
      repeat
        j = j + 1
        solve for z_{i,j}:  (K*K + α_j I) z_{i,j} = (α_j − α_i) w_i
      until ‖z_{i,j}‖ ≤ 4μ^{−j} and j < i
    until ‖z_{i,j}‖ ≤ 4μ^{−j}
  k = i − 1
  m = 0
  repeat
    m = m + 1
  until r d_0^{2^{m−1}} / 2^m > 1/μ^k
  n_k = m
  for l = 1 to n_k
    solve for u_{l−1}:  F'(x^δ_{l−1,α_k}) u_{l−1} = F(x^δ_{l−1,α_k}) − z^δ_{α_k}
    x^δ_{l,α_k} := x^δ_{l−1,α_k} − u_{l−1}
end
```

Remark 5.2. We have considered an iterative regularization method, which is a combination of Newton iterative method with a Tikhonov regularization method, for obtaining approximate solution for a nonlinear Hammerstein-type operator equation $AF(x) = y$, with the available data y^δ in place of the exact data y. If the operator K is a positive self-adjoint bounded linear operator on a Hilbert space, then one may consider Newton Lavrentiev regularization method for obtaining an approximate solution for $KF(x) = y$. It is, assumed that the Fréchet derivative $F'(x)$ of the nonlinear operator F has a continuous inverse, in a neighborhood of some initial guess x_0 of the actual solution $\hat{x}$. The procedure involves solving the equation

$$(K^*K + \alpha I)u_\alpha^\delta = K^*(y^\delta - KF(x_0))$$

and finding the fixed point of the function

$$G(x) = x - F'(x)^{-1}(F(x) - F(x_0) - u_\alpha^\delta)$$

in an iterative manner. For choosing the regularization parameter α and the stopping index for the iteration, we made use of the adaptive method suggested in [21]. Further we observe that since $d_0 < 1$, the quantity $\eta d_0^{2^{n-1}}/2^n$ in Theorem 4.1 converges rapidly to zero.

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Author information

S. George, Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, Surathkal-575 025, India.
Email: sgeorge@nitk.ac.in

M. Kunhanandan, Department of Mathematics, Goa University, Goa-403 206, India.
Email: kunha@unigoa.ac.in